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PHILOSOPHICAL
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ROYAL SOCIETY OF LONDON.

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CONTAINING PAPERS OF A MATHEMATICAL OR PHYSICAL CHARACTER.

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A D V E R T I S E M E N T.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society ; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices ; which in some instances have been too lightly credited, to the dishonour of the Society.

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PHILOSOPHICAL TRANSACTIONS.

I. *Mathematical Contributions to the Theory of Evolution.*—XI. *On the Influence of Natural Selection on the Variability and Correlation of Organs.*

By KARL PEARSON, *F.R.S., University College, London.*

Received December 20, 1901,—Read January 23, 1902.

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(1.) *Introductory. On the Influence of Selection upon Correlation.*

AT an earlier stage in the development of the statistical theory of evolution it was suggested that the coefficient of correlation (GALTON'S function) might be found constant for all races of the same species—in fact, it was considered possible that this coefficient might be the long-sought-for criterion of identity in species. Professor WELDON, following up this suggestion of Mr. GALTON'S, then made the elaborate series of measurements on crabs with which his name will always be closely associated. To a first approximation these researches seemed to confirm the possibility of GALTON'S function being a true criterion of species. When, however, a finer mathematical test was applied to Professor WELDON'S observations as well as to other statistical series for organs in man,* it became clear that the coefficient of correlation varied from local race to local race, and could not be used as a criterion of species. A slight investigation undertaken in the summer of 1896 convinced me that the coefficient of correlation between any two organs, is just as much peculiar and characteristic of a local race as the means and variations of those organs. In fact, if local races be the outcome of natural selection, then their coefficients of correlation must in general differ. The object of the present paper is to show, not only that natural selection must determine the amount of correlation, but that it is probably the chief factor in the production of correlation. If selection, natural or artificial, be capable of producing correlation, then it seems impossible to regard all correlation as evidence of a causal nexus,† although the converse proposition that all causal nexus denotes correlation, is undoubtedly the most philosophical method of regarding causality.

In dealing with the influence of selection on correlation, I shall suppose the distribution of complex groups of organs to follow the normal correlation surface—the generalised Gaussian law of frequency. I shall further assume the selection surfaces to be normal in character. Neither of these assumptions is absolutely true, but the Gaussian law in a good many cases describes the frequency sufficiently closely to enable us to obtain fair numerical results by its application. Probably in all cases it will enable us to reach *qualitative* if not accurate quantitative theoretical deductions. I have the less hesitation in asserting this, as Mr. G. U. YULE has recently succeeded in deducing the chief formulæ for correlation and regression as given by the Gaussian law from general principles, which make no appeal to a special law of frequency.‡

* “Mathematical Contributions to the Theory of Evolution.—III.,” ‘Phil. Trans.,’ A, vol. 189, pp. 266 and 280. See also E. WARREN, “Variation in *Portunus depurator*,” ‘Roy. Soc. Proc.,’ vol. 60, pp. 233–4.

† See a series of letters in ‘Nature,’ vol. 54, 1896, arising from a discussion upon a paper by A. R. WALLACE.

‡ ‘Roy. Soc. Proc.,’ vol. 60, p. 477.

In particular, if a selected group be not given with very great accuracy by a normal frequency surface, still we may, I think, consider ourselves justified in supposing that the effects of the actual selection, and those of a normal selection with the same means, the same amounts of variation, and with correlations of the same intensity will be at least qualitatively alike in character, if they be not indeed exactly the same quantitatively within the limits fixed by the probable errors of the constants.

My plan in this memoir will be as follows:—I shall first state the fundamental theorem in multiple correlation with a new proof, so that the formulæ required may be once for all collected for reference.* I shall then give the algebraic investigation of the new formulæ for selection. I shall afterwards consider special simple cases, and illustrate them by examples. Finally I shall draw attention to the nature of the selective death-rate as indicated in cases of this kind, and consider at length its algebraic theory. Throughout I shall endeavour to illustrate the somewhat complex algebra by arithmetical examples.

(2.) *On the Fundamental Theorem in Multiple Correlation.*

I have shown in my memoir on “Regression, Heredity, and Panmixia” (‘Phil. Trans.’ A, vol. 187, p. 261) that if the n variables of a complex be functions of m ($m > n$) independent variables with frequency distributions following the normal law, and such that the principle of superposition holds for the deviations from the means supposed small; then the frequency of the complex with deviations from the means of the n variables lying between x_1 and $x_1 + \delta x_1$, x_2 and $x_2 + \delta x_2 \dots x_n$ and $x_n + \delta x_n$ will be $z \delta x_1 \delta x_2 \dots \delta x_n$ where z :

$$z = z_0 \text{ expt. } - \frac{1}{2} (S_1 (c_{pp} x_p^2) + 2S_2 (c_{pq} x_p x_q)) \dots \dots \dots (i).$$

Here z_0 , c_{pp} , $c_{pq} \dots$ are constants, and S_1 denotes a summation for every value of p , and S_2 for every pair of values of p and q in the series from 1 to n .

In the same memoir (p. 302) I have determined the values of z_0 , c_{pp} , c_{pq} in terms of the correlations r_{pq} and the standard deviations σ_p and σ_q of the n variables. This point had already been considered by Professor EDGEWORTH (‘Phil. Mag.’ vol. 34, p. 201, 1892), and some further results by Mr. A. BLACK, reached before his death in 1893, were published in the ‘Camb. Phil. Trans.’ (vol. 16, p. 219, 1897). The present investigation is, I think, novel, and adds to results already reached others required in the present memoir, so that it thus places together with a fairly simple proof all the fundamental results to which I shall have occasion to appeal later.

* We have used these formulæ for several years, but they do not appear to have been hitherto published in a collected form.

Let us consider the quadric of the n^{th} order

$$Q = S_1(c_{pp}x_p^2) + 2S_2(c_{pq}x_px_q) = \text{constant},$$

and fix our attention on two of the variables, say the first two, x_1 and x_2 . If these be considered constants, the quadric of the remaining $n - 2$ variables will not now be referred to its "centre." But its centre, $\bar{x}_3, \bar{x}_4 \dots \bar{x}_n$, will be given by the equations:

$$\begin{aligned} c_{13}x_1 + c_{23}x_2 + c_{33}\bar{x}_3 + c_{43}\bar{x}_4 + \dots + c_{n3}\bar{x}_n &= 0. \\ c_{14}x_1 + c_{24}x_2 + c_{34}\bar{x}_3 + c_{44}\bar{x}_4 + \dots + c_{n4}\bar{x}_n &= 0. \\ \dots & \\ c_{1n}x_1 + c_{2n}x_2 + c_{3n}\bar{x}_3 + c_{4n}\bar{x}_4 + \dots + c_{nn}\bar{x}_n &= 0. \end{aligned} \quad \text{(ii).}$$

The following expressions will not be zero, but will be written α and β :

$$\begin{aligned} c_{11}x_1 + c_{21}x_2 + c_{31}\bar{x}_3 + c_{41}\bar{x}_4 + \dots + c_{n1}\bar{x}_n &= \alpha. \\ c_{12}x_1 + c_{22}x_2 + c_{32}\bar{x}_3 + c_{42}\bar{x}_4 + \dots + c_{n2}\bar{x}_n &= \beta. \end{aligned} \quad \text{(iii).}$$

Now, if Δ be the discriminant,

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \dots & c_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{vmatrix} \dots \dots \dots \text{(iv).}$$

and C_{pq} the minor corresponding to c_{pq} , we have by solving the above n linear equations,

$$x_1 = \frac{\alpha C_{11} + \beta C_{12}}{\Delta}, \quad x_2 = \frac{\alpha C_{12} + \beta C_{22}}{\Delta},$$

$$\text{whence:} \quad \alpha = \frac{\Delta(x_1 C_{22} - x_2 C_{12})}{C_{11}C_{22} - C_{12}^2}, \quad \beta = \frac{\Delta(x_2 C_{11} - x_1 C_{12})}{C_{11}C_{22} - C_{12}^2} \dots \dots \dots \text{(v).}$$

Generally also:

$$\begin{aligned} \bar{x}_q &= \frac{\alpha C_{1q} + \beta C_{2q}}{\Delta} \\ &= \frac{(C_{1q}C_{22} - C_{2q}C_{12})x_1 + (C_{11}C_{2q} - C_{1q}C_{12})x_2}{C_{11}C_{22} - C_{12}^2} \\ &= D_{1q}x_1 + D_{2q}x_2, \quad \text{say,} \dots \dots \dots \text{(vi).} \end{aligned}$$

must be the correlation surface for x_1 and x_2 , for all values of $x_3, x_4, \dots, x_n$. We may write it in the form

$$z = \zeta_0 \text{ expt. } - \frac{1}{2} \Delta \frac{r_1^2 C_{22} - 2C_{12} r_1 x_2 + C_{11} x_2^2}{C_{11} C_{22} - C_{12}^2} \dots \dots \dots \text{ (viii.)}$$

Comparing this with the known form of the correlation surface for two variables,*

$$z = \frac{N}{2\pi\sigma_1\sigma_2\sqrt{1-r_{12}^2}} \text{ expt. } - \frac{1}{2} \frac{1}{1-r_{12}^2} \left(\frac{x_1^2}{\sigma_1^2} - \frac{2r_{12}x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2} \right) \dots \dots \text{ (ix.)}$$

we have at once

$$\sigma_1^2(1-r_{12}^2) = \frac{C_{11}C_{22}-C_{12}^2}{\Delta C_{22}}, \quad \sigma_2^2(1-r_{12}^2) = \frac{C_{11}C_{22}-C_{12}^2}{\Delta C_{11}},$$

$$r_{12}\sigma_1\sigma_2 = C_{12}/\Delta \dots \dots \dots \text{ (x.)}$$

Whence

$$r_{12} = C_{12}/\sqrt{C_{11}C_{22}} \dots \dots \dots \text{ (xi.)}$$

$$\sigma_1^2 = C_{11}/\Delta, \quad \sigma_2^2 = C_{22}/\Delta \dots \dots \dots \text{ (xii.)}$$

Or, generally :

$$r_{pq} = C_{pq}/\sqrt{C_{pp}C_{qq}} \dots \dots \dots \text{ (xiii.)}$$

$$\sigma_p = \sqrt{C_{pp}/\Delta} \dots \dots \dots \text{ (xiv.)}$$

Thus correlations and variations are fully determined in terms of the discriminant and its minor for the constants $c_{11}, c_{22}, c_{12} \dots c_{pp}, c_{qq}, c_{pq}, \dots c_{nn}$.

We have next the inverse proposition to find the c 's in terms of the r 's and σ 's.

We have, by well-known propositions in the theory of determinants :

$$c_{11}C_{11} + c_{12}C_{12} + c_{13}C_{13} + \dots \dots + c_{1n}C_{1n} = \Delta.$$

$$c_{11}C_{21} + c_{12}C_{22} + c_{13}C_{23} + \dots \dots + c_{1n}C_{2n} = 0.$$

$$c_{11}C_{31} + c_{12}C_{32} + c_{13}C_{33} + \dots \dots + c_{1n}C_{3n} = 0.$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$c_{11}C_{n1} + c_{12}C_{n2} + c_{13}C_{n3} + \dots \dots + c_{1n}C_{nn} = 0.$$

Or,

$$c_{11}\sigma_1^2 + c_{12}r_{12}\sigma_1\sigma_2 + c_{13}r_{13}\sigma_1\sigma_3 + \dots \dots + c_{1n}r_{1n}\sigma_1\sigma_n = 1,$$

$$c_{11}r_{n1}\sigma_1\sigma_2 + c_{12}\sigma_2^2 + c_{13}r_{23}\sigma_2\sigma_3 + \dots \dots + c_{1n}r_{2n}\sigma_2\sigma_n = 0,$$

$$c_{11}r_{31}\sigma_1\sigma_3 + c_{12}r_{32}\sigma_2\sigma_3 + c_{13}\sigma_3^2 + \dots \dots + c_{1n}r_{3n}\sigma_3\sigma_n = 0.$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$c_{11}r_{n1}\sigma_1\sigma_n + c_{12}r_{n2}\sigma_2\sigma_n + c_{13}r_{n3}\sigma_3\sigma_n + \dots \dots + c_{1n}\sigma_n^2 = 0.$$

Hence solving

$$c_{11} = S_{11}/S, \quad c_{1q} = S_{1q}/S \dots \dots \dots \text{ (xv.)}$$

* PEARSON, 'Phil. Trans.,' A, vol. 187, p. 264.

where $S =$

$$\begin{vmatrix} \sigma_1^2, & r_{12}\sigma_1\sigma_2, & r_{13}\sigma_1\sigma_3, & . & . & . & r_{1n}\sigma_1\sigma_n \\ r_{21}\sigma_1\sigma_2, & \sigma_2^2, & r_{23}\sigma_2\sigma_3, & . & . & . & r_{2n}\sigma_2\sigma_n \\ r_{31}\sigma_1\sigma_3, & r_{32}\sigma_2\sigma_3, & \sigma_3^2, & . & . & . & r_{3n}\sigma_3\sigma_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ r_{n1}\sigma_1\sigma_n, & r_{n2}\sigma_2\sigma_n, & r_{n3}\sigma_3\sigma_n, & . & . & . & \sigma_n^2 \end{vmatrix}$$

and S_{pq} is the minor corresponding to the constituent $r_{pq}\sigma_p\sigma_q$.

But

$$S = \sigma_1^2 \sigma_2^2 \sigma_3^2 \dots \sigma_n^2 R,$$

$$S_{pq} = \sigma_1^2 \sigma_2^2 \sigma_3^2 \dots \sigma_p \sigma_q \dots \sigma_n^2 R_{pq},$$

where R is the determinant,

$$\begin{vmatrix} 1, & r_{12}, & r_{13}, & . & . & . & r_{1n} \\ r_{21}, & 1, & r_{23}, & . & . & . & r_{2n} \\ r_{31}, & r_{32}, & 1, & . & . & . & r_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ r_{n1}, & r_{n2}, & r_{n3}, & . & . & . & 1 \end{vmatrix} \dots \dots \dots \quad (\text{xvi.}),$$

and R_{pq} is the minor corresponding to the constituent r_{pq} . Thus we have

$$c_{11} = \frac{1}{\sigma_1^2} R_{11}/R, \quad c_{1q} = \frac{1}{\sigma_1 \sigma_q} R_{1q}/R \quad \dots \dots \dots \quad (\text{xvii.}),$$

or, generally,

$$c_{pp} = \frac{1}{\sigma_p^2} \frac{R_{pp}}{R}, \quad c_{pq} = \frac{1}{\sigma_p \sigma_q} \frac{R_{pq}}{R} \quad \dots \dots \dots \quad (\text{xviii.}).$$

Thus z may be written*

$$z = z_0 \text{ expt. } - \frac{1}{2} \left\{ S_1 \left(\frac{R_{pp}}{R} \frac{x'_p{}^2}{\sigma_p^2} \right) + 2S_2 \left(\frac{R_{pq}}{R} \frac{x'_p x'_q}{\sigma_p \sigma_q} \right) \right\} \quad \dots \dots \dots \quad (\text{xix.}).$$

It remains to determine z_0 from the fact that the volume of the surface = N .

Or,

$$N = z_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \text{expt. } (-\frac{1}{2}Q) dx_1 dx_2 \dots dx_n$$

$$= z_0 \sigma_1 \sigma_2 \dots \sigma_n \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \text{expt. } -\frac{1}{2} \left\{ S_1 \left(\frac{R_{pp}}{R} x'_p{}^2 \right) + 2S_2 \left(\frac{R_{pq}}{R} x'_p x'_q \right) \right\}$$

$$dx'_1 dx'_2 \dots dx'_n \quad \dots \dots \dots \quad (\text{xx.}),$$

by writing $x_p/\sigma_p = x'_p$, &c.

Now, integrate first with regard to x'_1 writing first

* 'Phil. Trans.,' A, vol. 187, p. 302.

Clearly, when $x'_2, x'_3, \dots, x'_n$ are constants, the distribution of x'_1 is of the form

$$\text{constant} \times \text{expt.} - \frac{1}{2} \frac{R_{11}}{R} \left\{ x'_1 + \Sigma_1 \left(\frac{R_{1p}}{R_{11}} x'_p \right) \right\}^2.$$

Or, re-introducing the $\sigma_1, \sigma_2, \dots, \sigma_n$, we have a distribution about the point given by

$$\bar{x}_1 = - \sigma_1 \left(\frac{R_{12}}{R_{11}} \frac{x_2}{\sigma_2} + \frac{R_{13}}{R_{11}} \frac{x_3}{\sigma_3} + \dots + \frac{R_{1n}}{R_{11}} \frac{x_n}{\sigma_n} \right), \quad \dots \quad (\text{xxiv}).$$

with standard deviation $\sigma'_1 = \sigma_1 \sqrt{R/R_{11}}$

(b) Given n variables, what are the mean values $m_1 + \bar{x}_1, m_2 + \bar{x}_2$, the standard deviations $\bar{\sigma}''_1, \bar{\sigma}''_2$, and the correlation $\bar{r}_{12}$ of two of them, when we give definite values $m_3 + h_3, m_4 + h_4, \dots, m_n + h_n$ to the remaining $(n - 2)$ variables?

In this case we have from (i.)

$$\begin{aligned} z = \text{expt.} - \frac{1}{2} \{ & c_{11}x_1^2 + 2c_{12}x_1x_2 + c_{22}x_2^2 \\ & + 2(c_{13}h_3 + c_{14}h_4 + \dots + c_{1n}h_n)x_1 \\ & + 2(c_{23}h_3 + c_{24}h_4 + \dots + c_{2n}h_n)x_2 \\ & + \text{terms not involving } x_1 \text{ and } x_2 \} \quad \dots \quad (\text{xxv}). \end{aligned}$$

Writing K_1 for the coefficient of x_1 , and K_2 for that of x_2 , we have for the centre

$$\begin{aligned} \bar{x}_1 &= - \frac{(K_1c_{22} - K_2c_{12})}{c_{11}c_{22} - c_{12}^2}, & \bar{x}_2 &= - \frac{(-K_1c_{12} + c_{11}K_2)}{c_{11}c_{22} - c_{12}^2}, \\ \bar{x}_1 &= - \Sigma_1 \left(\frac{c_{1p}c_{22} - c_{2p}c_{12}}{c_{11}c_{22} - c_{12}^2} h_p \right), & \bar{x}_2 &= - \Sigma_1 \left(\frac{c_{2p}c_{11} - c_{1p}c_{12}}{c_{11}c_{22} - c_{12}^2} h_p \right), \\ \bar{x}_1 &= - \sigma_1 \Sigma_1 \left(\frac{R_{1p}R_{22} - R_{2p}R_{12}}{R_{11}R_{22} - R_{12}^2} \frac{h_p}{\sigma_p} \right), & \bar{x}_2 &= - \sigma_2 \Sigma_1 \left(\frac{R_{2p}R_{11} - R_{1p}R_{12}}{R_{11}R_{22} - R_{12}^2} \frac{h_p}{\sigma_p} \right), \end{aligned}$$

by transferring to the minors of R and the σ 's. Or,

$$\bar{x}_1 = - \sigma_1 \Sigma_1 \left(\frac{\rho''_{1p}}{R''} \frac{h_p}{\sigma_p} \right), \quad \bar{x}_2 = - \sigma_2 \Sigma_2 \left(\frac{\rho'_{2p}}{R''} \frac{h_p}{\sigma_p} \right) \quad \dots \quad (\text{xxvi}).$$

Here R'' is the determinant formed by striking out the first two rows and columns of R ; ρ''_{1p} is the minor obtained by striking out the second row and column from R , and then the first row and p^{th} column; ρ'_{2p} the minor obtained by striking out the first row and column, and then the second row and p^{th} column. But a comparison with (xxiv.) shows us that these values for $\bar{x}_1$ and $\bar{x}_2$ are precisely what we should have obtained for the regression equations of the 1st and 2nd variables respectively *alone* on the other $n - 2$ variables. Thus the existence and the correlations of x_2 have no effect on the value of $\bar{x}_1$, nor those of x_1 on the value of $\bar{x}_2$.

Returning to (xxv.), we remark that the terms of the second order in x_1 and x_2 , on which the correlation and variations depend, are not altered by a transfer to the centre $\bar{x}_1$ and $\bar{x}_2$ of the array.

Hence by (ix.) and (xvii.) we have

$$\bar{r}_{12} = -\frac{c_{12}}{\sqrt{c_{11}c_{22}}} = -R_{12}/\sqrt{R_{11}R_{22}} \quad \dots \quad \text{(xxvii.)}$$

This is the *partial* correlation of the 1st and 2nd organs for the remaining $n-2$ organs with constant values.

Again,

$$c_{11} = \frac{1}{\bar{\sigma}''_1{}^2(1 - \bar{r}_{12}{}^2)}, \quad c_{22} = \frac{1}{\bar{\sigma}''_2{}^2(1 - \bar{r}_{12}{}^2)}.$$

Whence we easily find from (xviii.) and (xxvii.)

$$\bar{\sigma}''_1{}^2 = \sigma_1{}^2 \frac{R R_{22}}{R_{11}R_{22} - R_{12}{}^2}, \quad \bar{\sigma}''_2{}^2 = \sigma_2{}^2 \frac{R R_{11}}{R_{11}R_{22} - R_{12}{}^2},$$

or,

$$\bar{\sigma}''_1 = \sigma_1 \sqrt{R_{22}/R''}, \quad \bar{\sigma}''_2 = \sigma_2 \sqrt{R_{11}/R''} \quad \dots \quad \text{(xxviii.)}$$

where R'' is, as before, the determinant R without its first two rows and columns. These by (xxiv.), are what we should have reached by ignoring x_2 in finding $\bar{\sigma}'_1$, and x_1 in finding $\bar{\sigma}'_2$.

(3.) *General Theorem in Selection.*

To find the selected means, the selected variations and selected correlations, when q organs are selected, naturally or artificially, out of a complex of n organs.

Let the selected group of q organs have their means raised $h_1, h_2, h_3, \dots, h_q$ (some of these quantities may be negative); their standard deviations changed from $\sigma_1, \sigma_2, \dots, \sigma_q$ to $s_1, s_2, s_3, \dots, s_q$, and their mutual correlations from $r_{12}, r_{13}, \dots, r_{1q}, r_{23}, r_{24}, \dots, r_{2q}, \dots, r_{q-1,q}$ to $\rho_{12}, \rho_{13}, \dots, \rho_{1q}, \rho_{23}, \rho_{24}, \dots, \rho_{2q}, \dots, \rho_{q-1,q}$.

The whole system of n organs before selection will be defined by the means as origin of measurement for each organ, by the standard deviations $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n$, and by the coefficients of correlation $r_{12}, r_{13}, \dots, r_{1n}, r_{23}, r_{24}, \dots, r_{2n}, \dots, r_{n-1,n}$. Let R be the determinant

$$\begin{vmatrix} 1 & r_{12} & r_{13} & \dots & r_{1n} \\ r_{21} & 1 & r_{23} & \dots & r_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ r_{n1} & r_{n2} & r_{n3} & \dots & 1 \end{vmatrix}$$

and R_{uv} the minor corresponding to the constituent r_{uv} . Then the unselected population is given by the frequency surface of equation (xix.),

$$z = \text{constant} \times \text{expt.} - \frac{1}{2} \left\{ S_1 \left(\frac{R_{pp}}{R} \frac{x_p^2}{\sigma_p^2} \right) + 2S'_2 \left(\frac{R_{pq}}{R} \frac{x_p x_q}{\sigma_p \sigma_q} \right) \right\} \quad \text{. . . (xxix.)}$$

For brevity, we can also write this in the form

$$z = \text{constant} \times \text{expt.} - \frac{1}{2} \{ S_1 (c_{pp} x_p^2) + 2S_2 (c_{pq} x_p x_q) \} \quad \text{. . . (xxix.)}^{\text{bis.}}$$

Now consider for the time only $q + 1$ organs—namely, the first q organs and the m^{th} organ ($m > q$), and let us write $R(u)$, if u be $> q$, for the determinant :

$$R(u) = \begin{vmatrix} 1, & r_{12}, & r_{13}, & . & . & . & r_{1q}, & r_{1u} \\ r_{21}, & 1, & r_{23}, & . & . & . & r_{2q}, & r_{2u} \\ r_{31}, & r_{32}, & 1, & . & . & . & r_{3q}, & r_{3u} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ r_{q1}, & r_{q2}, & r_{q3}, & . & . & . & 1, & r_{qu} \\ r_{u1}, & r_{u2}, & r_{u3}, & . & . & . & r_{uq}, & 1 \end{vmatrix} \quad \text{. . . (xxx.)}$$

Then if $R(u)_{p'p''}$ be the minor corresponding to the constituent $r_{p'p''}$, and if $b_{p'p''} = \frac{R(u)_{p'p''}}{R(u)} \frac{1}{\sigma_{p'} \sigma_{p''}}$, the distribution of the $q + 1$ organs will be given by the frequency distribution

$$z' = \text{constant} \times \text{expt.} - \frac{1}{2} \{ S_1 (b_{p'p'} x_{p'}^2) + 2S_2 (b_{p'p''} x_{p'} x_{p''}) \} \quad \text{. . . (xxxi.)}$$

S_1 being a sum for every value of p' throughout the $q + 1$ organs, and S_2 for every pair of values.

Now let the first q organs be given values $h_1, h_2, \dots, h_q$, then the mean value of x_u will be given by

$$\begin{aligned} x'_u &= - \left(\frac{b_{1u}}{b_{uu}} h_1 + \frac{b_{2u}}{b_{uu}} h_2 + \dots + \frac{b_{qu}}{b_{uu}} h_q \right), \\ &= - \left(\frac{R(u)_{1u}}{R(u)_{uu}} \frac{\sigma_u}{\sigma_1} h_1 + \frac{R(u)_{2u}}{R(u)_{uu}} \frac{\sigma_u}{\sigma_2} h_2 + \dots + \frac{R(u)_{qu}}{R(u)_{uu}} \frac{\sigma_u}{\sigma_q} h_q \right) \quad \text{. . . (xxxii.)} \end{aligned}$$

Now these coefficients can be found at once if q be known.

For example :

$$q = 1, \quad - \frac{R(u)_{qu}}{R(u)_{uu}} = r_{1u},$$

$$q = 2, \quad - \frac{R(u)_{1u}}{R(u)_{uu}} = \frac{r_{1u} - r_{2u} r_{12}}{1 - r_{12}^2}, \quad - \frac{R(u)_{2u}}{R(u)_{uu}} = \frac{r_{2u} - r_{1u} r_{12}}{1 - r_{12}^2},$$

$$q = 3, \quad - \frac{R(u)_{1u}}{R(u)_{uu}} = \frac{r_{1u}(1 - r_{23}^2) - r_{12} r_{2u} - r_{13} r_{3u} + r_{23}(r_{12} r_{3u} + r_{13} r_{2u})}{1 - r_{12}^2 - r_{23}^2 - r_{13}^2 + 2r_{12} r_{23} r_{13}}$$

and $-\frac{R(u)_{2u}}{R(u)_{uu}}$ and $-\frac{R(u)_{3u}}{R(u)_{uu}}$ can be written down by symmetry . . . (xxxiii.).

$$\begin{aligned} \Sigma_v \Sigma_u r_{vu} = -s_v \sigma_v \left\{ \frac{R(u)_{1u}}{R(u)_{uu}} \rho_{1v} \frac{s_1}{\sigma_1} + \frac{R(u)_{2u}}{R(u)_{uu}} \rho_{2v} \frac{s_2}{\sigma_2} + \dots + \frac{R(u)_{vu}}{R(u)_{uu}} \frac{s_v}{\sigma_v} + \dots \right. \\ \left. + \frac{R(u)_{qu}}{R(u)_{uu}} \rho_{qv} \frac{s_q}{\sigma_q} \right\} \quad \dots \quad (\text{xli.})^{\text{bis}}. \end{aligned}$$

The next stage in our work is to find α_{vu} when both v and u are $> q$, and also α_{uu} when $v = u$ and both are $> q$. This is done at once by substituting the minors $\alpha_{u1}, \alpha_{u2}, \dots, \alpha_{uq}, \dots, \alpha_{un}$ in the equations formed from the last $p-q$ lines of the determinant (xxxvii.).

We obtain the following system :

$$\begin{aligned} c_{1,q+1}\alpha_{u1} + c_{2,q+1}\alpha_{u2} + c_{3,q+1}\alpha_{u3} + \dots + c_{q,q+1}\alpha_{uq} + c_{q+1,q+1}\alpha_{u,q+1} + \dots + c_{n,q+1}\alpha_{un} &= 0 \\ c_{1,q+2}\alpha_{u1} + c_{2,q+2}\alpha_{u2} + c_{3,q+2}\alpha_{u3} + \dots + c_{q,q+2}\alpha_{uq} + c_{q+1,q+2}\alpha_{u,q+1} + \dots + c_{n,q+2}\alpha_{un} &= 0 \\ \dots & \\ c_{1,n}\alpha_{u1} + c_{2,n}\alpha_{u2} + c_{3,n}\alpha_{u3} + \dots + c_{q,n}\alpha_{uq} + c_{q+1,n}\alpha_{u,q+1} + \dots + c_{n,n}\alpha_{un} &= 1 \\ \dots & \\ c_{1,n}\alpha_{u1} + c_{2,n}\alpha_{u2} + c_{3,n}\alpha_{u3} + \dots + c_{q,n}\alpha_{uq} + c_{q+1,n}\alpha_{u,q+1} + \dots + c_{n,n}\alpha_{un} &= 0. \end{aligned}$$

These are identical with equations (xl.), except that the equation with the coefficients $c_{1n}, c_{2n}, \dots, c_{nn}$ has unity instead of zero on the right-hand side. Hence we see that α_{uv} ($v > q$) will be the same function of $\alpha_{u1}, \alpha_{u2}, \dots, \alpha_{uq}$ that x'_v is of $h_1, h_2, \dots, h_q$ in equation (xxxvi.), but it will add to this a function of the last $p-q$ system of c 's, i.e., the c 's

$$\begin{aligned} c_{q,q+1}, \quad c_{q+1,q+1}, \quad \dots \quad c_{n,q+1} \\ c_{q,q+2}, \quad c_{q+1,q+2}, \quad \dots \quad c_{n,q+2} \\ \dots \\ c_{q,n}, \quad c_{q+1,n}, \quad \dots \quad c_{n,n}. \end{aligned}$$

Whatever this function may be we will represent it for the time by γ_{uv} ; we notice that it is independent of the selected variations $s_1, s_2, \dots, s_q$, the selected means $h_1, h_2, \dots, h_q$, and the selected correlation coefficients $\rho_{12}, \rho_{23}, \dots, \rho_{q-1,q}$. It depends only on the characters *before* selection.

We thus have

$$\alpha_{uv} = \gamma_{uv} - \left\{ \frac{R(v)_{1v}}{R(v)_{vv}} \frac{\sigma_v}{\sigma_1} \alpha_{u1} + \frac{R(v)_{2v}}{R(v)_{vv}} \frac{\sigma_v}{\sigma_2} \alpha_{u2} + \dots + \frac{R(v)_{qv}}{R(v)_{vv}} \alpha_{uq} \right\}.$$

Now the system $\alpha_{u1}, \alpha_{u2}, \dots, \alpha_{uq}$ can be found from (xli.), since $\alpha_{u'u''} = \alpha_{u''u'}$ whatever u' and u'' be.

Hence we have

$$\alpha_{uv} = \gamma_{uv} + \sigma_v \sigma_u \left[S_1 \left\{ \frac{R(v)_{pv} R(u)_{pv} \left(\frac{s_p}{\sigma_p} \right)^2}{R(v)_{rv} R(u)_{ru} \left(\frac{s_p}{\sigma_p} \right)^2} \right\} \right. \\ \left. + S_2 \left\{ \left(\frac{R(v)_{p'v} R(u)_{p'u}}{R(v)_{rv} R(u)_{ru}} + \frac{R(v)_{p'v} R(u)_{p'u}}{R(v)_{rv} R(u)_{ru}} \right) \rho_{p'p''} \frac{s_{p'} s_{p''}}{\sigma_{p'} \sigma_{p''}} \right\} \right].$$

Here S_1 denotes a sum from $p = 1$ to $p = q$, and S_2 a sum for every pair of values of p' and p'' out of $1, 2, 3, \dots q$.

When $u = v$ we have simply

$$\alpha_{vv} = \gamma_{vv} + \sigma_v^2 \left[S_1 \left\{ \left(\frac{R(v)_{pv}}{R(v)_{rv}} \right)^2 \left(\frac{s_p}{\sigma_p} \right)^2 + 2S_2 \left(\frac{R(v)_{p'v} R(v)_{p'v}}{(R(v)_{rv})^2} \rho_{p'p''} \frac{s_{p'} s_{p''}}{\sigma_{p'} \sigma_{p''}} \right) \right\} \right].$$

It only remains to determine γ_{uv} and γ_{vv} . This we can do by putting all the s 's zero, or selecting our q organs of one size only. We see at once that γ_{uv} and γ_{vv} are the values of α_{uv} and α_{vv} , that is, of $\Sigma_p \Sigma_v r_{pvu}$, and Σ_v^2 , when we select q organs of definite values and seek the correlation and the variabilities of two others, the u^{th} and the v^{th} . These values have already been found on p. 10. Or :

$$\begin{aligned} \Sigma_u \Sigma_v r_{vu} &= \gamma_{uv} = -\sigma_u \sigma_v R(uv)_{uv} / R(q), \\ \Sigma_v^2 &= \gamma_{vv} = \sigma_v^2 R(uv)_{vv} / R(q), \\ \Sigma_u^2 &= \gamma_{uu} = \sigma_u^2 R(uv)_{uu} / R(q), \end{aligned} \quad \text{. (xlii).}$$

The notation of that page has been changed so that $R(uv)$ now stands for the determinant

$$\begin{vmatrix} 1, & r_{12}, & r_{13}, & . & . & . & r_{1q}, & r_{1u}, & r_{1v} \\ r_{21}, & 1, & r_{23}, & . & . & . & r_{2q}, & r_{2u}, & r_{2v} \\ r_{31}, & r_{32}, & 1, & . & . & . & r_{3q}, & r_{3u}, & r_{3v} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ r_{q1}, & r_{q2}, & r_{q3}, & . & . & . & 1, & r_{qu}, & r_{qv} \\ r_{u1}, & r_{u2}, & r_{u3}, & . & . & . & r_{uq}, & 1, & r_{uv} \\ r_{v1}, & r_{v2}, & r_{v3}, & . & . & . & r_{vq}, & r_{vu}, & 1 \end{vmatrix} \quad \text{. (xliii).}$$

$R(uv)_{uv}$ is the minor corresponding to the constituent r_{uv} ; $R(uv)_{uu}$, the minor corresponding to the constituent at the meet of the u^{th} column and u^{th} row; and $R(q)$ the determinant with the last two rows and two columns struck out.

For example :

$$\begin{aligned}
q = 1: \quad & \gamma_{vv} = \sigma_v^2 (1 - r_{1v}^2) \quad \gamma_{uu} = \sigma_u^2 (1 - r_{1u}^2), \\
& \gamma_{uv} = \sigma_v \sigma_u (r_{1v} - r_{1v} r_{1u}), \\
q = 2: \quad & \gamma_{vv} = \sigma_v^2 (1 - r_{12}^2 - r_{1v}^2 - r_{2v}^2 + 2r_{12}r_{1v}r_{2v}) / (1 - r_{12}^2), \\
& \gamma_{uu} = \sigma_u^2 (1 - r_{12}^2 - r_{1u}^2 - r_{2u}^2 + 2r_{12}r_{1u}r_{2u}) / (1 - r_{12}^2), \\
& \gamma_{uv} = \sigma_u \sigma_v \{ r_{1v} (1 - r_{12}^2) - r_{1v}r_{1u} - r_{2v}r_{2u} + r_{12}(r_{1u}r_{2v} + r_{2u}r_{1v}) \} / (1 - r_{12}^2), \\
q = 3: * \quad & \gamma_{vv} = \sigma_v^2 \times \begin{vmatrix} 1, & r_{12}, & r_{13}, & r_{1v} \\ r_{21}, & 1, & r_{23}, & r_{2v} \\ r_{31}, & r_{32}, & 1, & r_{3v} \\ r_{v1}, & r_{v2}, & r_{v3}, & 1 \end{vmatrix} \div (1 - r_{12}^2 - r_{23}^2 - r_{31}^2 + 2r_{12}r_{23}r_{31}) \\
& \gamma_{uu} = \sigma_u^2 \times \begin{vmatrix} 1, & r_{12}, & r_{13}, & r_{1u} \\ r_{21}, & 1, & r_{23}, & r_{2u} \\ r_{31}, & r_{32}, & 1, & r_{3u} \\ r_{u1}, & r_{u2}, & r_{u3}, & 1 \end{vmatrix} \div (1 - r_{12}^2 - r_{23}^2 - r_{31}^2 + 2r_{12}r_{23}r_{31}) \\
& \gamma_{uv} = \sigma_v \sigma_u \begin{vmatrix} 1, & r_{12}, & r_{13}, & r_{1v} \\ r_{21}, & 1, & r_{23}, & r_{2v} \\ r_{31}, & r_{32}, & 1, & r_{3v} \\ r_{v1}, & r_{v2}, & r_{v3}, & r_{uv} \end{vmatrix} \div (1 - r_{12}^2 - r_{23}^2 - r_{31}^2 + 2r_{12}r_{23}r_{31}) \quad \dots \dots \dots \text{(xliv.)}
\end{aligned}$$

We can now collect our complete results.

The variability of a non-selected organ $v > q$ is after the selection of q organs given by

$$\Sigma_v^2 = \sigma_v^2 \left\{ \frac{R(ur)_{vv}}{R(q)} + S_1 \left[\left(\frac{R(v)_{pr}}{R(v)_{vv}} \right)^2 \left(\frac{s_p}{\sigma_p} \right)^2 \right] + 2S_2 \left[\frac{R(v)_{pr} R(v)_{p'v}}{(R(v)_{vv})^2} \rho_{p'p''} \frac{s_p s_{p''}}{\sigma_p \sigma_{p''}} \right] \right\}. \quad \text{(xlv.)}$$

The correlation of two non-selected organs v and u both $> q$ is after the selection of q organs given by

$$\begin{aligned}
\Sigma_v \Sigma_u r_{uv} = \sigma_u \sigma_v \left[- \frac{R(ur)_{uv}}{R(q)} + S_1 \left\{ \frac{R(v)_{pr} R(u)_{pr}}{R(v)_{vv} R(u)_{uu}} \left(\frac{s_p}{\sigma_p} \right)^2 \right\} \right. \\
\left. + S_2 \left\{ \left(\frac{R(v)_{pr} R(u)_{p'u}}{R(v)_{vv} R(u)_{uu}} + \frac{R(v)_{p'v} R(u)_{p'u}}{R(v)_{vv} R(u)_{uu}} \right) \rho_{p'p''} \frac{s_p s_{p''}}{\sigma_p \sigma_{p''}} \right\} \right] \quad \dots \quad \text{(xlvi.)}
\end{aligned}$$

The correlation between a non-selected organ $u > q$ and a selected organ $v < q + 1$ is given by (xli.)^{bis} or,

$$s_v \Sigma_u r_{vu} = - s_v \sigma_u \left[S_1 \left(\frac{R(u)_{pu}}{R(u)_{uu}} \rho_{pv} \frac{s_p}{\sigma_p} \right) \right] \quad \dots \quad \text{(xlvii.)}$$

* The expanded values of these determinants are given, 'Phil. Trans,' A, vol. 187, p. 294.

Here in S_1 : p takes every value from 1 to q , and in S_2 : p' and p'' every possible pair of values from 1 to q . Equations (xlv.)–(xlvii.) fully determine all the required quantities and form the full solution of the problem of selection. Before we see the remarkably simple forms they take for simpler cases, we may draw some general conclusions of a most important character.

In the first place we must distinguish between directly selected and what we have termed non-selected organs. It would be better to term the latter *indirectly* selected organs. Suppose the recruiting sergeant were to pay attention only to stature and seek to form a regiment of men of *about* 5 feet 10 inches. He might have a real range of stature about 6 or 7 inches, but he would strive to get men of *about* this height from the population. We will suppose that he did not consider chest-breadth, head-length, foot-length, lungs or any other character. The distribution of these “non-selected” characters in the regiment would not be the same as in the general population. Their means would have changed by (xxxvi.) and their variabilities and correlations be given by (xlv.)–(xlvii.). In other words, an indirect selection would have taken place. A selection by stature would change foot-length and head-length and indeed every other correlated organ. Much the same result must occur in natural selection. If it be advantageous for a species to have a certain group of its organs of definite size, falling within a definite range, and related to each other in a definite manner, then these changes cannot take place without modifying not only the size, but the variability and correlation of all the other organs correlated with these, although these organs themselves be not directly selected. Practically this means all the other organs, for so far one can hardly say with certainty that we have come across any two characters in an organism which are uncorrelated. Many of those investigated are highly correlated, all appear to have some correlation, even if it be very small or negative.

We may therefore conclude as follows:—

(a) The selection of any complex of characters or organs in an organism changes all the other characters and organs not directly selected.

(b) If the change in the complex be continuous and progressive, the other characters will continue to be modified until the change in them is so considerable that selection begins to act directly upon them also.

(c) The changes noted here are not confined to the average value of a non-directly selected character and to its variability; the correlations between non-directly selected characters and the correlations between directly and non-directly selected characters are also both changed.

(d.) If local races have been produced by selection from a common stock, it will be impossible to look upon correlation as a criterion for species. Every selection will modify such correlation, and it has no greater fixity than either type value (mean) or variability (standard deviation).

The whole of these statements will become more manifest as we apply our general

theorem to special cases, but we must note that if two organs were uncorrelated with each other, it still might be possible by selecting a third, or a third and fourth, to produce correlation between them. Further, by selection of one or more organs, two non-directly selected organs can have their existing correlation increased, lessened or even changed in sign.

(4.) A primary difficulty will of course arise in the case of natural selection. How are we to determine which are the directly and which are the indirectly selected organs? With artificial selection by man, we know which organs have been selected fairly well: attention has been paid to colour, size, proportion of parts, &c. Even in the case of the medical examination of the recruit, it is chest, lungs, heart, stature, &c., which form the basis of the acceptance or rejection. If the head or foot be not absolutely deformed, little if any attention is paid to them, so with hair-colour, probably eye-colour, and a mass of other details. No doubt the direct medical selection indirectly selects these, but we could roughly class the selected and non-selected organs or characters and investigate the changes in the correlations of the latter owing to the indirect selection. But how are we to form these classes in the case of natural selection?

The investigations may look difficult, and even from the standpoint of arithmetic appalling, but it seems to me that the differentiation of organs into directly and indirectly selected classes is the keynote to the problem of evolution by natural selection.

Let us look at a simple case and see whether it will throw any light on the problem of distinguishing between directly and indirectly selected organs. Suppose we have two organs only, with means m_1, m_2 , standard deviations σ_1, σ_2 , correlation r_{12} , and let the first be selected so as to have a mean value $m_1 + h_1$, and standard deviation s_1 . Let Σ_2 be the standard deviation of the second organ and r_{12} the correlation of the two organs after selection, and $m_2 + x'_2$ the mean of the non-selected organ.

Then by (xxxii.) :

$$x'_2 = r_{12} \frac{\sigma_2}{\sigma_1} h_1,$$

and it will be shown later (see p. 23) that

$$\Sigma_2^2 = \sigma_2^2 \left(1 - \left(1 - \frac{s_1^2}{\sigma_1^2} \right) r_{12}^2 \right),$$

and

$$\Sigma_2 r_{12} = \sigma_2 \frac{s_1}{\sigma_1} r_{12}.$$

Hence we have :

$$\frac{r_{12} \Sigma_2}{s_1} = \frac{r_{12} \sigma_2}{\sigma_1},$$

$$\frac{r_{12} s_1}{\Sigma_2} = \frac{s_1^2}{\sigma_1 \sigma_2} \frac{r_{12}}{1 - \left(1 - \frac{s_1^2}{\sigma_1^2} \right) r_{12}^2},$$

In other words the regression coefficient of the non-selected organ on the selected remains unchanged, while that of the selected organ on the non-selected will, as a rule, be widely modified.

Further, let X_2 be the mean value of the second organ before selection corresponding to a value H_1 of the first; let M_1 and M_2 be the means of the organs after selection, and Y_2 be the mean value of the second organ corresponding to a value K_1 of the first. Then the equation to the regression line before selection is

$$X_2 = r_{12} \frac{\sigma_2}{\sigma_1} H_1 + m_2 - r_1 \frac{\sigma_2}{\sigma_1} m_1,$$

and after selection it is

$$\begin{aligned} Y_2 &= r_{12} \frac{s_2}{s_1} K_1 + M_2 - r_{12} \frac{s_2}{s_1} M_1. \\ &= r_{12} \frac{\sigma_2}{\sigma_1} K_1 + m_2 + r_{12} \frac{\sigma_2}{\sigma_1} h_1 - r_{12} \frac{\sigma_2}{\sigma_1} (m_1 + h_1), \\ &= r_{12} \frac{\sigma_2}{\sigma_1} K_1 + m_2 - r_{12} \frac{\sigma_2}{\sigma_1} m_1. \end{aligned}$$

But this is identically the same line as the regression line before selection. Hence not only the slope (regression coefficient) of the line, but its position is identical, and we have the following result:—

*If two local races have been evolved from a single stock by the selection in different ways of one organ only, then the regression lines for the two races of any non-directly selected organ on the directly selected organ will be the same in direction and position; but the regression lines of the selected organ on any non-selected organ will differ for the two races.**

Of course the means, standard deviations and correlations, not only of the selected organ but of all the non-selected organs also, will probably have changed. It is only certain of the regression lines which remain unchanged and serve as a criterion to enable us to distinguish between directly and non-directly selected organs.

Of course the problem in Nature will not be as simple as this, for differentiation of the two local races may have arisen from the selection of more than one organ, or may have arisen from the selection of two different organs, but the illustration will, I think, indicate the nature of the investigation we are proposing.

We can easily generalise our theorem by considering the form of the selection surface given on p. 12. Any result obtained from (xxxv.) which does not involve any of the $\bar{c}$'s will be a result unaffected by the selection that has gone on. Now to obtain a regression equation we put any number of the x 's equal to constants, to h 's

* The geometrical interpretation in this simple case that the regression line is unchanged is quite obvious, and, indeed, may serve as a proof.

say, and find the "centre" of the quadric of the remaining x 's, the co-ordinates of this centre, expressed in terms of the h 's, are the regression equations. Now it will be clear, that if we put all the selected x 's equal to h 's, the differentials of the quadric with regard to the remaining or non-selected x 's can contain no $\bar{c}$'s or the coefficients of the regression equations thus found will not be modified by selection.

Further, we might have given not only the selected organs, but any number of the non-selected organs constant values, and the resulting regression equations would involve only the c 's and not the $\bar{c}$'s.

Hence we have the following general theorems :—

(i.) *If an organ has been modified only by indirect selection, then its partial regression coefficients on any complex of other organs, however large or small, provided it includes all the directly selected organs, will remain unchanged by the selection.*

(ii.) *The same organ in two different local races which have been derived from a common stock by the selection of two complexes of organs, some of which may or may not be common, will, if it has not been directly selected in either case, give the same partial regression coefficients for any group of organs which includes the members of both complexes and any number of non-directly selected organs besides.*

If the partial regression equations have changed coefficients, then we cannot at once determine whether—

(a) We are dealing with a non-directly selected organ, and have not included *all* the directly selected organs in the group upon which we are calculating the regression ; or

(b) We are really dealing with a directly selected organ. In this case, we have also certainly not included at least one directly selected organ in the regression group.

Theoretically, however, (i.) and (ii.) suffice to find out which, if any, are the non-directly selected organs in the differentiation of local races. Practically, however, the number of organs and characters may be so great, and our ignorance of those probably selected so complete, that the arithmetic of determining so extensive a series of partial regression coefficients may be quite beyond our powers. Still, where the divergence between local races is not too great, and the source of the differentiation not too obscure, it is probable that the above theorems will lead to results of great interest.*

Without laying too great weight on these theorems, I would still venture to suggest that if the criterion of a species be the discovery of any numerical constant

* Mr. L. BRAMLEY-MOORE has been working with this end in view at the long-bones in man. But even here the direct selection of parts of the vertebral column—for which, at present, we have no correlation values either among themselves or with the long-bones—and of the hand and foot, which Dr. W. R. MACDONELL has just shown, are very highly correlated with the long-bones, may render nugatory all attempt to ascertain which, if any, long-bone has been only indirectly, or, at any rate, least directly selected.

or group of constants, which is the same for all local races, then these constants must not be sought in the values of mean characters, degrees of variability, or of correlation. but in a system of partial regression coefficients, and the discovery of these is therefore of first class biological importance; it is the classification of the characters into directly and non-directly selected groups, *i.e.*, it is the discovery of the *modus operandi* of the factors by which the differentiation has taken place. We are a long way from solution yet, but we may venture, perhaps, to admit a faint glimmer of light in the direction of what might seem the culminating problem of the mathematical method as applied to evolution--the piecing together by quantitative analysis of the stages of descent.

(5.) I will take now the application of the above results to simple cases; but for the benefit of those who cannot easily follow the main principles of our investigation through the stages of determinant analysis, I will prove directly the proposition that: *the selection of an organ A alters the mean and variability of a correlated organ B, and also the correlation between A and B.*

Let the correlation surface for the two organs be

$$z = \frac{N}{2\pi\sigma_1\sigma_2} \frac{1}{\sqrt{1-r_{12}^2}} e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2(1-r_{12}^2)} - \frac{2r_{12}x_1x_2}{\sigma_1\sigma_2(1-r_{12}^2)} + \frac{x_2^2}{\sigma_2^2(1-r_{12}^2)}\right)},$$

where N is the number of individuals in the general population before selection, and the subscripts 1 and 2 refer to the organs A and B respectively.

Let the distribution of the population after selection of the A organ be

$$z' = \frac{n}{\sqrt{2\pi}s_1} e^{-\frac{(x_1-h_1)^2}{2s_1^2}},$$

where $N - n$ is the total destruction, h_1 the mean and s_1 the variability of the population with regard to A after selection. Before selection this distribution was

$$z_1 = \frac{N}{\sqrt{2\pi}\sigma_1} e^{-\frac{x_1^2}{2\sigma_1^2}}.$$

Hence, the selection being random with regard to the array of B's corresponding to any A, we have for the surface after selection

$$Z = z \times z'/z_1,$$

for each array must be altered in the ratio of the corresponding z' to z_1 .

This gives for the surface in full

$$Z = \frac{n}{2\pi\sigma_2s_1} \text{expt.} - \frac{1}{2} \left\{ x_1^2 \left(\frac{1}{s_1^2} - \frac{1}{\sigma_1^2} + \frac{1}{\sigma_1^2(1-r_{12}^2)} \right) - \frac{2r_{12}x_1x_2}{\sigma_1\sigma_2(1-r_{12}^2)} + x_2^2 \frac{1}{\sigma_2^2(1-r_{12}^2)} - \frac{2h_1x_1}{s_1^2} \right\} \quad \text{(xlvi.iii.).}$$

Integrate this for every possible value of x_1 and we shall have the distribution h_2 and s_2 of x_2 or B after the selection of A. After some reductions we find for the frequency $\zeta \delta x_2$

$$\zeta = \frac{n}{\sqrt{2\pi\Sigma_2}} \cdot e^{-\frac{1}{2}\frac{1}{\Sigma_2^2}\left(x_2 - \frac{h_1 r_{12} \sigma_2}{\sigma_1}\right)^2},$$

where

$$\Sigma_2^2 = \sigma_2^2 \left(1 - \left(1 - \frac{s_1^2}{\sigma_1^2}\right) r_{12}^2\right). \quad \dots \quad \text{(xlix.)}$$

which gives the standard deviation of the indirectly selected organ, and the "centre" of this organ is given by

$$h_2 = \frac{r_{12} \sigma_2}{\sigma_1} h_1. \quad \dots \quad \text{(l.)}$$

These are special cases of our results (xlv.) and (xxxvi.) above respectively.

Further, returning to the correlation surface (xlviii.), the coefficient of correlation r_{12} is the coefficient of $x_1 x_2$ divided by the product of the coefficients of x_1^2 and x_2^2 . Hence we find

$$r_{12} = \frac{s_1}{\sigma_1} \frac{r_{12}}{\sqrt{1 - \left(1 - \frac{s_1^2}{\sigma_1^2}\right) r_{12}^2}}. \quad \dots \quad \text{(li.)}$$

Let $r_{12} = \sin \theta_{12}$, $r_{12} = \sin \phi_{12}$, then we have

$$\tan \phi_{12} = \frac{s_1}{\sigma_1} \tan \theta_{12}. \quad \dots \quad \text{(lii.)}$$

This shows us that ϕ_{12} decreases with s_1 , or that r_{12} decreases with s_1 , that is to say, the more intense the selection the less is the correlation. This in broad terms demonstrates the general principle that intensity of selection connotes a lessening of correlation. It is this principle which very possibly accounts for the fact that the more civilized races of man appear to be not only more variable but more highly correlated than the less civilized, among whom the struggle for existence is more intense. It may, perhaps, also account for the skeletons of women of the civilized races having their parts more highly correlated than the parts of those of men.* Lastly, it may well throw some light on the markedly plastic character of races which have been stringently selected with regard to one or a few organs only.

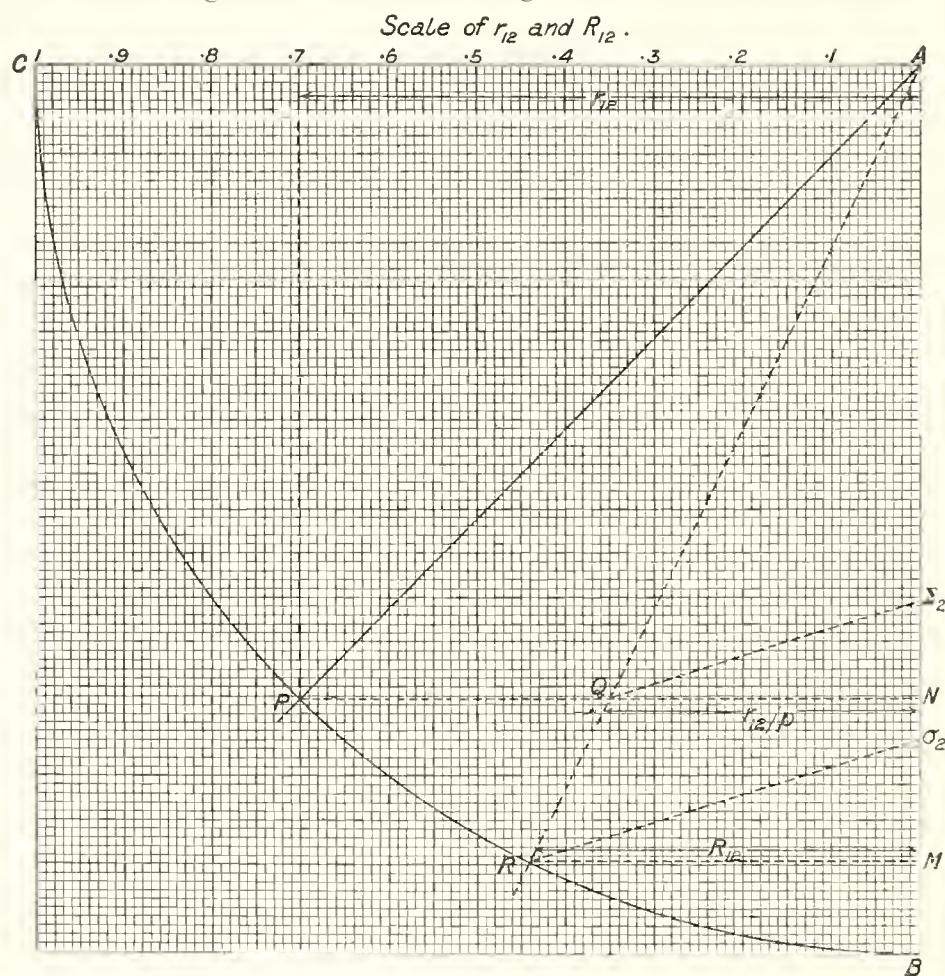
As an illustration, suppose that the correlation between femur and tibia were .7, and let us investigate what would be the effect of reducing the variability of the tibia by direct selection 50 per cent. We find at once on substituting $s_1/\sigma_1 = .5$ and $r_{12} = .7$ in (li.) above that $r_{12} = .44$, or a reduction of about 37 per cent. This will, perhaps, be sufficient to indicate what immense changes must be made in the correlation of highly correlated organs whenever selection, artificial or natural, is stringent. It is important to notice that the change in the size of the organ in no

* See LEE and PEARSON, 'Roy. Soc. Proc.' vol. 61, p. 354: and LEE, 'Phil. Trans.,' A, vol. 196, p. 231.

way influences the change in the correlation between organs, if the distribution be normal,* the change depends only on the stringency of the selection. Breeders who select by the size of an organ only are in that case very likely to reduce the variability of the organ in the selected group by far more than 50 per cent. Accordingly, it is not to be wondered at if they, to a great extent, destroy the correlation between the selected organ and other organs. This destruction would appear as a want of relationship between parts, possibly as a want of fixity in type.

By means of equation (lii.) r_{12} can easily be found from r_{12} and $p = s_1/\sigma_1$. All we need is a table of trigonometrical functions. We observe that r_{12} is always of the same sign and less than r_{12} . For many biological purposes the following graphical construction gives quite sufficiently accurate results. Let CAB be a quadrant, say of 10 centims. radius, and take the point P on this quadrant distant $PN = 10r_{12}$ from AB. Take $QN = \frac{1}{p} PN$, and let AQ meet the quadrant in R, then RM the distance of R from AB $= 10r_{12}$, and consequently determines r_{12} . If the figure be drawn on decimal paper the determination of r_{12} is peculiarly easy.

Graphical method of finding correlation between organs A and B after selection has acted on A.



In the above example $r_{12} = PN = .7$; $s_1/\sigma = \frac{1}{p} = \frac{1}{2}$, and therefore $QN = r_{12} p = .35$.

$r_{12} = R_{12} = RM = .44$, as before. See p. 23.

* It will do so if the distribution be skew, see 'Phil. Trans.,' A, vol. 191, p. 231.

Further, taking AP as our unit, $AN^2 = 1 - r_{12}^2$, and $QN^2 = r_{12}^2 \times \frac{s_1^2}{\sigma_1^2}$. Hence, from (lxix.) :

$$\Sigma_2^2 = \sigma_2^2 (AN^2 + QN^2) = \sigma_2^2 AQ^2.$$

Therefore if $A\sigma_2$ in the diagram be taken equal to σ_2 and $Q\Sigma_2$ be drawn parallel to $R\sigma_2$, we shall have $A\Sigma_2 = \Sigma_2$, or we can scale off the reduced variability.

Thus the diagram enables us to see at a glance the reduction in correlation and variability.

(6.) Let us now write down the results when an organ A is selected out of a group of three organs, A, B, C, whose constants are marked by the subscripts 1, 2, 3, respectively. Let $\mu_1 = s_1/\sigma_1$, and be represented, when required, by $\cos \chi_1$. Then we find from (xlv.) (xlvii.) :

$$x'_2 = r_{12} \frac{\sigma_2}{\sigma_1} h_1, \quad x'_3 = r_{13} \frac{\sigma_3}{\sigma_1} h_1. \quad \dots \quad \text{(liii.),}$$

$$\left. \begin{aligned} \Sigma_2 &= \sigma_2 \left\{ 1 - \left(1 - \left(\frac{s_1}{\sigma_1} \right)^2 \right) r_{12}^2 \right\}^{\frac{1}{2}} = \sigma_2 \{ 1 - \sin^2 \chi_1 \cos^2 \theta_{12} \}^{\frac{1}{2}} \\ \Sigma_3 &= \sigma_3 \left\{ 1 - \left(1 - \left(\frac{s_1}{\sigma_1} \right)^2 \right) r_{13}^2 \right\}^{\frac{1}{2}} = \sigma_3 \{ 1 - \sin^2 \chi_1 \cos^2 \theta_{13} \}^{\frac{1}{2}} \end{aligned} \right\} \quad \text{(liv.),}$$

$$\left. \begin{aligned} r_{12} &= \frac{\mu_1 r'_{12}}{\{ 1 - (1 - \mu_1^2) r_{12}^2 \}^{\frac{1}{2}}} = \frac{\cos \chi_1 \cos \theta_{12}}{\sqrt{1 - \sin^2 \chi_1 \cos^2 \theta_{12}}} \\ r_{13} &= \frac{\mu_1 r'_{13}}{\{ 1 - (1 - \mu_1^2) r_{13}^2 \}^{\frac{1}{2}}} = \frac{\cos \chi_1 \cos \theta_{13}}{\sqrt{1 - \sin^2 \chi_1 \cos^2 \theta_{13}}} \end{aligned} \right\} \quad \dots \quad \text{(lv.),}$$

$$\left. \begin{aligned} r_{23} &= \frac{(r_{23} - r_{13} r'_{12}) \left(1 - \frac{s_1^2}{\sigma_1^2} \right) + \frac{s_1^2}{\sigma_1^2} r_{23}}{\sqrt{\left\{ 1 - \left(1 - \left(\frac{s_1}{\sigma_1} \right)^2 \right) r_{12}^2 \right\} \left\{ 1 - \left(1 - \left(\frac{s_1}{\sigma_1} \right)^2 \right) r_{13}^2 \right\}}} \\ &= \frac{\cos \theta_{23} + \sin^2 \chi_1 \cos \theta_{13} \cos \theta_{12}}{\sqrt{(1 - \sin^2 \chi_1 \cos^2 \theta_{12}) (1 - \sin^2 \chi_1 \cos^2 \theta_{13})}} \end{aligned} \right\} \quad \dots \quad \text{(lvi.),}$$

where, as before, we write $r_{pq} = \cos \theta_{pq}$. Let us also write $r_{pq} = \cos \Theta_{pq}$, and

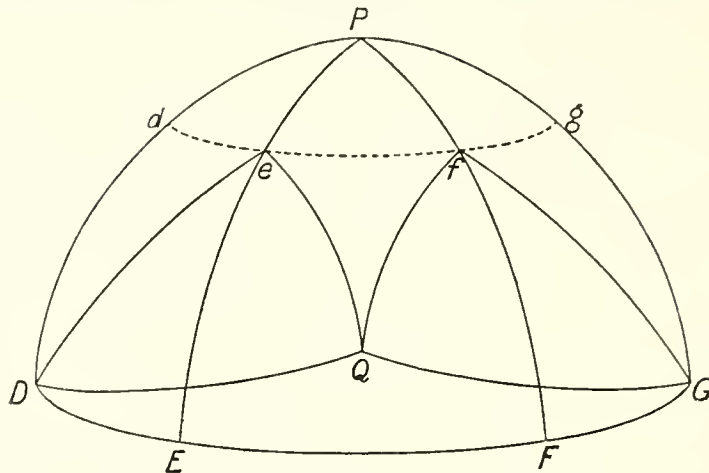
$$\sin \chi_1 \cos \theta_{12} = \cos a_{12}, \quad \sin \chi_1 \cos \theta_{13} = \cos a_{13}.$$

Then we can replace the above results by

$$\begin{aligned} \Sigma_2 &= \sigma_2 \sin a_{12}, & \Sigma_3 &= \sigma_3 \sin a_{13}, \\ \cos \Theta_{12} &= \cot \chi_1 \cot a_{12}, & \cos \Theta_{13} &= \cot \chi_1 \cot a_{13}, \end{aligned}$$

$$\cos \Theta_{23} = \frac{\cos \theta_{23} - \cos a_{12} \cos a_{13}}{\sin a_{12} \sin a_{13}} \quad \dots \quad \text{(lvii.).}$$

These equations admit of easy interpretation by spherical geometry.



Let P be the pole of the great circle $DEFG$. Take $DG = \theta_{23}$, $DE = \theta_{12}$, $GF = \theta_{13}$. Join P to E and F ; let the small circle of radius χ_1 round P meet PD , PE , PF , and PG in d , e , f , g respectively. Draw the arcs De and Gf . Let the small circles with centres D and G and radii De and Gf respectively meet in Q . Join DQ and GQ . Then the quantities required are :

$$\Sigma_2/\sigma_2 = \sin DQ, \quad \Sigma_3/\sigma_3 = \sin GQ.$$

$$r_{12} = \cos DeE, \quad r_{13} = \cos FfG, \quad r_{23} = \cos DQG.$$

For $DE = \theta_{12}$, $Ee = \frac{\pi}{2} - \chi_1$, $\angle DEe = \frac{\pi}{2}$;

hence : $\cos De = \cos \theta_{12} \cos Ee = \sin \chi_1 \cos \theta_{12} = \cos a_{12}$, or $a_{12} = De$;

similarly $a_{13} = Gf$.

Next, $\cos DeE = \cot De \tan eE = \cot a_{12} \cot \chi_1$, or $\angle DeE = \Theta_{12}$; similarly $\angle FfG = \Theta_{13}$. Lastly, from the triangle DQG : $DQ = a_{12}$, $QG = a_{13}$, and $DG = \theta_{23}$, but

$$\cos DG = \cos DQ \cos QG + \sin DQ \sin QG \cos DQG,$$

or, $\cos DQG = \frac{\cos \theta_{23} - \cos a_{12} \cos a_{13}}{\sin a_{12} \sin a_{13}} = \cos \Theta_{23}$; or $\angle DQG = \Theta_{23}$.

Thus all the relations can be expressed in terms of the sides and angles of a simple system of spherical triangles. For the degree of accuracy generally possible in biological and sociological investigations these triangles can be solved by a spherical trigonometer, such as that sold by KREIDL, of Prague.* The changes, however, which r_{23} undergoes for various values of r_{12} , r_{13} , r_{23} are, indeed, far more difficult to appreciate as a whole than those of r_{12} or r_{13} . In order that they may be followed easily, and in order to solve directly to a degree of approximation sufficient for many practical purposes problems in the influence of selection on correlation, my assistant, Dr. L. N. G. FILON, has kindly drawn up the tables which accompany this memoir.

* It will suffice fairly well for all but a few special values of r_{12} , r_{23} , r_{31} .

Of course to bring them within any reasonable compass we have had to limit the values taken. In the first place we have considered only eleven grades of selective stringency given by

$$s_1/\sigma_1 = 0, 1/10, 2/10, 3/10, 4/10, 5/10, 6/10, 7/10, 8/10, 9/10, 1,$$

the corresponding values of r_{23} in the tables are entered as

$$R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}.$$

The tables are calculated for r_{23} both positive and negative, but r_{12} and r_{13} are always supposed *positive*. If r_{12} and r_{13} be both *negative*, then r_{23} will be the same as if they were both positive. If r_{12} and r_{13} be of opposite signs, then all we have to do is to look out r_{23} in the table in which r_{23} has a sign the reverse of its actual value, and having found the corresponding value of r_{23} , then change its sign to obtain the actual coefficient of correlation after selection. This follows, if r_{13} be the negative coefficient, by writing :

$$r_{23} = \frac{r_{23} + r_{13}r_{12} \sin^2 \chi_1}{\sqrt{\{1 - r_{12}^2 \sin^2 \chi_1\} \{1 - r_{13}^2 \sin^2 \chi_1\}}} = - \frac{(-r_{23}) + (-r_{13})(r_{12}) \sin^2 \chi}{\sqrt{\{1 - r_{12}^2 \sin^2 \chi\} \{1 - (-r_{13})^2 \sin^2 \chi_1\}}}.$$

Lastly, it would clearly be very laborious to tabulate r_{23} for a very great series of values of r_{12} , r_{13} , r_{23} . Accordingly a selection had to be made of these coefficients of correlation. They were given the values 0, .25, .5, .75, and 1. These may be spoken of as zero, small, medium, large, and perfect correlations, and the ranges 0 to .25, .25 to .5, .5 to .75, and .75 to 1, as the ranges of little, moderate, considerable, and high correlation respectively. There would thus appear to be 15 combinations of values for r_{12} , r_{13} ; these are given in the key to the tables as (a), (b), (c), (d) . . . (m), (n), (p), see p. 63. If these 15 values had to be combined with the 10 values (5 positive and 5 negative) of r_{23} and the 11 values of s_1/σ_1 , we should have 1650 entries in our tables. But this number is much reduced by the consideration that the expression $1 - r_{23}^2 - r_{13}^2 - r_{12}^2 + 2r_{23}r_{13}r_{12}$ has for the real correlation of three characters to be always positive. r_{23} can also never be greater than unity. Accordingly all values of r_{23} , r_{13} , r_{12} , which do not satisfy these conditions, have been excluded from the tables; they cannot arise in nature. A few impossible values of r_{23} have been included in the tables, but these are placed there solely for the purpose of finding by interpolation values of r_{23} , which are less than unity. The following *purely hypothetical* illustrations of formula (lvi.) and the tables will serve to indicate their use.

Illustration I.—Suppose the correlation of tibia and femur with each other to be .8, and of both with the stature to be .6. How would their correlation be altered if the variation in stature were reduced by selection to half its present value?

Let $s_1/\sigma_1 = \mu_1$ as before, and suppose $r_{23} = R$; then let μ_1 , r_{12} , r_{13} , r_{23} be the values of the constants next *below* the required values occurring in the tables, and giving

$r_{23} = R$; let R' be the true value of r_{23} , corresponding to the values $\mu_1 + \delta\mu_1$, $r_{12} + \delta r_{12}$, $r_{13} + \delta r_{13}$, $r_{23} + \delta r_{23}$. Thus we have, as far as first differences:

$$R' = R + 10 (\Delta_{\mu_1} R) \delta\mu_1 + 4 \{ (\Delta_{r_{12}} R) \delta r_{12} + (\Delta_{r_{13}} R) \delta r_{13} + (\Delta_{r_{23}} R) \delta r_{23} \}.$$

In our case $\mu_1 = \frac{2}{10}$, $r_{12} = .5$, $r_{13} = .5$, $r_{23} = .75$, $\delta\mu_1 = .05$, $\delta r_{12} = \delta r_{13} = .1$, $\delta r_{23} = .05$. Further, we look up Table IV. (a), and the nearest case is (j) under R_2 , which gives $R = .67105$. We then see that $(\Delta_{r_{12}} R) + (\Delta_{r_{13}} R) =$ difference between (j) and (m) cases $= -.21455$; $(\Delta_{r_{23}} R) =$ difference between result in IV. (a) and V. (a) $= .32895$; and lastly $(\Delta_{\mu_1} R) =$ difference between R_3 and R_2 columns of (j) row of Table IV. (a) $= .00535$. Thus we find:

$$R' = .671050 + .002675 - .04291 + .06578 = .6966.$$

The value by straightforward calculation of formula (lvi.) is .6981, the two results giving substantially the same value .7. Thus we see that such a selection would reduce the correlation of tibia and femur by 12.5 per cent.

Illustration (II).—Suppose the correlation of humerus and femur to be .5, and of those with stature to .7 and .8 respectively. How would the correlation of humerus and femur be modified by a selection of stature given by $s_1/\sigma_1 = .5$?

In this case, $\mu_1 = .5$, $r_{12} = .5$, $r_{13} = .75$, $r_{23} = .5$, $\delta\mu_1 = 0$, $\delta r_{12} = .2$, $\delta r_{13} = .05$, $\delta r_{23} = 0$. We turn to Table III. (a) and take out (k) under R_5 , which gives us $R = .3192$. We have $\Delta_{r_{12}} R = -.1841$ and $\Delta_{r_{13}} R = -.04185$, whence we find $R' = .1636$, but the differences of the table are too great at this point for the result to be very trustworthy.* Suppose we take μ_1 and r_{23} as before, but $r_{12} = .75$, $r_{13} = .75$, and therefore R , to be found from (m), $= .1351$; then $\delta r_{12} = -.05$, $\delta r_{13} = .05$ and $\Delta_{r_{12}} R = -.1841$ as before, $\Delta_{r_{13}} R = -.2995$. Hence we deduce $R' = .1120$. The mean of these two values of R' is .1377, and the true value calculated from (lvi.) is $R' = .1395$. Taking .14 for the practical value, we see that the correlation of humerus and femur has been reduced by this comparatively moderate selection of stature upwards of 70 per cent.!

Illustration (III).—Suppose a case in which humerus and femur were not correlated, but that both were correlated .7 with stature. What would be the effect of selecting stature with the same intensity, i.e., $s_1/\sigma_1 = .5$?

Our best results from the tables will be to take R_5 (m) from Table I. (a), which gives $R = -.7297$. We have then $r_{12} = r_{13} = .75$, hence $\delta r_{12} = -.05$, $\delta r_{13} = -.05$. $\Delta_{r_{12}} R$ is to be found from (m) and (h) and $= -.3193 = \Delta_{r_{13}} R$, and

$$\begin{aligned} R' &= -.7297 + (\Delta_{r_{12}} R) \delta r_{12} + (\Delta_{r_{13}} R) \delta r_{13}, \\ &= -.7297 + .1277 = -.6020. \end{aligned}$$

The actual value by formula is $-.5810$.

* Second differences ought to be used, and the process indicated is practically equivalent to using them.

Now this again is a remarkable result; by selecting an organ correlated with two others, neither of which are correlated with each other, we have produced a considerable correlation, and what is more, one of a *negative* sign.

In other words, if humerus and femur were unrelated to each other, but were related to stature, then a selection of stature would result in men of long femur having a short humerus, and *vice versa*.

Illustration IV.—Suppose the correlation between greatest length and breadth of the skull to be .25, and between both and the auricular height to be .5. Now let a stringent selection, $\frac{s_1}{\sigma_1} = \frac{1}{10}$, of height take place. What modification will there be of the length and breadth correlation?

Table II. (a), R_1 , case (j) gives us at once the result—

$$r_{23} = .0033.$$

In other words, the correlation between length and breadth would be sensibly destroyed by such a selection. Thus correlation can be created or destroyed or reversed by selection.

The above illustrations, hypothetical though they may be, will suffice to indicate how entirely dependent correlation is upon selection. We must look upon coefficients of correlation, in fact, as just as much the outcome of selection as coefficients of variation, standard-deviations, or even the mean size of organs. No selection can take place, in the sense in which it has usually been understood to take place—*i.e.*, by a change of mean and of variability, without at the same time the means, variabilities and the *correlations* of all correlated, but not directly selected, organs being varied. This is true whether the non-selected organs be initially correlated or not among themselves. We must always bear in mind this all-important fundamental conception, that natural or artificial selection, or even random sampling, are in themselves active factors in the modification (*i.e.*, creation, destruction, or reversal) of correlation. Thus not only is the impossibility of the constancy of correlation for local races obvious, but the primary importance of insuring that our samples are *representative*, and not accidentally selected samples, in all observations or experiments on heredity, homotyposis, or organic correlation becomes more and more manifest. We must not lay too much stress on two heredity constants—differing, for example, by more than the probable error of their difference—unless we are convinced, which practically it will be difficult to be, that all modification of correlation by unintentional and unmarked selection has really been avoided.

(7.) Let us now take the next most simple case. *If A, B, C, D be four mutually correlated organs (in either the same or different individuals), and a selection take place of A and B, to find the changes in the characters of the non-selected organs.*

Let subscripts 1, 2, 3, 4 mark the organs A, B, C, D respectively; let $h_1, h_2, s_1, s_2, \rho_{12}$ be the constants which determine the selection of A and B; and let us apply (xlv.) to (xlvii.). Here the coefficients will be obtained from the set for $q = 2$ in (xxxiii.) and (xlv.). Hence we find:

$$\Sigma_3^2 = \sigma_3^2 \left\{ \frac{1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23}}{1 - r_{12}^2} + \left(\frac{r_{13} - r_{23}r_{12}}{1 - r_{12}^2} \right)^2 \left(\frac{s_1}{\sigma_1} \right)^2 + \left(\frac{r_{23} - r_{13}r_{12}}{1 - r_{12}^2} \right)^2 \left(\frac{s_2}{\sigma_2} \right)^2 + 2\rho_{12} \frac{r_{13} - r_{23}r_{12}}{1 - r_{12}^2} \frac{r_{23} - r_{13}r_{12}}{1 - r_{12}^2} \frac{s_1 s_2}{\sigma_1 \sigma_2} \right\} \quad \text{. (lviii.)}$$

$$\Sigma_4^2 = \sigma_4^2 \left\{ \frac{1 - r_{12}^2 - r_{14}^2 - r_{24}^2 + 2r_{12}r_{14}r_{24}}{1 - r_{12}^2} + \left(\frac{r_{14} - r_{24}r_{12}}{1 - r_{12}^2} \right)^2 \left(\frac{s_1}{\sigma_1} \right)^2 + \left(\frac{r_{24} - r_{14}r_{12}}{1 - r_{12}^2} \right)^2 \left(\frac{s_2}{\sigma_2} \right)^2 + 2\rho_{12} \frac{r_{14} - r_{24}r_{12}}{1 - r_{12}^2} \frac{r_{24} - r_{14}r_{12}}{1 - r_{12}^2} \frac{s_1 s_2}{\sigma_1 \sigma_2} \right\} \quad \text{. (lix.)}$$

$$\Sigma_3 \Sigma_4 r_{34} = \sigma_3 \sigma_4 \left\{ \frac{r_{34}(1 - r_{12}^2) - r_{13}r_{14} - r_{23}r_{24} + r_{12}(r_{13}r_{24} + r_{14}r_{23})}{1 - r_{12}^2} + \frac{r_{13} - r_{23}r_{12}}{1 - r_{12}^2} \frac{r_{14} - r_{24}r_{12}}{1 - r_{12}^2} \left(\frac{s_1}{\sigma_1} \right)^2 + \frac{r_{23} - r_{13}r_{12}}{1 - r_{12}^2} \frac{r_{24} - r_{14}r_{12}}{1 - r_{12}^2} \left(\frac{s_2}{\sigma_2} \right)^2 + \rho_{12} \left(\frac{r_{13} - r_{23}r_{12}}{1 - r_{12}^2} \frac{r_{24} - r_{14}r_{12}}{1 - r_{12}^2} + \frac{r_{14} - r_{24}r_{12}}{1 - r_{12}^2} \frac{r_{23} - r_{13}r_{12}}{1 - r_{12}^2} \right) \frac{s_1 s_2}{\sigma_1 \sigma_2} \right\} \quad \text{. . . (lx.)}$$

$$s_1 \Sigma_3 r_{13} = s_1 \sigma_3 \left\{ \frac{r_{13} - r_{12}r_{23}}{1 - r_{12}^2} \frac{s_1}{\sigma_1} + \frac{r_{23} - r_{12}r_{13}}{1 - r_{12}^2} \rho_{12} \frac{s_2}{\sigma_2} \right\} \quad \text{. (lxi.)}$$

$$s_1 \Sigma_4 r_{14} = s_1 \sigma_4 \left\{ \frac{r_{14} - r_{12}r_{24}}{1 - r_{12}^2} \frac{s_1}{\sigma_1} + \frac{r_{24} - r_{12}r_{14}}{1 - r_{12}^2} \rho_{12} \frac{s_2}{\sigma_2} \right\} \quad \text{. (lxii.)}$$

$$s_2 \Sigma_3 r_{23} = s_2 \sigma_3 \left\{ \frac{r_{23} - r_{12}r_{13}}{1 - r_{12}^2} \frac{s_2}{\sigma_2} + \frac{r_{13} - r_{12}r_{23}}{1 - r_{12}^2} \rho_{12} \frac{s_1}{\sigma_1} \right\} \quad \text{. (lxiii.)}$$

$$s_2 \Sigma_4 r_{24} = s_2 \sigma_4 \left\{ \frac{r_{24} - r_{12}r_{14}}{1 - r_{12}^2} \frac{s_2}{\sigma_2} + \frac{r_{14} - r_{12}r_{24}}{1 - r_{12}^2} \rho_{12} \frac{s_1}{\sigma_1} \right\} \quad \text{. (lxiv.)}$$

Finally, for the change of means of the non-directly selected organs, we have:

$$x'_3 = \frac{r_{13} - r_{12}r_{23}}{1 - r_{12}^2} \frac{\sigma_3}{\sigma_1} h_1 + \frac{r_{23} - r_{12}r_{13}}{1 - r_{12}^2} \frac{\sigma_3}{\sigma_2} h_2 \quad \text{. (lxv.)}$$

$$x'_4 = \frac{r_{14} - r_{12}r_{24}}{1 - r_{12}^2} \frac{\sigma_4}{\sigma_1} h_1 + \frac{r_{24} - r_{12}r_{14}}{1 - r_{12}^2} \frac{\sigma_4}{\sigma_2} h_2 \quad \text{. (lxvi.)}$$

If we write (lxv.) and (lxvi.) in the form—

$$\left. \begin{aligned} x'_3 &= \beta_{13} \frac{\sigma_3}{\sigma_1} h_1 + \beta_{23} \frac{\sigma_3}{\sigma_2} h_2 \\ x'_4 &= \beta_{14} \frac{\sigma_4}{\sigma_1} h_1 + \beta_{24} \frac{\sigma_4}{\sigma_2} h_2 \end{aligned} \right\} \dots \dots \dots \text{(lxvii.)}$$

the β 's are the partial regression coefficients, and the whole solution can be expressed in terms of them. Thus :

$$\Sigma_3^2 = \sigma_3^2 \left\{ 1 - \beta_{13} r_{13} - \beta_{23} r_{23} + \beta_{13}^2 \left(\frac{s_1}{\sigma_1} \right)^2 + \beta_{23}^2 \left(\frac{s_2}{\sigma_2} \right)^2 + 2\rho_{12} \beta_{13} \beta_{23} \frac{s_1 s_2}{\sigma_1 \sigma_2} \right\}. \text{(lxviii.)}$$

$$\begin{aligned} \Sigma_3 \Sigma_4 r_{34} &= \sigma_3 \sigma_4 \left\{ r_{34} - \beta_{13} r_{14} - \beta_{23} r_{24} + \beta_{13} \beta_{14} \left(\frac{s_1}{\sigma_1} \right)^2 + \beta_{23} \beta_{24} \left(\frac{s_2}{\sigma_2} \right)^2 \right. \\ &\quad \left. + \rho_{12} (\beta_{13} \beta_{24} + \beta_{23} \beta_{14}) \frac{s_1 s_2}{\sigma_1 \sigma_2} \right\} \dots \dots \dots \text{(lxix.)} \end{aligned}$$

$$s_1 \Sigma_3 r_{13} = s_1 \sigma_3 \left\{ \beta_{13} \frac{s_1}{\sigma_1} + \rho_{12} \beta_{23} \frac{s_2}{\sigma_2} \right\} \dots \dots \dots \text{(lxx.)}$$

Thus the whole series of results can be easily calculated, if the regression coefficients are first calculated.

I may make some remarks upon these results. A formula equivalent to (lxviii.) was first given by me in my memoir on "Heredity, Panmixia, and Regression" ('Phil. Trans.' A, vol. 187, p. 303), and used for certain problems of inheritance, and conclusions drawn from (lxix.) or (lxx.) have been cited or indicated in other memoirs.

Some interesting results follow at once. If the selection be very stringent, s_1/σ and $s_2/\sigma = 0$ sensibly, then all correlation between a selected and non-selected organ is destroyed. But

$$r_{34} = \frac{r_{34}(1 - r_{12}^2) - r_{13}r_{14} - r_{23}r_{24} + r_{12}(r_{13}r_{24} + r_{14}r_{23})}{\sqrt{(1 - r_{12}^2 - r_{23}^2 - r_{13}^2 + 2r_{12}r_{13}r_{23})} \sqrt{(1 - r_{12}^2 - r_{24}^2 - r_{14}^2 + 2r_{12}r_{14}r_{24})}}. \text{(lxxi.)}$$

This is what I have termed a *partial* correlation coefficient—i.e., the correlation between C and D when fixed values are given to A and B. So far as I am aware, such coefficients were first directly used by Mr. G. U. YULE in certain economic problems.* They are of very considerable interest, but for natural or artificial selection are not quite so important as the generalised form (lxix.), for we generally select *about* a mean value, and not absolutely at it.

It will be noticed that the coefficient of correlation of two non-selected organs differs from the corresponding partial correlation coefficient by terms of the square order in s/σ , but the coefficient of correlation of a selected and non-selected organ

* 'Roy. Soc. Proc.,' vol. 60, pp. 485, 488; 'Economic Journal,' December, 1895, and December, 1896.

differs from zero by terms of the first order in s/σ . Hence, when selection is intense (s/σ small), we may neglect the former as compared with the latter, and we have thus the basis of a method of approximation very useful in some cases.

I have not yet succeeded in giving a geometrical interpretation to the above formulæ, but have little doubt that it would be quite easy if the "spherical" geometry of four dimensioned space were more familiar to me. It will suffice to observe that it is easy to find cases in which the correlation of a non-directly selected organ with a directly selected organ, or with another of its own class, can be reduced, destroyed, increased, or reversed. In fact, all our previous warnings as to the caution necessary in avoiding unintentional selection in collecting material for testing correlation remain in force, and, indeed, are emphasised.

The following illustrations will indicate the kind of problems which may be attacked by such formulæ as the above :—

(8). ILLUSTRATION I.—*Influence of a Selection of the Long Bones of the Leg on the Long Bones of the Arm, and on the Relation of the Leg to the Arm.*

A numerical example will throw light on the application of the above formulæ, and effectively illustrate the manner in which a selection influences size, variation, and correlation.

Consider the long bones femur, tibia, humerus, and radius, indicated by the subscripts 1, 2, 3, 4 respectively, and let m_1 , m_2 , m_3 , m_4 be the mean values in centimetres. Then the following numerical values are given in a memoir by Miss ALICE LEE and myself :—*

	French ♂.	Aino ♂.
m_1	45·23	40·77
m_2	36·81	33·89
m_3	33·01	29·50
m_4	24·39	21·55
σ_1	2·37	1·90
σ_2	1·80	1·67
σ_3	1·54	1·34
σ_4	1·17	1·06
r_{12}	·806	·827
r_{13}	·842	·858
r_{14}	·744	·789
r_{23}	·860	·745
r_{24}	·780	·865
r_{34}	·845	·776

* 'Roy. Soc. Proc.' vol. 61, p. 343 *et seq.* The correlations are not worked out for exactly the same lengths in the case of the two races, but the numbers will serve quite well for the purposes of illustration.

Now let us select from the French population a group having the same characteristics of the long bones of the leg as the Aino population, and then compare the characteristics of the arm bones of this selected group with those of the Aino population.

Our selection is given by :

$$\begin{aligned}h_1 &= 40.77-45.23 = -4.46, \\h_2 &= 33.89-36.81 = -2.92, \\s_1 &= 1.90 \quad \rho_{12} = .827, \\s_2 &= 1.67.\end{aligned}$$

The following constants must now be determined arithmetically :

$$\begin{aligned}s_1/\sigma_1 &= 1.90/2.37 = .802, \\s_2/\sigma_2 &= 1.67/1.80 = .928,\end{aligned}$$

$$\frac{r_{13} - r_{12}r_{23}}{1 - r_{12}^2} = .426, \quad \frac{r_{23} - r_{12}r_{13}}{1 - r_{12}^2} = .518,$$

$$\frac{r_{14} - r_{12}r_{24}}{1 - r_{12}^2} = .329, \quad \frac{r_{24} - r_{12}r_{14}}{1 - r_{12}^2} = .515,$$

$$\frac{1 - r_{12}^2 - r_{23}^2 - r_{13}^2 + 2r_{12}r_{13}r_{23}}{1 - r_{12}^2} = .196,$$

$$\frac{1 - r_{12}^2 - r_{24}^2 - r_{14}^2 + 2r_{12}r_{24}r_{14}}{1 - r_{12}^2} = .354,$$

$$\frac{r_{34}(1 - r_{12}^2) - r_{13}r_{14} - r_{23}r_{24} + r_{12}(r_{13}r_{24} + r_{14}r_{23})}{1 - r_{12}^2} = .125.$$

If x'_3 and x'_4 be the mean humerus and radius of a femur-tibia selection from the French population, we have from (lxv.)—

$$\begin{aligned}x'_3 &= 33.01 + .277h_1 + .443h_2, \\x'_4 &= 24.39 + .162h_1 + .335h_2.\end{aligned}$$

These would give the effect of selecting any femur and tibia defined by h_1 and h_2 from the mean values of the humerus and radius. For the particular selection indicated above :

$$x'_3 = 30.48, \quad x'_4 = 22.69,$$

both of which are about a centimetre in excess of the Aino population. By selecting, therefore, from the French, a population with a mean leg like the Aino, we should still find the average arm of this population some two centimetres greater in length

than the Aino. The variabilities Σ_3 and Σ_4 of humerus and radius for a population selected from the French by femur and tibia are obtained from (lxiii.) and (lix.).

We have :

$$\Sigma_3^2/\sigma_3^2 = .196 + .181 \left(\frac{s_1}{\sigma_1} \right)^2 + .268 \left(\frac{s_2}{\sigma_2} \right)^2 + 2 \frac{s_1}{\sigma_1} \frac{s_2}{\sigma_2} \rho_{12} \times .221,$$

$$\Sigma_4^2/\sigma_4^2 = .354 + .108 \left(\frac{s_1}{\sigma_1} \right)^2 + .265 \left(\frac{s_2}{\sigma_2} \right)^2 + 2 \frac{s_1}{\sigma_1} \frac{s_2}{\sigma_2} \rho_{12} \times .169.$$

These give for the particular case :

$$\Sigma_3 = 1.39, \quad \Sigma_4 = 1.11.$$

Turning to the correlation of humerus and radius, we have by (lx.) :

$$r_{34} = \frac{\sigma_3}{\Sigma_3} \frac{\sigma_4}{\Sigma_4} \left\{ .125 + .140 \left(\frac{s_1}{\sigma_1} \right)^2 + .267 \left(\frac{s_2}{\sigma_2} \right)^2 + .389 \rho_{12} \frac{s_1}{\sigma_1} \frac{s_2}{\sigma_2} \right\},$$

giving for the particular case :

$$r_{34} = .799.$$

It will thus be seen that if we selected from the French a group with the same variabilities and correlation of femur and tibia as the Aino, the variabilities and correlation of the humerus and radius of this group would not be very different from those of the Aino. On the other hand, the correlations between upper and lower members would be very significantly different.

Generally we have by (lxi.) for selection from the French :

$$r_{13} = \frac{\sigma_3}{\Sigma_3} \left\{ .426 \frac{s_1}{\sigma_1} + .518 \rho \frac{s_2}{\sigma_2} \right\},$$

$$r_{14} = \frac{\sigma_4}{\Sigma_4} \left\{ .329 \frac{s_1}{\sigma_1} + .515 \rho \frac{s_2}{\sigma_2} \right\},$$

$$r_{23} = \frac{\sigma_3}{\Sigma_3} \left\{ .518 \frac{s_2}{\sigma_2} + .426 \rho \frac{s_1}{\sigma_1} \right\},$$

$$r_{24} = \frac{\sigma_4}{\Sigma_4} \left\{ .515 \frac{s_2}{\sigma_2} + .329 \rho \frac{s_1}{\sigma_1} \right\}.$$

These yield for our particular case :

$$r_{13} = .819, \quad r_{14} = .694,$$

$$r_{23} = .845, \quad r_{24} = .768.$$

These are all smaller than the corresponding French values, the selection has reduced the correlation, but the Aino population has in all the cases but r_{23} a greater

correlation than the French. We must accordingly conclude that by a leg selection from the French aimed at reproducing the proportions of the Aino leg, we should not obtain an arm equivalent to the Aino arm. The divergences are indicated in the accompanying table :—

Selection from the French.		Aino.	Unselected French.
Mean humerus	30·48	29·50	33·01
„ radius	22·69	21·55	24·39
Variability of humerus	1·39	1·34	1·54
„ radius	1·11	1·06	1·17
Correlation of humerus and radius .	·799	·776	·845
„ „ femur and humerus .	·819	·858	·842
„ „ femur and radius .	·694	·789	·744
„ „ tibia and humerus .	·845	·745	·860
„ „ tibia and radius . .	·768	·865	·780

There is, of course, no special reason for supposing that the French and Aino differ merely by an evolution which has acted by selection of femur and tibia. We might have obtained a race out of the French more nearly akin to the Aino by a selection of femur and humerus, but the process would numerically be exactly similar. The particular illustration here chosen is taken merely as an instance, to indicate how the methods developed in this memoir enable us to ascertain with quantitative certainty how far racial differences may be due to the more or less stringent selection of a limited number of organs in the one race.

If we consider that local races have been differentiated from a parent stock by the selection of the chief or more markedly divergent organs, then we have in processes such as that just illustrated a method of ascertaining, at least tentatively, whether two races are to be considered as merely local varieties, and further the particular organs through selection of which the differentiation has taken place.

ILLUSTRATION II.—*Influence of a Selection of Femur and Humerus in Modifying Stature.*

The following data have been calculated for me by Miss ALICE LEE from ROLLET'S measurements on the French :—*

* They have been undertaking, with the view of determining more scientifically than appears to me yet to have been done, the mean stature of a race from a measurement of the long bones found in burial mounds, &c. ROLLET'S measurements are given in 'De la Mensuration des Os longs des Membres,' Lyons, 1889. I hope shortly to publish a memoir on the subject. [The memoir in question was published in 'Phil. Trans.,' A, vol. 192, pp. 169–244, 1898.]

	♂	♀
Mean stature, m_1	166·26 cms.	154·02 cms.
Variation of stature, σ_1	5·50 „	5·45 „
Mean femur, m_2	45·23 „	41·57 „
Variation of femur, σ_2	2·37 „	2·26 „
Mean humerus m_3	33·01 „	29·77 „
Variation of humerus, σ_3	1·54 „	1·53 „
Correlation, femur and humerus, r_{23} . .	·842	·872
„ stature and humerus, r_{13} . .	·809	·771
„ stature and femur, r_{12} . .	·811	·805

Now let us select from among French males a group having the same variability, correlation, and mean size of humerus and femur as French females, and let us ask how this would alter the variability (Σ_1), mean size (M_1) of stature in French males, and also the correlation between stature and humerus (r_{13}) and stature and femur (r_{12}).

We have at once from the second column—

$$h_2 = -3\cdot66, \qquad h_3 = -3\cdot24,$$
$$s_2 = 2\cdot26, \qquad s_3 = 1\cdot53,$$
$$\rho_{23} = \cdot872,$$

whence we find,

$$\frac{r_{12} - r_{23}r_{13}}{1 - r_{23}^2} = \cdot447, \qquad \frac{r_{13} - r_{23}r_{12}}{1 - r_{23}^2} = \cdot433.$$

From (lxv.) we deduce,

$$M_1 = 166\cdot26 + 1\cdot037h_2 + 1\cdot546h_3.$$

This formula gives the stature of any group of males selected from the French, and having their femur and humerus respectively h_2 and h_3 centims. longer than the average.

For the special selection referred to above, $h_2 = -3\cdot66$ and $h_3 = -3\cdot24$, hence

$$M_1 = 166\cdot26 - 3\cdot79 - 5\cdot01 = 157\cdot46.$$

This example shows us that if we selected French men with the same femur and humerus as French women, it would be the selection of the humerus which would contribute mostly to the reduction of stature—a somewhat singular result. Further, such a selected group of French men would be still some $3\frac{1}{2}$ centims. taller than the average of French women (instead of about $12\frac{1}{4}$ centims.). Probably had we selected the tibia as well, the greater portion of this remaining advantage in height would have disappeared.

To find the variability in stature of the selected group we must use (lviii.). We deduce :

$$\Sigma_1^2 = \sigma_1^2 \left\{ .287 + .200 \left(\frac{s_2}{\sigma_2} \right)^2 + .187 \left(\frac{s_3}{\sigma_3} \right)^2 + .387 \frac{s_2}{\sigma_2} \frac{s_3}{\sigma_3} \rho \right\}.$$

In our particular case this gives :

$$\Sigma_1 = \sigma_1 \times .987 = 5.43.$$

The actual variability in stature of French women is measured by a standard deviation of 5.45. Hence our selected group of men would be sensibly equally variable with French women, as far as absolute variation is concerned.

Lastly, from (lxi.) :

$$r_{12} = \frac{\sigma_1}{\Sigma_1} \left\{ .447 \frac{s_2}{\sigma_2} + .433 \frac{s_3}{\sigma_3} \rho_{23} \right\},$$

$$r_{13} = \frac{\sigma_1}{\Sigma_1} \left\{ .433 \frac{s_3}{\sigma_3} + .447 \frac{s_2}{\sigma_2} \rho_{23} \right\},$$

which give in our particular case :

$$r_{12} = \frac{\sigma_1}{\Sigma_1} \times .8011 = .811,$$

$$r_{13} = \frac{\sigma_1}{\Sigma_1} \times .8017 = .812.$$

Such a selection, therefore, would accordingly only increase insensibly the correlation between stature and humerus, while leaving that between stature and femur the same. The sensible reduction of correlation between stature and humerus (.809 to .771), which is found as we pass from male to female, does not arise when we select a group of males with their femur and humerus of the same length, variation, and correlation as those of the females.

ILLUSTRATION III.—*Influence of a Selection of Stature in Modifying Femur and Humerus.*

Let us select a group of French men having the same height and variability in height as French women, and calculate the changes which will arise in their femur and humerus.

Here the selection is given by :

$$h_1 = -12.24, \quad s_1 = 5.45.$$

We now need only the earlier formulæ of this memoir. From (l.) we find

$$M_2 = m_2 + .349h_1, \quad M_3 = m_3 + .227h_1.$$

These give for our case :

$$M_2 = 40.95, \quad M_3 = 30.24.$$

Thus a group of males, selected to have the same stature as the females, would have a slightly shorter femur and a slightly longer humerus. A slightly longer femur in woman and a slightly longer humerus in man would thus appear to be sexual characters.

Turning to the variations, these are given by (xlix.). We find :

$$\Sigma_2 = 2.36, \quad \Sigma_3 = 1.53.$$

This shows us that while the selection would give the same variability of humerus to the men that women have, it would fail to produce the reduction of variability in the femur, which is characteristic of the women.

From (li.) we deduce

$$r_{12} = .808, \quad r_{13} = .806.$$

while from (lvi.) we have

$$r_{23} = .840.$$

Thus we see that very small changes would be made in the correlations, stature and femur, stature and humerus, and femur and humerus, if we selected French men to have the same size and variability of stature as French women. The explanation of this lies in the nearly equal absolute variability of the two sexes with regard to stature, for, as we have seen, it is the selection of variability which modifies correlation. Looking at the table of values on p. 36, we see that the largest difference of variability in the two sexes lies in the femur, and accordingly it is from a selection of femur that we should expect the greatest differences in the variability and correlation of the two sexes to have arisen, but even this difference alone would not account for the observed sexual differences in the correlation. Indeed, it would be surprising if it did, for the selection of other organs, notably the pelvis, must have played a considerable part in the differentiation of sex.*

(9.) I shall now proceed to a series of problems, which will show the application of results, such as those obtained in this memoir, to questions which arise in dealing with inheritance and selection. If we suppose a general population to have statistical "constants," which remain constant at any rate for a moderate interval, we still want to know not only the error which may arise from a *random* sampling, but also the sort of effect which results from our sample being too much drawn from one kind of environment, from a rather limited class, or from any other practically necessary or unconsciously introduced limitation of the random character of our sample.

* "Primitive man and woman are more nearly equal in size, variability, and correlation than highly civilized man and woman" ('Roy. Soc. Proc.,' vol. 61, p. 354).

ILLUSTRATION I.—*To find the Influence on the Intensity of Parental Heredity of the Selection of Parents.*

Let the subscripts 1, 2, 3 refer respectively to father, mother, child. Let us first select one parent—say, the father—very stringently, *i.e.*, $s_1/\sigma_1 = \mu_1$ is small. Then we need only equations (xlix.) and (li.). These give us :

$$r_{13} = \frac{\mu_1 r'_{13}}{\sqrt{1 - (1 - \mu_1^2) r_{13}^2}},$$

and

$$\Sigma_2^2 = \sigma_2^2 (1 - (1 - \mu_1^2) r_{13}^2).$$

The first may be written

$$r_{13} = \frac{r'_{13}}{\sqrt{r_{13}^2 + \frac{1 - r_{13}^2}{\mu_1^2}}},$$

or, we see that r_{13} will decrease, as μ_1 decreases. Thus if $r_{13} = .4$ we have for

$\mu_1 = 1/2,$	$r_{13} = .2132,$	$\Sigma_2/\sigma_2 = .8367,$
$\mu_1 = 1/4,$	$r_{13} = .1085,$	$\Sigma_2/\sigma_2 = .7906,$
$\mu_1 = 1/8,$	$r_{13} = .0545,$	$\Sigma_2/\sigma_2 = .7786.$

It is clear, therefore, that the correlation of parent and child will be much reduced by such a selection. On the other hand, the regression coefficient will not be altered, *i.e.*, $\Sigma_2 r_{13}/s_1 = \sigma_2 r'_{13}/\sigma_1$, as we have seen. Hence in problems of heredity, where we suspect a parent to have been highly selected, we should seek for the regression of son on parent rather than for the correlation. Thus in the case of Basset Hounds,* some if not all the reduction in correlation between sire and offspring may be due to selection of the sire. A test of whether the reduction in correlation is due to selection of a parent ought to be given by a comparison of s_1 and Σ_2 . We cannot, I think, suppose, unless natural selection be very stringent, that σ_2 differs much from σ_1 . Hence it follows that Σ_2/s_1 ought generally to be large, if there be selection of a parent. We can hardly test this point effectively in the case of the Basset Hounds, owing to the nature of the classification. In racehorses, although the sire appears to be far more selected than the dam, there is not a great reduction of the coefficient of correlation between sire and offspring,† s_1 appears to be less than Σ_2 , but not so greatly and certainly less, that we can be surprised that the correlation of sire and offspring is not much less than we have found it for material in which selection of the father is certainly far less marked. We must accept the warning as to the reduction

* 'Roy. Soc. Proc.,' vol. 66, p. 157.

† See 'Phil. Trans.,' A, vol. 195, p. 93.

in correlation produced by the stringent selection of one parent, but we must remember the complexity of the factors—the variety of other influences at work in selecting and modifying selection—before we lay much stress on this source of alteration in parental correlation.

Now let us deal with the case of both parents selected, and suppose their selection given by $s_1/\sigma_1 = \mu_1$, $s_2/\sigma_2 = \mu_2$, and the change of their coefficient of assortative mating from r_{12} to ρ_{12} . We have from formulæ (lxviii.) and (lxx.) by a little rearranging

$$\begin{aligned}\Sigma_3^2 &= \sigma_3^2 \{1 - \beta_{13}^2 (1 - \mu_1^2) - \beta_{23}^2 (1 - \mu_2^2) - 2(r_{12} - \rho_{12}\mu_1\mu_2) \beta_{13}\beta_{23}\} \\ \Sigma_3 r_{13} &= \sigma_3 \{r_{13} - (1 - \mu_1) \beta_{13} - (r_{12} - \rho_{12}\mu_2) \beta_{23}\} \quad \dots \quad \text{(lxxii.)}\end{aligned}$$

where $\beta_{13} = (r_{13} - r_{23}r_{12}) / (1 - r_{12}^2)$, $\beta_{23} = (r_{23} - r_{13}r_{12}) / (1 - r_{12}^2)$. Now let us take special cases to bring out points. Let us suppose $r_{12} = 0$, or no assortative mating to exist, and let us enquire what change would be made in parental correlation if we selected parents who had assortatively mated, without altering their variability, *i.e.*, let us take $\mu_1 = 1$, $\mu_2 = 1$, we have at once

$$\Sigma_3^2 = \sigma_3^2 (1 + 2\rho_{12}r_{13}r_{23}) \quad \dots \quad \text{(lxxiii.)},$$

$$\Sigma_3 r_{13} = \sigma_3 (r_{13} + \rho_{12}r_{23}) \quad \dots \quad \text{(lxxiv.)},$$

whence we find for $r_{13} = r_{23} = \cdot 4$ and

$\rho_{12} = \cdot 1,$	$\Sigma_3/\sigma_3 = 1\cdot 0149,$	$r_{13} = \cdot 4335,$
$\rho_{12} = \cdot 2,$	$= 1\cdot 0315,$	$= \cdot 4653,$
$\rho_{12} = \cdot 3,$	$= 1\cdot 0469,$	$= \cdot 4967,$
$\rho_{12} = \cdot 4,$	$= 1\cdot 0621,$	$= \cdot 5271,$
$\rho_{12} = \cdot 5,$	$= 1\cdot 0770,$	$= \cdot 5571,$
$\rho_{12} = 1,$	$= 1\cdot 1489,$	$= \cdot 6963.$

Thus the general effect of assortative pairing of parents is to increase the correlation between parent and offspring sensibly, but not to very rapidly increase the variability of the offspring. Thus marriages within a class would, if heredity statistics were collected for a class, tend to show increased parental correlation. Very high assortative mating no doubt occurs with some forms of breeding, and we may well find in such cases higher values of the parental heredity than we should obtain for a population of the same species with random mating. I think this may be an effective factor in the raising of the parental correlation in the case of coat-colour in thoroughbred horses.

Now let us see what happens if we select both parents moderately. As test cases, let us take $\mu_1 = \mu_2 = \cdot 8$ and $\cdot 5$, and for the extreme $= 0$.

We have at once

$$\Sigma_3 = \sigma_3 \sqrt{\cdot 68 + \cdot 32 \mu_1^2 (1 + \rho_{12})} \quad \dots \quad (lxxv.),$$

$$\Sigma_3 r_{13} = \sigma_3 \{ \cdot 4 \mu_1 (1 + \rho_{12}) \} \quad \dots \quad (lxxvi.).$$

Hence we deduce

ρ_{12} ·	$\mu_1 = \mu_2 = \cdot 8.$		$\mu_1 = \mu_2 = \cdot 5.$		$\mu_1 = \mu_2 = 0.$	
	$\Sigma_3/\sigma_3.$	$r_{13}.$	$\Sigma_3/\sigma_3.$	$r_{13}.$	$\Sigma_3/\sigma_3.$	$r_{13}.$
·1	·9515	·3700	·8764	·2865	·8246	0
·2	·9622	·3991	·8809	·3093	·8246	0
·3	·9727	·4277	·8854	·3316	·8246	0
·4	·9832	·4556	·8899	·3535	·8246	0
·5	·9936	·4831	·8944	·3750	·8246	0
1	1·0438	·6131	·9165	·4762	·8246	0

This table is very instructive. It shows us that selection and assortative mating are factors of opposite influence; that if selection be only moderate, then with considerable assortative mating the coefficient of parental correlation may be increased, but if selection be stringent, then assortative mating cannot counteract, even if as high as $\cdot 5$, its destructive influence on parental correlation.

For example, if we take parents remarkable for some intellectual or physical character, say with a variability only a very small fraction of that of the general population, then, however proportionately we might pair them, we should find their relationship to their children, as measured by the coefficient of correlation, very sensibly reduced below that of the general population. I think we have here the reason why Mr. GALTON's Family Data, which were drawn from a rather narrow class, and had only a small coefficient of assortative mating, give so much smaller parental correlation than my own Family Data, which seem to me drawn from a wider class, and have a considerably higher assortative mating.*

It will be clear that with factors like assortative mating, natural selection, artificial selection of breeders, unconscious selection of material from one class or one environment, modifying our coefficients of heredity in one or another direction, we can hardly hope for more in practical statistics than an approximation to the strength of the pure inheritance factor by dealing with the average of as many races and characters as possible.

* The work for Mr. GALTON's Family Data is given, 'Phil. Trans.,' A, vol. 187, p. 270. My own results are as yet unpublished. The average value is about $\cdot 45$, as compared with Mr. GALTON's $\cdot 34$

ILLUSTRATION II.—*To Find the Influence of Parental Selection on Modifying Fraternal Correlation.*

Let the subscripts 1, 2 represent the parents, and 3 and 4 two of their offspring. Let us first select one parent only, the selection being given as before by $s_1/\sigma_1 = \mu_1$. Our formulæ will now be (liv.) and (lvi.). So far as the change in variability is concerned, we have already discussed it under our first illustration, so we need only consider :

$$r_{34} = \frac{(r_{34} - r_{13}r_{14})(1 - \mu_1^2) + \mu_1^2 r_{34}}{\sqrt{\{1 - (1 - \mu_1^2)r_{13}^2\}\{1 - (1 - \mu_1^2)r_{14}^2\}}} \quad \text{. (lxxvii.).}$$

Now $r_{13} = r_{14}$, if the offspring are of one sex ; hence :

$$r_{34} = \frac{r_{34} - r_{13}^2(1 - \mu_1^2)}{1 - r_{13}^2(1 - \mu_1^2)} \quad \text{. (lxxviii.).}$$

If we take $r_{13} = .4$ and $r_{34} = .5$ as reasonable values, we have

$$r_{34} = \frac{.34 + .16\mu_1^2}{.84 + .16\mu_1^2} \quad \text{. (lxxix.).}$$

Thus r_{34} will be greatest when μ_1 is greatest, *i.e.*, when there is no selection, and will decrease with μ_1 until it reaches .4048, when $\mu_1 = 0$, or there is selection of fathers of one value of the character only.*

The selection of one parent only does not, therefore, immensely modify the correlation of brothers. Still, if we work sensibly with one class of the community—say, men of genius—we should expect to find their sons rather less like each other than if we worked with the general population of brothers.

Now let us select both parents. Here again the variability of the offspring has already been dealt with. We are concerned with equation (lxix.), and we shall put $r_{13} = r_{14} = r_{23} = r_{24} = r$, or make parental influence equipotent for the two sexes. Hence

$$\beta_{13} = \beta_{23} = \beta_{14} = \beta_{24} = \beta = \frac{r}{1 + r_{12}},$$

where r is the parental correlation, and r_{12} the coefficient of assortative mating. Hence we find

$$r_{34} = \frac{r_{34} - \beta^2 \{1 - \mu_1^2 + 1 - \mu_2^2 + 2(r_{12} - \rho_{12}\mu_1\mu_2)\}}{1 - \beta^2 \{1 - \mu_1^2 + 1 - \mu_2^2 + 2(r_{12} - \rho_{12}\mu_1\mu_2)\}} \quad \text{. (lxxx.).}$$

To reduce to numbers, suppose $\mu_1 = \mu_2$, and $r_{12} = 0$ for the general population. We have

$$r_{34} = \frac{r_{34} - 2r^2 \{1 - \mu_1^2(1 + \rho_{12})\}}{1 - 2r^2 \{1 - \mu_1^2(1 + \rho_{12})\}} \quad \text{. (lxxx.).}$$

* In general the value of r_{34} ranges from r_{34} down to $\frac{r_{34} - r_{13}^2}{1 - r_{13}^2}$.

Hence if we put $r_{34} = \cdot 5$, and $r = \cdot 4$,

$$r_{34} = \frac{\cdot 18 + \cdot 32\mu_1^2(1 + \rho_{12})}{\cdot 68 + \cdot 32\mu_1^2(1 + \rho_{12})} \dots \dots \dots (lxxxii.).$$

The following table will suffice to indicate the changes which take place, when we give a series of values to μ_1 and ρ_{12} . Thus the first row gives the influence of selecting parents without any assortative mating. We see that with increasing stringency of selection the reduction of correlation is very considerable, and that with such selection the influence of assortative mating becomes less and less. Nevertheless, assortative mating can produce quite sensible results, if there be little or no selection. I am, indeed, inclined to think that a good deal of the high values found for the fraternal colour correlation in the thoroughbred foals* is due to much assortative colour mating in sire and dam. Of course it cannot be all due to this source.

VALUES of Fraternal Correlation with Parental Selection.

	$\mu_1 = 1.$	$\mu_1 = \cdot 8.$	$\mu_1 = \cdot 6.$	$\mu_1 = \cdot 4.$	$\mu_1 = \cdot 2.$	$\mu_1 = 0.$
$\rho_{12} = 0$	$\cdot 5000$	$\cdot 4349$	$\cdot 3712$	$\cdot 3162$	$\cdot 2783$	$\cdot 2647$
$\rho_{12} = \cdot 1$	$\cdot 5155$	$\cdot 4477$	$\cdot 3802$	$\cdot 3209$	$\cdot 2796$	$\cdot 2647$
$\rho_{12} = \cdot 2$	$\cdot 5301$	$\cdot 4599$	$\cdot 3889$	$\cdot 3256$	$\cdot 2809$	$\cdot 2647$
$\rho_{12} = \cdot 3$	$\cdot 5438$	$\cdot 4716$	$\cdot 3974$	$\cdot 3303$	$\cdot 2823$	$\cdot 2647$
$\rho_{12} = \cdot 5$	$\cdot 5690$	$\cdot 4935$	$\cdot 4137$	$\cdot 3392$	$\cdot 2849$	$\cdot 2647$
$\rho_{12} = 1$	$\cdot 6212$	$\cdot 5411$	$\cdot 4508$	$\cdot 3609$	$\cdot 2914$	$\cdot 2647$

On the whole, I think, we may conclude, so far as the relative influences of sexual selection in the form of assortative mating and natural selection go, that :

Both sexual and natural selection can sensibly modify the intensity of inheritance as measured by the coefficient of correlation, the former tends to raise, the latter to lower, its intensity. But the effect of the latter, if at all stringent, is to completely mask the effect of the former.

In fact, we may write

$$r_{34} = 1 - \frac{1 - r_{34}}{1 - 2r^2 + 2r^2\mu_1^2(1 + \rho_{12})}.$$

Hence the smaller $\mu_1^2(1 + \rho_{12})$, the smaller will be fraternal correlation. This varies as the square of μ_1 and only as the linear power of $1 + \rho_{12}$. Thus we see at once why stringency of selection is far more potent than assortative mating.

ILLUSTRATION III.—*To find the influence of selecting two organs A and B in a parent, on the correlation of the like organs A' and B' in the offspring.*

Let the organs in the parent be denoted by 1 and 2, and in the offspring by 3 and 4. Suppose the organic correlation of the two organs in the general population to be r' ,

* ‘Phil. Trans.,’ A, vol. 195, p. 93.

so that $r_{12} = r_{34} = r'$ before any selection takes place. Let r , the correlation of the organs in the parent and offspring be supposed to be the same for both organs; then $r = r_{13} = r_{24}$. Finally we have the coefficients of cross-heredity, r_{14} and r_{23} . These must vanish if there be no heredity and no organic correlation, and should be perfect if both these are perfect. Hence we will take $r_{14} = r_{23} = rr'$ as a probable hypothesis.* With these values of the correlation coefficients we easily find

$$\beta_{13} = r = \beta_{24}, \quad \beta_{23} = \beta_{14} = 0.$$

Hence from (lxviii.) and (lxix.) we have :

$$\Sigma_3^2 = \sigma_3^2 \{1 - r^2 + r^2 \mu_1^2\}, \quad \Sigma_1^2 = \{1 - r^2 + r^2 \mu_2^2\} \quad \dots \quad (\text{lxxxiii.}),$$

$$r_{34} = \frac{r'(1 - r^2) + \rho_{12} r^2 \mu_1 \mu_2}{\sqrt{\{1 - r^2 + r^2 \mu_1^2\} \{1 - r^2 + r^2 \mu_2^2\}}} \quad \dots \quad (\text{lxxxiv.}).$$

For simplicity, suppose the stringency of the selection to be the same for both organs, then :

$$r_{34} = \frac{r' + \rho_{12} \mu_1^2 \frac{r^2}{1 - r^2}}{1 + \mu_1^2 \frac{r^2}{1 - r^2}}.$$

If $r = .4$, and $\rho_{12} = \gamma r'$,

$$r_{34} = r' \times \frac{1 + \frac{\gamma}{2.1} \mu_1^2}{1 + \frac{\gamma}{2.1} \mu_1^2} \quad \dots \quad (\text{lxxxv.}).$$

The following table indicates the value of r_{34}/r' :—

	$\mu_1 = 1.$	$\mu_1 = .8.$	$\mu_1 = .6.$	$\mu_1 = .4.$	$\mu_1 = .2.$	$\mu_1 = 0.$
$\gamma = 1$	1	1	1	1	1	1
$\gamma = 0.8$	.9680	.9783	.9872	.9941	.9985	1
$\gamma = 0.6$	.9360	.9565	.9743	.9882	.9970	1
$\gamma = 0.4$	.9040	.9348	.9615	.9823	.9955	1
$\gamma = 0.2$	.8720	.9131	.9487	.9763	.9940	1
$\gamma = 0.0$	.8400	.8913	.9358	.9704	.9924	1

It will be clear from this table that if the selection be at all stringent, no *reduction* of organic correlation in the parents will affect substantially the organic correlation in the offspring.

On the other hand, if γ be > 1 , we can have considerable modifications in the value of the correlation, even if the selection be stringent.

* See 'Roy. Soc. Proc.,' vol. 62, p. 411. I have a good deal of data on the value of these cross-heredity correlations now reduced and soon to be published.

Thus we have the following values of r_{31}/r' , if:—

	$\mu_1 = 1.$	$\mu_1 = .8.$	$\mu_1 = .6.$	$\mu_1 = .4.$	$\mu_1 = .2.$	$\mu_1 = 0.$
$\gamma = 2$	1.1600	1.1087	1.0642	1.0296	1.0076	1
$= 10$	2.4400	1.9779	1.5775	1.2662	1.0681	1
$= 50$	8.8400	6.3243	4.1448	2.4492	1.3705	1

Lastly, if $r' = 0$:

$$r_{31} = \rho_{12} \frac{1}{1 + \frac{21}{4} \frac{1}{\mu_1^2}},$$

or, r_{31}/ρ_{12} is given by:

	$\mu_1 = 1.$	$\mu_1 = .8.$	$\mu_1 = .6.$	$\mu_1 = .4.$	$\mu_1 = .2.$	$\mu_1 = 0.$
r_{31}/ρ_{12}	.1600	.1087	.0642	.0296	.0076	0

Thus, even if there were no correlation between the organs A and B in the general population, still a selection of parents in which such organs were correlated would lead to offspring with correlated organs A' and B'. The amount of such correlation would only be .1600, if the variability of the parent were not selected, and would diminish rapidly with stringent selection of variability. Still .1600 is quite sensible, and would, if the selection continued for a few generations, continue to increase. Thus we see how selection of a pair of organs in a parent may increase or even create correlation between the like organs in the offspring.

The reader will find other interesting illustrations in tracing the influence of an absolute selection of one parent only on the correlation of the offspring, *e.g.*, relation between pairs of foals which all have a common sire, the influence of selecting an organ A in the sire and an organ B in the dam on the correlation of the organs A and B in the offspring, the influence on assortative mating of selecting parents of men of genius,* and in many other problems.

(10.) It is not without value to consider how ρ_{12} arises in the case of natural or artificial selection. Suppose we have two organs, A and B, then we shall endeavour (i.) to give these definite values, say $\bar{x}$ and $\bar{y}$, but we shall not be able to get all our individuals with such absolute values, we shall select with certain deviations from $\bar{x}$ and $\bar{y}$, given by $x = \bar{x} + x'$ and $y = \bar{y} + y'$, say. Further, we shall endeavour to

* This is a peculiarly interesting case, for if we select men of remarkable intellectual ability, we should expect to find both parents above the average of the general population, but with a *negative* correlation between them amounting at a maximum to $-.1905$.

make y some function of x , say $y = f(x)$ or $\bar{y} + y' = f(\bar{x}) + x' f'(\bar{x}) + \frac{x'^2}{1.2} f''(\bar{x}) + \dots$ by TAYLOR'S theorem. But $\bar{y} = f(\bar{x})$ and x' is small, so that our attempted relation will be of the form :

$$y' = mx'.$$

Here m is the slope of a line to which we endeavour to confine the selected organs. But we shall not be quite able to exactly hit this relation off; when $x' = \epsilon$, we shall find that $y' = m\epsilon + \eta$, where η is a small unavoidable error in selection of y' uncorrelated with ϵ . Thus, if s_1 and s_2 be the selected variabilities, we shall have :

$$s_2^2 = \frac{1}{n} S(y'^2) = \frac{1}{n} S(m\epsilon + \eta)^2, \quad s_1^2 = \frac{1}{n} S(\epsilon^2).$$

Therefore :

$$s_2^2 = m^2 s_1^2 + \frac{1}{n} S(\eta^2).$$

Further :

$$\rho_{12} = \frac{S(y'\epsilon)}{ns_1s_2} = \frac{S\{(m\epsilon + \eta)\epsilon\}}{ns_1s_2} = \frac{ms_1^2}{s_1s_2} = \frac{ms_1}{s_2},$$

and therefore :

$$\frac{1}{n} S(\eta^2) = s_2^2(1 - \rho_{12}^2).$$

Or, ρ_{12} is at once obtained from the slope of the line m , by which we endeavour to fix the relationship of the organs A and B. Or, again, we may look upon $s_2\sqrt{1 - \rho_{12}^2}$ as a quantity measuring the mean divergence of the B organ from that absolute fulfilment of the relationship between A and B which we are striving to attain. Thus ρ_{12} is a quantity which naturally arises in every attempt, whether artificial or natural, to select organs having a definite relationship to each other.

Much the same considerations arise when we select three or more organs. In each case the selected coefficients of correlation are constants which enable us to express (i.) to a first approximation the form of relationship we are aiming at, and (ii.) the average degree of divergence from absolute fulfilment of this relationship.

Thus, without regard to any particular distribution of frequency, the s 's and the ρ 's are the appropriate constants to express approximately the nature of any form of natural or artificial selection.

(11.) *On the Probability of Survival and the Surface of Survival Rates.*

In the course of the present paper I have assumed that when measurements are made on any population for a complex of n organs, the frequency surface may be taken as approximately normal. If this holds for the population before and after selection, and measurements made on many groups at different periods of life seem to indicate that it is approximately true, it follows that we can determine the form of the probability of survival as a function of the means, variations, and correlations of the selected and unselected populations.

Let the unselected population be given by

$$Z = Z_0 \text{ expt. } - \frac{1}{2} \{ c_{11}x_1^2 + c_{22}x_2^2 + \dots + c_{nn}x_n^2 \\ + 2c_{12}x_1x_2 + \dots + 2c_{n-1,n}x_{n-1}x_n \} \dots \quad (\text{lxxxvi}).$$

Let the probability of survival be given by

$$p = p_0 f(x_1 - k_1, \quad x_2 - k_2, \quad x_3 - k_3, \dots x_n - k_n) \dots \quad (\text{lxxxvii}),$$

where f is at present an unknown function, which is to be a maximum for

$$x_1 = k_1, \quad x_2 = k_2, \quad x_3 = k_3, \dots x_n = k_n,$$

and, if the selection be at all stringent, to take rapidly decreasing values as

$$x_1 - k_1, \quad x_2 - k_2, \quad x_3 - k_3, \dots x_n - k_n$$

take increasing large negative or positive values. It will be clear then that the individuals who are "fittest to survive," i.e., have the smallest death-rate, are those whose organs are defined by :

$$x_1 = k_1, \quad x_2 = k_2, \dots x_n = k_n,$$

and fitness generally will be measured by the closeness of the individual to these "fittest" individuals.

In order to find the surface of survivors, immediately after the selection if growth be taking place,* or at any later stage if growth have ceased, we have only to multiply Z by p , or :

$$z = Z \times p \dots \dots \dots (\text{lxxxviii}).$$

is what in the earlier part of this memoir I have termed the selection surface. Now if this selection surface be itself normal, it will be of the form :

$$z = z_0 \text{ expt. } - \frac{1}{2} \{ b_{11}(x_1 - h_1)^2 + b_{22}(x_2 - h_2)^2 + \dots \\ + b_{nn}(x_n - h_n)^2 + 2b_{12}(x_1 - h_1)(x_2 - h_2) \\ - \dots + 2b_{n-1,n}(x_{n-1} - h_{n-1})(x_n - h_n) \dots \quad (\text{lxxxix}).$$

Here, as in the value of Z , all the constants $b_{11}, b_{22}, \dots b_{nn}, b_{12} \dots b_{n-1,n}$ are known in terms of the variations and correlations. If there be selection of q organs only out of the n , then $b_{q+1,q+1} \dots b_{nn}, b_{1,q+1}, b_{2,q+1}, \dots b_{n-1,n}$, will all be zero. Since by Equation (lxxxviii.) $p = z/Z$, it follows that the function f which defines the probability of survival must be of the normal exponential type, or

* I propose to deal in another memoir with the important problems of slow selection during rapid growth, and of secular selection during several generations.

$$f(x_1 - k_1, x_2 - k_2, \dots, x_n - k_n) = \text{expt.} - \frac{1}{2} \{ a_{11}(x_1 - k_1)^2 + a_{22}(x_2 - k_2)^2 + \dots \\ + 2a_{12}(x_1 - k_1)(x_2 - k_2) - \dots + 2a_{n-1,n}(x_{n-1} - k_{n-1})(x_n - k_n) \} \quad (\text{xc.}).$$

Thus, to determine the probability of survival, we require to know the values of the a 's and k 's in terms of the b 's, h 's, and c 's. The shortest method of finding p_0 is to put $x_1 = k_1, x_2 = k_2, \dots, x_n = k_n$, and then note that :

$$p_0 = \frac{z(x_1 = k_1, x_2 = k_2, \dots, x_n = k_n)}{Z(x_1 = k_1, x_2 = k_2, \dots, x_n = k_n)} \quad (\text{xc.}).$$

Since $p = z/Z$, and $z = p/Z^{-1}$,

we see that the relations for p , given z and Z , and for z , given p and Z , are cyclicly interchangeable if at the same time we change c_{11} to $-c_{11}$, c_{22} to $-c_{22}$ $\dots$ c_{12} to $-c_{12}$ $\dots$ $c_{n-1,n}$ to $-c_{n-1,n}$. If $\sigma_1, \sigma_2, \dots, \sigma_n$ be the standard deviations of the unselected population, this amounts to changing $\sigma_1, \sigma_2, \dots, \sigma_n$ to $\sqrt{-1}\sigma_1, \sqrt{-1}\sigma_2, \dots, \sqrt{-1}\sigma_n$ respectively. Thus the results which give the probability of survival in terms of unselected and selected populations can always by an easy interchange be used to obtain the selected population from a knowledge of the unselected population and of the probability of survival.

Let the unselected population be defined by $m_1, m_2, \dots, m_n, \sigma_1, \sigma_2, \dots, \sigma_n$, and $r_{12}, r_{13}, r_{23}, \dots, r_{n-1,n}$.

Let the selected population be defined by $m_1 + H_1, m_2 + H_2, \dots, m_n + H_n, \Sigma_1, \Sigma_2, \dots, \Sigma_n$, and $r_{12}, r_{13}, r_{23}, \dots, r_{n-1,n}$.

Let the constants of the probability of survival function, or $a_{11}, a_{22}, \dots, a_{nn}, a_{12}, \dots, a_{n-1,n}$, be expressed by $\bar{s}_1, \bar{s}_2, \dots, \bar{s}_n, \bar{\rho}_{12}, \bar{\rho}_{13}, \dots, \bar{\rho}_{n-1,n}$ as if it were a normal correlation surface.*

Then the problem will be solved, if we know the k 's, $\bar{s}$'s, and $\bar{\rho}$'s in terms of the σ 's, r 's, H 's, Σ 's, and r 's.

Equating the squares, products, and linear terms in the x 's in the equation $p = z/Z$, we have at once the system :

$$\left. \begin{aligned} a_{uu} &= b_{uu} - c_{uu}, \\ a_{uv} &= b_{uv} - c_{uv}, \end{aligned} \right\} \dots \dots \dots (\text{xcii.}),$$

for all values of u and v from 1 to n .

$$\begin{aligned} -a_{e1}k_1 - a_{e2}k_2 - \dots - a_{ev}k_v - \dots - a_{en}k_n \\ = -b_{e1}H_1 - b_{e2}H_2 - \dots - b_{ev}H_v - \dots - b_{en}H_n \end{aligned} \quad (\text{xciii.}),$$

for all values of e from 1 to n .

* These must not be confused with the $s_1, s_2, \dots, s_q, \rho_{12}, \rho_{13}, \rho_{23}, \dots, \rho_{q,q-1}$ constants of the q selected organs of the previous discussion. The new quantities may be in part *imaginary*.

If we now substitute from the first two equations for the α 's we find:

$$\begin{aligned} & (c_{r1} - b_{r1})k_1 + (c_{r2} - b_{r2})k_2 + \dots + (c_{rv} - b_{rv})k_v + \dots + (c_{rn} - b_{rn})k_n \\ & = -b_{r1}H_1 - b_{r2}H_2 - \dots - b_{rv}H_v - \dots - b_{rn}H_n. \quad (\text{xiv}). \end{aligned}$$

Now let Δ be the determinant

$$\begin{array}{ccccccc} c_{11} - b_{11}, & c_{12} - b_{12}, & c_{13} - b_{13}, & . & . & . & c_{1n} - b_{1n} \\ c_{21} - b_{21}, & c_{22} - b_{22}, & c_{23} - b_{23}, & . & . & . & c_{2n} - b_{2n} \\ c_{31} - b_{31}, & c_{32} - b_{32}, & c_{33} - b_{33}, & . & . & . & c_{3n} - b_{3n} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ c_{n1} - b_{n1}, & c_{n2} - b_{n2}, & c_{n3} - b_{n3}, & . & . & . & c_{nn} - b_{nn} \end{array}$$

and Δ_{uv} the minor of its uv^{th} constituent.

Then we have at once :

[illegible]

Or :

[illegible]

Thus k_v is determined in terms of $H_1, H_2, \dots, H_n$, which define the maximum frequency of survival.

In a similar manner by making the proper changes indicated above we have :

$$\begin{aligned} \Delta' \times H_r = & k_1(a_{11}\Delta'_{r1} + a_{12}\Delta'_{r2} + a_{13}\Delta'_{r3} + \dots + a_{1n}\Delta'_{rn}) \\ & + k_2(a_{21}\Delta'_{r1} + a_{22}\Delta'_{r2} + a_{23}\Delta'_{r3} + \dots + a_{2n}\Delta'_{rn}) \\ & + k_3(a_{31}\Delta'_{r1} + a_{32}\Delta'_{r2} + a_{33}\Delta'_{r3} + \dots + a_{3n}\Delta'_{rn}) \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & + k_n(a_{n1}\Delta'_{r1} + a_{n2}\Delta'_{r2} + a_{n3}\Delta'_{r3} + \dots + a_{nn}\Delta'_{rn}). \end{aligned} \quad (\text{xvii}),$$

where Δ' is the determinant

(12.) CASE (i).—*Selection of a Single Organ only.*

The original population is given by

$$Z = Z_0 e^{-c^2 \langle 2\sigma^2 \rangle},$$

and the curve of survivors by

$$z = z_0 e^{-(x-H)^2/2\Sigma^2}.$$

The probability of survival is

$$p = p_0 e^{-(x-k)^2/2\bar{s}^2},$$

where we easily find, if $\Sigma/\sigma = \lambda$,

$$k = H/(1 - \lambda^2) (\text{xvii}),$$

$$\bar{s} = \Sigma / \sqrt{1 - \lambda^2}. \quad (\text{xviii}),$$

and

$$p_0 = \frac{z_0}{Z_0} e^{\frac{H^2}{2\sigma^2(1-\lambda^2)}} \quad (\text{xcx}).$$

As an illustration consider a selection from modern French peasants, which should reduce the mean and variability of their cephalic index to those of the Libyan race.

French peasants :—

$$m = 79.786, \quad \sigma = 3.841.$$

Libyans :—

$$m + H = 72.938, \quad \Sigma = 2.885.$$

Hence :

$$H = -6.848, \quad \lambda = .7511.$$

These give :

$$k = -15.712, \quad \bar{s} = 4.370.$$

Thus for such a change as 7 points in the cephalic index to take place by selection* we should have to make the “fittest to survive” of such a ridiculously low cephalic index as 64·074, and such a high variation as 4·370.

We find

$$p_0 = 51.0474 \text{ n/N},$$

and accordingly the probability of survival given by

$$p = 51.0474 \frac{n}{N} e^{-(x+15.712)^2/(38.1939)},$$

where N are the number of Frenchmen converted into n Libyans so far as cephalic index is concerned.

I have purposely taken a somewhat extreme case of selection in order to illustrate how widely the most frequently surviving individual can diverge from the fittest.

In this case, if the chances of survival (i.) of the fittest, (ii.) of the individuals most frequent after selection, and (iii.) of the individuals most frequent before selection, be C_1 , C_2 , and C_3 respectively, we have :

* This is, of course, supposing the change to occur by catastrophic selection and not by a continuous secular selection, see footnote preceding page.

$$C_1 : C_2 : C_3 :: 1 : \cdot 12782 : \cdot 00156,$$

or, the chances of survival of an individual of the fittest type would be about eight times as great as those of an individual of the most frequent type after selection and about 700 times as great as those of an individual of the most frequent type before selection. If ν_1, ν_2, ν_3 be the numbers after selection in the three classes* of the fittest to survive, the most frequent after selection and the most frequent before selection, we find

$$\nu_1 : \nu_2 : \nu_3 :: \cdot 00892 : 1 : \cdot 05978.$$

In other words, the most numerous type before selection is still after selection about 6·7 times as numerous as the type with the least mortality, and this latter type is only about $\frac{1}{111}$ as numerous as the type to be most frequently met with after selection has taken place.

Thus, although there would have been a very great evolution in cephalic index. due to a fairly stringent selection, the fittest to survive would always have formed but a small fraction of the dominant type. Even if we were to replace the selection here considered by a gradual evolution spread over several generations, we should still reach in the main the same conclusion, *i.e.*, that *natural selection never proceeds by the survival of the fittest, or the survival of those with the least death-rate.* These will always remain a small fraction of the community—they are the goal, but often the very distant goal, to which selection tends to shift the population.

(13.) CASE (ii.)—*Selection of Two Organs.*

In this case let the surface of survivors be :

$$z = z_0 \text{ expt. } - \frac{1}{2} \left\{ \frac{(x_1 - H_1)^2}{\Sigma_1^2 (1 - r_{12}^2)} - \frac{2r_{12}(x_1 - H_1)(x_2 - H_2)}{\Sigma_1 \Sigma_2 (1 - r_{12}^2)} + \frac{(x_2 - H_2)^2}{\Sigma_2^2 (1 - r_{12}^2)} \right\} \quad \text{(c.)},$$

the original population :

$$Z = Z_0 \text{ expt. } - \frac{1}{2} \left\{ \frac{x_1^2}{\sigma_1^2 (1 - r_{12}^2)} - \frac{2r_{12}x_1x_2}{\sigma_1\sigma_2 (1 - r_{12}^2)} + \frac{x_2^2}{\sigma_2^2 (1 - r_{12}^2)} \right\} \quad \text{(ci.)},$$

and the curve of probability of survival :

$$p = p_0 \text{ expt. } - \frac{1}{2} \left\{ \frac{(x_1 - k_1)^2}{\bar{s}_1^2 (1 - \bar{r}_{12}^2)} - \frac{2\bar{r}_{12}(x_1 - k_1)(x_2 - k_2)}{\bar{s}_1\bar{s}_2 (1 - \bar{r}_{12}^2)} + \frac{(x_2 - k_2)^2}{\bar{s}_2^2 (1 - \bar{r}_{12}^2)} \right\} \quad \text{(cii.)}.$$

Since :

$$p = z/Z,$$

* By individuals of a type or class is meant here, as elsewhere in this section, all the group falling within some small definite range of variation lying round a particular value of the organ (*e.g.*, $m, m + H$, or $m + k$), which defines the type or class.

we find at once

$$\frac{1}{\bar{s}_1^2(1 - \bar{\rho}_{12}^2)} = \frac{1}{\Sigma_1^2(1 - r_{12}^2)} - \frac{1}{\sigma_1^2(1 - r_{12}^2)} \quad \dots \quad \text{(ciii.)}$$

$$\frac{1}{\bar{s}_2^2(1 - \bar{\rho}_{12}^2)} = \frac{1}{\Sigma_2^2(1 - r_{12}^2)} - \frac{1}{\sigma_2^2(1 - r_{12}^2)} \quad \dots \quad \text{(civ.)}$$

$$\frac{1}{\bar{s}_1\bar{s}_2} \frac{\bar{\rho}_{12}}{1 - \bar{\rho}_{12}^2} = \frac{r_{12}}{\Sigma_1\Sigma_2(1 - r_{12}^2)} - \frac{r_{12}}{\sigma_1\sigma_2(1 - r_{12}^2)} \quad \dots \quad \text{(cv.)}$$

$$\frac{k_1}{\bar{s}_1^2(1 - \bar{\rho}_{12}^2)} - \frac{k_2\bar{\rho}_{12}}{\bar{s}_1\bar{s}_2(1 - \bar{\rho}_{12}^2)} = \frac{H_1}{\Sigma_1^2(1 - r_{12}^2)} - \frac{H_2r_{12}}{\Sigma_1\Sigma_2(1 - r_{12}^2)} \quad \dots \quad \text{(cvi.)}$$

$$- \frac{k_1\bar{\rho}_{12}}{\bar{s}_1\bar{s}_2(1 - \bar{\rho}_{12}^2)} + \frac{k_2}{\bar{s}_2^2(1 - \bar{\rho}_{12}^2)} = - \frac{r_{12}H_1}{\Sigma_1\Sigma_2(1 - r_{12}^2)} + \frac{H_2}{\Sigma_2^2(1 - r_{12}^2)} \quad \dots \quad \text{(cvii.)}$$

Let $\Sigma_1/\sigma_1 = \lambda_1$, $\Sigma_2/\sigma_2 = \lambda_2$ measure the stringency of the selection, and $\mu = \sqrt{\frac{1 - r_{12}^2}{1 - r_1^2}}$ measure the change in correlation.* Then solving the above equations we find :

$$\bar{\rho}_{12} = \frac{r_{12} - r_{12}\mu^2\lambda_1\lambda_2}{\sqrt{1 - \mu^2\lambda_1^2}\sqrt{1 - \mu^2\lambda_2^2}} \quad \dots \quad \text{(cviii.)}$$

$$\bar{s}_1 = \frac{\Sigma_1\sqrt{1 - r_{12}^2 - (1 - r_{12}^2)\lambda_2^2}}{\sqrt{1 - r_{12}^2 - \lambda_1^2 - \lambda_2^2 + (1 - r_{12}^2)\lambda_1^2\lambda_2^2 + 2r_{12}r_{12}\lambda_1\lambda_2}} \quad \dots \quad \text{(cix.)}$$

$$\bar{s}_2 = \frac{\Sigma_2\sqrt{1 - r_{12}^2 - (1 - r_{12}^2)\lambda_1^2}}{\sqrt{1 - r_{12}^2 - \lambda_1^2 - \lambda_2^2 + (1 - r_{12}^2)\lambda_1^2\lambda_2^2 + 2r_{12}r_{12}\lambda_1\lambda_2}} \quad \dots \quad \text{(cx.)}$$

$$\frac{k_1}{\sigma_1} = \frac{H_1}{\sigma_1} \frac{1 - r_{12}^2 - \lambda_2^2 + r_{12}r_{12}\lambda_1\lambda_2}{\beta} + \frac{h_2}{\sigma_2} \frac{r_{12}\lambda_2\lambda_1 - r_{12}\lambda_1^2}{\beta} \quad \dots \quad \text{(cxi.)}$$

$$\frac{k_2}{\sigma_2} = \frac{h_1}{\sigma_1} \frac{r_{12}\lambda_1\lambda_2 - r_{12}\lambda_2^2}{\beta} + \frac{h_2}{\sigma_2} \frac{1 - r_{12}^2 - \lambda_1^2 + r_{12}r_{12}\lambda_1\lambda_2}{\beta} \quad \dots \quad \text{(cxii.)}$$

where

$$\beta = 1 - r_{12}^2 - \lambda_1^2 - \lambda_2^2 + (1 - r_{12}^2)\lambda_1^2\lambda_2^2 + 2r_{12}r_{12}\lambda_1\lambda_2.$$

Similarly, if the original population and the curve of probability of surviving or of survival rates be given, we have to find the selected population :

$$r_{12} = \frac{\bar{\rho}_{12} + r_{12}v^2\kappa_1\kappa_2}{\sqrt{1 + v^2\kappa_1^2}\sqrt{1 + v^2\kappa_2^2}} \quad \dots \quad \text{(cxiii.)}$$

* If $r_{12} = \cos d$, $r_{12} = \cos D$, $\mu = \sin D / \sin d$. The quantity D has been conveniently termed the "divergence" by Mr. SHEPPARD. Hence μ is the ratio of the sines of the selected and unselected divergences. The above formula for ρ_{12} can be at once changed into one suitable for trigonometrical logarithmic calculation. Let $\sin \alpha_1 = \mu\lambda_1$, $\sin \alpha_2 = \mu\lambda_2$, and $\bar{\rho}_{12} = \cos \delta$; then, if Δ be the side of the spherical triangle, of which α_1 , α_2 are the other sides and δ the included angle :

$$\sin \frac{1}{2}\delta = \sqrt{\left\{ \frac{\sin \frac{1}{2}(D - \Delta) \sin \frac{1}{2}(D + \Delta)}{\cos \alpha_1 \cos \alpha_2} \right\}}.$$

where

$$\nu = \sqrt{\frac{1 - \bar{\rho}_{12}^2}{1 - r_{12}^2}}, \quad \kappa_1 = s_1/\sigma_1 \quad \text{and} \quad \kappa_2 = s_2/\sigma_2,$$

$$\Sigma_1 = \frac{\bar{s}_1 \sqrt{1 - r_{12}^2 + (1 - \bar{\rho}_{12}^2) \kappa_2^2}}{\sqrt{1 - r_{12}^2 + \kappa_1^2 + \kappa_2^2 + (1 - \bar{\rho}_{12}^2) \kappa_1^2 \kappa_2^2 - 2\bar{\rho}_{12} r_{12} \kappa_1 \kappa_2}} \quad \dots \quad (\text{cxiv.}),$$

$$\Sigma_2 = \frac{\bar{s}_2 \sqrt{1 - r_{12}^2 + (1 - \bar{\rho}_{12}^2) \kappa_1^2}}{\sqrt{1 - r_{12}^2 + \kappa_1^2 + \kappa_2^2 + (1 - \bar{\rho}_{12}^2) \kappa_1^2 \kappa_2^2 - 2\bar{\rho}_{12} r_{12} \kappa_1 \kappa_2}} \quad \dots \quad (\text{cxv.}),$$

$$\frac{H_1}{\sigma_1} = \frac{k_1}{\sigma_1} \frac{1 - r_{12}^2 + \kappa_2^2 - \bar{\rho}_{12} r_{12} \kappa_1 \kappa_2}{\gamma} + \frac{k_2}{\sigma_2} \frac{r_{12} \kappa_1^2 - \bar{\rho}_{12} \kappa_1 \kappa_2}{\gamma} \quad \dots \quad (\text{cxvi.}),$$

$$\frac{H_2}{\sigma_2} = \frac{k_1}{\sigma_1} \frac{r_{12} \kappa_2^2 - \bar{\rho}_{12} \kappa_1 \kappa_2}{\gamma} + \frac{k_2}{\sigma_2} \frac{1 - r_{12}^2 + \kappa_1^2 - \bar{\rho}_{12} r_{12} \kappa_1 \kappa_2}{\gamma} \quad \dots \quad (\text{cxvii.}),$$

where

$$\gamma = 1 - r_{12}^2 + \kappa_1^2 + \kappa_2^2 + (1 - \bar{\rho}_{12}^2) \kappa_1^2 \kappa_2^2 - 2\bar{\rho}_{12} r_{12} \kappa_1 \kappa_2.$$

(14.) *Illustration.*—The following results are taken from the paper by Miss ALICE LEE and myself already cited:—

	French ♂.	Aino ♂.
Femur	$m_1 = 45.228$ centims.	40.770 centims. $= m_1 + H.$
	$\sigma_1 = 2.372$ „	1.898 „ $= \Sigma_1.$
Humerus	$m_2 = 33.010$ „	29.502 „ $= m_2 + H_2.$
	$\sigma_2 = 1.538$ „	1.343 „ $= \Sigma_2.$
Correlation of		
Femur and	$r_{12} = .8421$	$.8584 = r_{12}.$
Humerus		

As indicated by the symbols above, let us select from the French a population having the same femur and humerus relations as the Aino.

We have at once:

$$\begin{aligned} H_1 &= -4.458, & H_2 &= -3.508, \\ \lambda_1 &= .80017, & \lambda_2 &= .87321, \\ \mu^2 &= .9047. \end{aligned}$$

Whence we find:

$$\begin{aligned} \bar{\rho}_{12} &= .9027, & \bar{s}_1 &= 3.4870, & \bar{s}_2 &= 2.8736, \\ k_1 &= -17.8464, & k_2 &= -15.8547, \\ p_0 &= 208,425 \frac{n}{N}, \end{aligned}$$

where n are the number of Ainos which can be obtained from N Frenchmen.

We have accordingly the following form for the surface of fitness to survive:

$$p = 208,425 \frac{n}{N} \text{ expt. } - \frac{1}{2} \{ .222056 (x + 17.8464)^2 + .326,960 (y + 15.8547)^2 \\ - .486,450 (x + 17.8464) (y + 15.8547) \}.$$

Now it is clear that if we wanted by a "catastrophic" selection to convert the French into something resembling the Aino, we should have to give the least death-rate to those French with femur corresponding to $m_1 + k_1$ and humerus to $m_2 + k_2$, or to the dwarfs with femur = 27.382 centims. and humerus = 17.155 centims. ! By no other means could we shift the modal value of the French population down as low as the Aino modal value. The physical meaning of this is that we have been compelled to put on an excessive death-rate for the bigger Frenchmen.

An interesting point of our work is that

$$k_1 = 1.2514 H_1 + 3.4971 H_2, \\ k_2 = -1.0266 H_1 + 5.8242 H_2,$$

whence we see that while a selective reduction of humerus is far more effective in reducing both femur and humerus centres of survival than a reduction of femur, a selective reduction of femur occurring contemporaneously with that of the humerus actually tends to *raise* the centre of the humerus, *i.e.*, the coefficient of H_1 is *negative*.

Now let us consider the frequency of survivors per unit length, say centimetre of femur and humerus, at different points. The surface of survivors, *i.e.*, the Aino population, is

$$z = \frac{n}{2\pi\Sigma_1\Sigma_2\sqrt{1-r_{12}^2}} e^{-\frac{1}{2}\frac{1}{(1-r_{12}^2)}\left\{\frac{(x-H_1)^2}{\Sigma_1^2} - \frac{2r_{12}(x-H_1)(y-H_2)}{\Sigma_1\Sigma_2} + \frac{(y-H_2)^2}{\Sigma_2^2}\right\}}.$$

If we put $x = 0$, $y = 0$ we have the frequency after selection of the original population type ; if we put $x = H_1$, $y = H_2$ we have the frequency after selection of the new population type ; and if we put $x = k_1$, $y = k_2$, we shall have the frequency after selection of those best fitted to survive. If these frequencies be ν_3 , ν_2 , ν_1 respectively, we find on substituting the numerical values that

$$\nu_1 : \nu_2 : \nu_3 :: .117/10^{18} : 1 : .032289.$$

Thus the most frequent type of the new population is now about thirty times as frequent as the old most frequent type, while the type most fitted to survive has practically no existence at all. It probably lies outside the actual boundary of the French population.

Here really arises the question as to how we are, in any actual problem, to fix the ratio of n to N , or, what amounts to the same thing, to fix a practical boundary to a given population. Such a boundary must be *conventional*, but I think that for

practical purposes we are quite safe if we assume that an individual who occurs only once per thousand can produce no effect on the physical evolution of the population as a whole.

Now the form of a correlation-surface for two organs, x and y , is

$$z = \frac{N}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{2}\frac{1}{1-r^2}\left\{\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right\}}.$$

Let us write $\kappa^2 = \frac{1}{1-r^2}\left\{\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right\}$; then $\kappa = a$ constant gives a series of similar ellipses which are the contour lines of the surface, or lines of equal frequency, *i.e.*, giving individuals with equal probability of occurrence. Let the equation to these contour lines referred to their principal axes be

$$\kappa^2 = \frac{X^2}{A^2} + \frac{Y^2}{B^2}.$$

Then we have at once :

$$\frac{1}{A^2} + \frac{1}{B^2} = \frac{1}{\sigma_1^2(1-r^2)} + \frac{1}{\sigma_2^2(1-r^2)} = \frac{1}{1-r^2} \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2},$$

$$\frac{1}{A^2B^2} = \frac{1}{\sigma_1^2\sigma_2^2(1-r^2)^2} - \frac{r^2}{\sigma_1^2\sigma_2^2(1-r^2)^2} = \frac{1}{(1-r^2)\sigma_1^2\sigma_2^2}$$

or,

$$AB = \sigma_1\sigma_2\sqrt{1-r^2}, \quad A^2 + B^2 = \sigma_1^2 + \sigma_2^2.$$

Further, if ϕ be the angle the A principal axis makes with the axis of x , we have :*

$$\tan 2\phi = 2r\sigma_1\sigma_2/(\sigma_1^2 - \sigma_2^2).$$

These fully determine the principal axes of the frequency surface. Now consider the frequency between the elliptic cylinders corresponding to κ and $\kappa + \delta\kappa$; we have it

$$= z \times 2\pi AB \kappa d\kappa = z \times 2\pi\sigma_1\sigma_2\sqrt{1-r^2} \kappa d\kappa = Ne^{-\frac{1}{2}\kappa^2} \kappa d\kappa.$$

Hence, if N_κ be the frequency outside any contour κ ,

$$N_\kappa = N \int_\kappa^\infty e^{-\frac{1}{2}\kappa^2} \kappa d\kappa = Ne^{-\frac{1}{2}\kappa^2} \quad \dots \quad \text{(cxviii.)}$$

For N_κ to be $\frac{1}{1000}$ of N we have simply

$$\kappa^2 = \frac{6}{\log e}, \quad \text{whence } \kappa = 3.716,923.$$

* For easy calculation put $\gamma = \sqrt{\sigma_1^2 + \sigma_2^2}$, $\tan \psi = \sigma_2/\sigma_1$. Then we have at once if $r = \cos D$:

where : $A = \gamma \cos \chi, \quad B = \gamma \sin \chi,$
 $\sin 2\chi = \sin 2\psi \sin D, \quad \tan 2\phi = \tan 2\psi \cos D.$

This will enable us to determine our conventional boundary to effective population. Now let us refer our non-selected and selected populations to their centres and principal axes.

We find for the contour curves :—

	Unselected Population (i.e., French).	Selected Population (i.e., Aino).	Surface of Survival (i.e., Rate of Survival).
<i>x</i> centre (femur)	45·228	40·770	27·382
<i>y</i> centre (humerus)	33·010	29·502	17·155
tan <i>φ</i> (slope to <i>x</i>)	·601,3775	·670,1454	·807,3371
Principal axes { <i>A</i>	2·7338	2·2514	4·4115
{ <i>B</i>	·7197	·5808	·9775
1 in 1000 limit { <i>κA</i>	10·1615	8·3682	—
{ <i>κB</i>	2·6750	2·1588	—

Referred to its principal axes, the rate of survival is now

$$p = 208,425 \frac{n}{N} e^{-\frac{1}{2} \left\{ \frac{X^2}{(4.4115)^2} + \frac{Y^2}{(.9775)^2} \right\}}.$$

Suppose we require to get at least 1000 Aino out of the French population, N, then *n* = 1000. Now suppose the Aino limiting ellipse drawn, then the French population must be sufficiently large to give the individuals inside this ellipse. Now *p* gets smaller as we go further from the centre of the survival surface. Hence the contour line of the survival surface corresponding to *p* = 1 must be touched *externally* by the limiting contour of the Aino population, in order that we may get at least 1000 Aino out of N Frenchmen. Now, by a graphical construction, I find the major axis of the elliptic contour line of the survival surface which touches the Aino limiting ellipse, is about 11·44. This gives for the parameter *κ*₁ of this ellipse, *κ*₁ × 4·1149 = 11·44, or *κ*₁ = 2·5932. Whence :

$$p_{\kappa_1} = 1 = 208,425 \frac{1000}{N} e^{-\frac{1}{2} (2.5932)^2}$$

gives the greatest possible value of *p* and the least possible value of N. Numerically this gives us N = 7,200,000 about, or we should want more than 7,000,000 of Frenchmen to obtain our 1000 Aino by a catastrophic selection. The actual bounding contour line of this least possible number of Frenchmen* has for its major axis 15·285 centims., and it touches the Aino limiting ellipse at the point where it is touched by the survival contour *p* = 1.

Now let us turn the problem round and ask what is the least population of Aino

* The least possible to reproduce the Aino, as far as femur and humerus are concerned, to 1 in a 1000 of the population.

from which we could produce 1000 Frenchmen by a catastrophic selection. In this case the surface of survivals is simply obtained by inverting p , and, if N' be the number of Aino, $n' = 1000 =$ required number of Frenchmen, we have :

$$p' = \frac{1}{208,425} \frac{n'}{N} e^{\frac{1}{2} \left\{ \frac{x^2}{(4.4115)^2} + \frac{y^2}{(.9775)^2} \right\}}.$$

Here p' gets larger as we go away from the centre of the surface of survivals, and we must therefore make the French limiting ellipse just touched *internally* by the contour line of the surface of survivors for which $p' = 1$. The major axis of this contour line for $p' = 1$ was found by a graphical process to be about 33.91. This gives for κ_2

$$\kappa_2 \times (4.4115) = 33.91, \quad \text{or} \quad \kappa_2 = 7.6867.$$

Thus :

$$p'_{\kappa_2} = 1 = \frac{1000}{208,425} \frac{1}{N'} e^{\frac{1}{2} (7.6867)^2},$$

leads to $N' = 32,460,000,000$ about, or we should want upwards of 32,460,000,000 of Aino to produce the 1000 Frenchmen. The bounding contour line of this number of Aino has a major axis of 15.890 centims. about, and touches the French limiting contour in the point in which it is touched by the $p' = 1$ contour of the surface of survivals.

Now the difference between these two unselected populations is very great. We see that to get the Aino a very great number of Frenchmen would have to be exterminated, about 7000 for each Aino selected; but to get the Frenchmen from the Aino an appalling number of Aino would have to be destroyed, upwards of 32,000,000 for each Frenchman selected. Even if the selection were not catastrophic but spread over centuries and centuries, we must recognise what a large consumption of life there must be—individuals destroyed without progeny*—if we are to suppose any highly civilized race like the French produced by selection from an apparently primitive type like the Aino. Indeed, the return journey in this case seems much easier than the upward ascent. Beyond all this we have only made French and Aino alike for two organs, and only for one character of each of them! Allowing for our conventional limit to the population, allowing for the fact that our Aino data are drawn from a very limited population of remarkably small variability, it seems very improbable that the French have ever been produced by selection from a primitive race at all resembling the Aino. The fact that the Aino could be so much more easily obtained by selection from the French seems to indicate that they are rather

* Of course, with a secular selection spread over many generations, it is largely the potentiality and not the actuality of life which is destroyed. Still, while the gross number killed among a small primitive community may not be large, the death-rate must still be immense. I hope to return to these points when dealing with secular selection as distinguished from catastrophic selection.

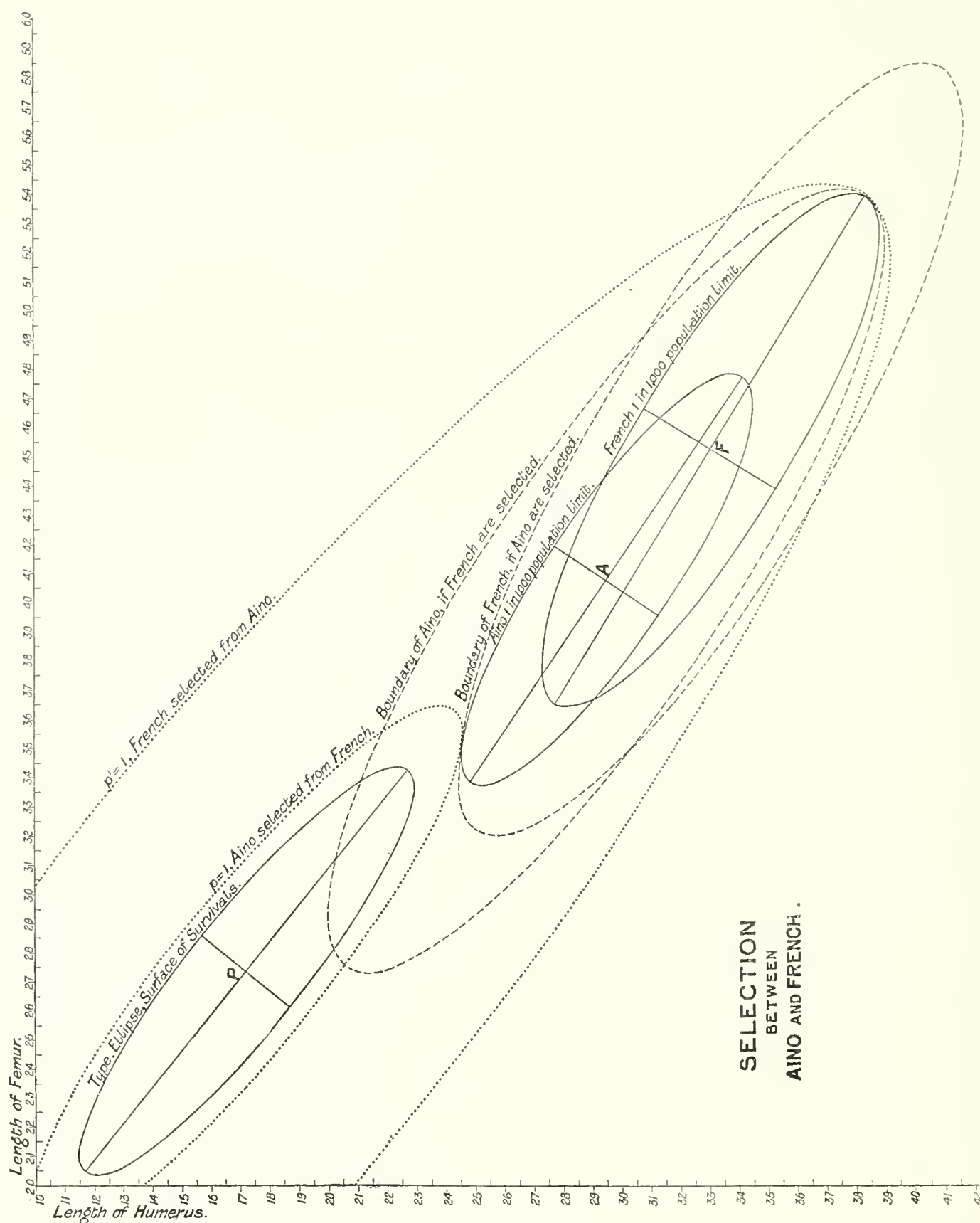
some degenerate offshoot of a race superior to themselves than a sample of the primitive people from which the Circassian races may be supposed to have sprung.

The whole of this discussion is, of course, very hypothetical; no stress whatever is to be laid upon it except as an illustration of method, and a rough appreciation of the vast amount of elimination which must be necessary to evolve one race from a second in the case of organs which we know by measurement to have continuity of variation, and only saltatory changes in pathological cases, which have, as far as we can judge, no influence on the mass-evolution which has produced the local races of man.

But given fair samples of material our method will enable us to determine whether a race A—for of course a limited number of characters—could with less destruction be deduced from a race B, than the race B from A. It will not therefore follow that the path of least selection is that which necessarily was used by Nature. Possibly both A and B have been reached by far less expenditure of material from C. Still it is something definite in the midst of our gropings after truth in problems of descent to have even a rough appreciation of the amount of selective destruction which would arise from alternative suggestions. That is why this special numerical illustration of the surface of survival has been given.

The reader will possibly find the matter rendered somewhat clearer by the diagram. The femur is measured along the horizontal and the humerus along the vertical. A is the type or mean femur-humerus of the Aino population. Within in the continuous ellipse round A the whole Aino population up to 1 in 1000 would fall. F is the type of the French population and the continuous ellipse round F gives the area within which up to 1 in 1000 of the French population fall. Since the diagram is drawn to centimetres of the bones, it will be seen how very small are the limits of variation within both populations. P is the centre of the surface of survivals; for the selection of Aino from French it makes the “fittest to survive.” In the case of the selection of the French from the Aino, P is no longer the centre of fitness, but the “centre of unfitness”; the Aino are killed off with an intensity which increases the closer we approach to P. Now it seems to me that these two cases, which are quite distinct in theory, ought to manifest themselves in Nature and require distinguishing names. A race may be modified because a complex of organs with a certain system of values is good for it, or because it is bad for it. The race may be modified because a certain element of it is fittest or because it is unfittest to survive. In the former case we select for survival round the centre, in the latter case we select for destruction. I propose to call these cases *positive* and *negative* selection respectively. It may be said that if there be positive selection in one part of the population there will be negative in another. But the kernel of the matter is in either case the existence of a centre, a definite set of most fit or of most unfit organs, while in positive selection the less fit organs, and in negative selection the more fit organs are distributed over wide areas of the field, and do not reach a maximum of unfitness or a maximum of fitness respectively for any definite individual.

In the diagram we have also drawn the contour line to which the French population must extend if we are to get at least a representative population of 1000 Aino from



it, and further the contour line to which the Aino population must extend if we are to get at least a representative population of 1000 French from it. A consideration of

the nature of the contour lines of the surface of survivals shows that the contour lines above referred to, and marked "boundary" in the diagram, must touch the Aino 1 in 1000 limit and the French 1 in 1000 limit respectively at the points in which they are touched by the contour lines $p = 1$ and $p' = 1$ of the corresponding surfaces of survivals. I have already indicated that the major axes of these boundaries are 15.285 for the French and 15.890 for the Aino. The corresponding values of the parameter κ are respectively given by

$$\kappa_F = \left(\frac{15.285}{2.7338} \right) = 5.5911, \quad \kappa_A = \frac{15.890}{2.2514} = 7.0578.$$

Hence by (cxviii.) we can easily find the frequency of population outside the contours κ_F and κ_A ; if these be ν_F and ν_A we have :

$$\nu_F = .000,000,163, \quad \nu_A = .000,000,000,015.$$

Thus the French population would have to be extended to a boundary in which only about 1 in six millions was excluded, and the Aino population to a boundary excluding only 15 in the billion! The boundaries of what we may thus term the selection populations are far larger than our conventional boundaries of 1 in 1000 for representative populations. In fact, it would be impossible to select a representative Aino population from a conventional representative French population and *vice versa*—in either case the very exceptional members of French or Aino populations are required to complete the conventional representative populations of Aino or French by selection.

(15.) I have devoted most of my consideration of the surface of survivals to a particular case in which two organs have been selected, and we consider the nature of p which determines the fraction of each group of individuals which survives. I have done this partly because normal surfaces are at best only an approximate representation of our selectable and selected distributions, and partly because I have thought a concrete case would best bring out the general points of investigations of this kind.

But some little indication of the properties of the surface of survival-rates ought to be indicated here, or it may appear that they have been overlooked. While the contour lines of the correlation frequency surfaces for two organs must be ellipses, this does not follow in the case of the surface of survival-rates. In our illustration they were ellipses, but they may be also parabolas, hyperbolas, or even straight lines. We must not therefore expect to find always a "centre" of positive or negative selection. We may come across a "saddle-back system" of contours with the rate of survival constant along two intersecting lines, but rising in one pair of opposite angles and falling in the other pair. In this case we have fields of negative and positive selection separated by two independent relations between the two organs,

which are linear and for which the survival-rate is the same, they may be termed the "critical lines." For one pair of angles the centre is now a "centre of fitness," for the other pair of angles a "centre of unfitness." It seems to me that these critical organic relations may possess considerable biological importance.

If the contour lines of the surface of survival-rates are parabolas, we have really only a limiting case of the centre at a very great distance. It is one in which the fittest (or most unfit) has no practical existence, but there is a direction towards which the rate of survival will be found to be always increasing or decreasing.

If the contour lines of the surface of survival-rates are parallel straight lines, then so long as the deviation in one organ has a certain definite relation to that in the other, the survival-rate will remain constant. In this case the survival-rate will fall uniformly in one direction and remain constant in the direction at right angles to it.

All the cases I have given here can occur just as easily as the elliptic contour system of our illustration and diagram. Each is marked by quite definite biological characteristics, and we may, perhaps, class them as elliptic, hyperbolic, parabolic, and linear selection. Even if the surface of survival-rates be not of the exponential quadric type discussed in this paper, yet to the neighbourhood of each part of it this classification of selection types will apply.

If we pass to more than two organs, then similar considerations will apply; we shall only be reproducing the geometry of quadric surfaces in space of three and higher dimensions. But before we allow ourselves excursions into the higher geometry of the surface of survival-rates, it seems desirable that we should obtain quantitative determinations of this surface by experiments in artificial selection. We shall then be better able to see what part of our geometry will really be of service for the problems of natural selection. The field is too large to be cultivated for merely theoretical interests. We must first determine what parts of it are likely to have practical application to life as we find it, but of death-rates in the case of any living form but man, we are at present sadly ignorant.

Key to Selective Correlation Tables.

Case.	Values of	
	r_{12} .	r_{13} .
(a)	0	0
(b)	0	.25
(c)	0	.5
(d)	0	.75
(e)	0	1
(f)	.25	.25
(g)	.25	.5
(h)	.25	.75
(i)	.25	1
(j)	.5	.5
(k)	.5	.75
(l)	.5	1
(m)	.75	.75
(n)	.75	1
(p)	1	1

Formula :

$$R' = R + 10 (\Delta_{\mu_1} R) \delta \mu_1 + 4 \{ (\Delta_{r_{12}} R) \delta r_{12} + (\Delta_{r_{13}} R) \delta r_{13} + (\Delta_{r_{23}} R) \delta r_{23} \}.$$

Occasionally second differences must be used.

SELECTIVE CORRELATION TABLES.

Values of r_{23} positive.

Table I (A).— $r_{23} = 0$. The possible values are (a), (b), (c), (d), (e) [limiting] (f), (g), (h), (j), (k), (i), (m), (l) are needed for interpolation.

[In one case, viz. (e), R_0 is indeterminate].

	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
(a), (b), (c), (d), (e)	0	0	0	0	0	0	0	0	0	0	0
(f)	·0667	·0660	·0638	·0603	·0554	·0492	·0417	·0329	·0230	·0120	0
(g)	·1491	·1473	·1420	·1333	·1214	·1065	·0891	·0694	·0477	·0245	0
(h)	·2928	·2879	·2737	·2515	·2228	·1894	·1531	·1151	·0765	·0379	0
(i)	∞	·25553	·12377	·7809	·5394	·3841	·2722	·1851	·1138	·0531	0
(j)	·3333	·3289	·3158	·2945	·2658	·2308	·1905	·1461	·0989	·0499	0
(k)	·6547	·6429	·6089	·5557	·4880	·4104	·3273	·2425	·1585	·07725	0
(l)	∞	·57063	·27530	·17256	·11813	·83205	·5819	·3900	·2359	·1082	0
(m)	·12857	·12567	·11739	·10487	·8957	·7297	·5625	·4023	·2539	·1197	0

Table II (A).— $r_{23} = .25$ the possible values are (a), (b), (c), (d), (e), (f), (g), (h), (i) [limiting] (j), (k), (m), (e), (l), (n) are needed for interpolation.

	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
(a)	·2500	·2500	·2500	·2500	·2500	·2500	·2500	·2500	·2500	·2500	·25
(b)	·2582	·2581	·25785	·2574	·2568	·2561	·2552	·2541	·2529	·2515	·25
(c)	·2887	·2882	·2868	·28445	·2813	·27735	·2728	·2676	·26205	·2562	·25
(d)	·3780	·37555	·3686	·3578	·3442	·3288	·3125	·2960	·28645	·2645	·25
(e)	∞	2·5000	1·2500	·8333	·6250	·5000	·4167	·3571	·3125	·2778	·25
(f)	·2000	·2005	·2021	·2048	·20845	·2131	·21875	·2253	·2328	·2410	·25
(g)	·1491	·1503	·1538	·1596	·1676	·1776	·1893	·20265	·21735	·2332	·25
(h)	·0976	·09985	·10645	·1170	·1308	·14735	·1659	·1858	·2067	·2282	·25
(i)	0	·0258	·0516	·0772	·1027	·1280	·1531	·1779	·2023	·2263	·25
(j)	0	·0033	·01315	·0291	·0506	·0769	·10715	·1404	·1758	·2126	·25
(k)	·2182	·2100	·18605	·1486	·10065	·0456	·0136	·0745	·1350	·1938	·25
(l)	∞	·28243	·13192	·7775	·4782	·27735	·1273	·0076	·0917	·1765	·25
(m)	·7143	·6925	·6304	·5365	·4217	·2973	·1719	·0517	·0596	·1602	·25
(n)	∞	·73985	·34649	·206345	·13080	·8220	·4792	·2241	·0280	·1264	·25

SELECTIVE CORRELATION TABLES.

Positive values of r_{23} .

Table III. (A). $r_{23} = .5$. The possible values are (a), (b), (c), (d), (e), (f), (g), (h), (i), (j), (k), (l), (m), (n) [limiting]. (e), (i), (n), are needed for interpolation.

	R ₀	R ₁	R ₂	R ₃	R ₄	R ₅	R ₆	R ₇	R ₈	R ₉	R ₁₀
(a)	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
(b)	.5164	.5162	.5157	.51485	.5137	.51215	.5103	.5082	.5057	.5030	.5000
(c)	.57735	.5764	.5735	.5689	.5625	.5547	.54555	.5353	.5241	.5123	.5000
(d)	.7559	.7511	.7372	.71565	.6884	.6576	.6250	.5924	.5599	.5291	.5000
(e)	+ ∞	5.0000	2.5000	1.6667	1.2500	1.0000	.8333	.7143	.6250	.5556	.5000
(f)	.4667	.4670	.4681	.46985	.4723	.4754	.4792	.4835	.4885	.4940	.5000
(g)	.4472	.4478	.4496	.4525	.45655	.46165	.4677	.4747	.4824	.49095	.5000
(h)	.48795	.4876	.4866	.48545	.48445	.4841	.4848	.4867	.4898	.4943	.5000
(i)	+ ∞	2.6069	1.34085	.9353	.7448	.6402	.57835	.5408	.5184	.5058	.5000
(j)	.3333	.33555	.3421	.35275	.3671	.3846	.4048	.4269	.45055	.4751	.5000
(k)	.2182	.2230	.2368	.2585	.2866	.3192	.3546	.3914	.4285	.46485	.5000
(l)	0	.0576	.1147	.1707	.2250	.27735	.3273	.3747	.4193	.4611	.5000
(m)	-.14285	-.12835	-.08695	-.0243	.0521	.1351	.2188	.2989	.37305	.44015	.5000
(n)	- ∞	-3.6429	-1.6219	-.8707	-.4475	-.1644	-.0417	.1988	.3219	.4203	.5000

Table IV. (A). $r_{23} = .75$. The possible values are (a), (b), (c), (d), (e), (f), (g), (h), (i), (j), (k), (l), (m), (n) [limiting]. (d), (i), (l), are needed for interpolation.

	R ₀	R ₁	R ₂	R ₃	R ₄	R ₅	R ₆	R ₇	R ₈	R ₉	R ₁₀
(a)	.7500	.7500	.7500	.7500	.7500	.7500	.7500	.7500	.7500	.7500	.7500
(b)	.7746	.7743	.7736	.7723	.7705	.7682	.7655	.76225	.7586	.7545	.7500
(c)	.8660	.8646	.8603	.8533	.8438	.83205	.8183	.8029	.7862	.7685	.7500
(d)	1.1339	1.1267	1.1058	1.0735	1.0327	.9864	.9375	.8881	.83985	.7936	.7500
(e)	.7333	.7335	.7340	.7349	.73615	.7377	.7396	.7418	.74425	.7470	.7500
(f)	.74536	.74536	.74537	.7454	.7455	.7457	.7461	.7467	.7476	.74865	.7500
(g)	.8783	.8753	.8668	.8539	.8381	.8209	.80375	.78755	.7730	.76045	.7500
(h)	+ ∞	5.1881	2.6301	1.7934	1.3869	1.1523	1.0036	.9038	.8345	.78525	.7500
(i)	.6667	.6678	.67105	.6764	.6835	.6923	.7024	.7135	.7253	.7376	.7500
(j)	.65465	.6559	.6596	.6656	.67385	.6839	.69555	.7084	.7220	.7359	.7500
(k)	+ ∞	2.9396	1.5486	1.1188	.9282	.83205	.78195	.75705	.7469	.7457	.7500
(l)	.4286	.4358	.4565	.4878	.5261	.5676	.6094	.6494	.6865	.7201	.7500
(m)	0	.1127	.2212	.32205	.4131	.4932	.5625	.6217	.6719	.71425	.7500
(n)											

Table V. (A). $r_{23} = 1$. The only possible values are $r_{12} = r_{13}$, i.e., (a), (f), (j), (m), (p). All these give $R = 1$ for all values of the selective intensity λ .

SELECTIVE CORRELATION TABLES.

Values of r_{23} negative.

Table I. (B). This is the same as Table I. (A). $r_{23} = 0$.

The possible values are (a), (b), (c), (d), (e) [limiting], (f), (g), (h), (i), (j), (k). Values which are impossible, but which are tabulated for purposes of interpolation, are (i), (m), (l).

Table II. (B). $r_{23} = -.25$.

The possible values are (a), (b), (c), (d), (f), (g), (h), (j) Values which are impossible, but are needed for purposes of interpolation, are (e), (i), (k). (a), (b), (c), (d), (e) are the same as in Table II. (A) with the sign changed.

	R ₀ .	R ₁ .	R ₂ .	R ₃ .	R ₄ .	R ₅ .	R ₆ .	R ₇ .	R ₈ .	R ₉ .	R ₁₀ .
(f)	-.3333	-.3324	-.3298	-.3254	-.3193	-.3115	-.3021	-.2911	-.2788	-.2650	-.2500
(g)	-.4472	-.4449	-.4377	-.4261	-.4103	-.3907	-.3675	-.3414	-.3128	-.2822	-.2500
(h)	-.6831	-.67565	-.6539	-.61995	-.5764	-.5262	-.4721	-.4160	-.3596	-.3040	-.2500
(i)	∞	-.51364	-.25270	-.16389	-.11814	-.8962	-.6974	-.5481	-.4299	-.3325	-.2500
(j)	-.6667	-.6611	-.6447	-.6181	-.5822	-.5384	-.4881	-.4327	-.3736	-.3123	-.2500
(k)	-.10911	-.10758	-.10317	-.9628	-.8753	-.7751	-.6683	-.5594	-.4519	-.3483	-.2500

Table III. (B). $r_{23} = -.5$.

The possible values are (a), (b), (c), (d), (f), (g), (i) [limiting]. (h) and (e) are tabulated for purposes of interpolation. (a), (b), (c), (d), (e) are the same as in Table III. (A) with the sign changed.

	R ₀ .	R ₁ .	R ₂ .	R ₃ .	R ₄ .	R ₅ .	R ₆ .	R ₇ .	R ₈ .	R ₉ .	R ₁₀ .
(f)	-.6000	-.5989	-.5958	-.5905	-.5831	-.5738	-.5625	-.5493	-.5345	-.5180	-.5
(g)	-.7453	-.7424	-.7335	-.7190	-.6993	-.6747	-.6459	-.6134	-.5778	-.5399	-.5
(h)	-.10735	-.10634	-.10341	-.9884	-.9300	-.8630	-.7910	-.7169	-.6427	-.5701	-.5
(i)	-.10000	-.9934	-.9737	-.9417	-.8987	-.8461	-.7858	-.7192	-.6484	-.5748	-.5

Table IV. (B). $r_{23} = -.75$.

The possible values are (a), (b), (c), (f). (d) and (g) are needed for interpolation. (a), (b), (c), (d) are the same as in Table IV. (A) with the sign changed.

	R ₀ .	R ₁ .	R ₂ .	R ₃ .	R ₄ .	R ₅ .	R ₆ .	R ₇ .	R ₈ .	R ₉ .	R ₁₀ .
(f)	-.8667	-.8654	-.8617	-.8555	-.8470	-.8361	-.8229	-.8076	-.7903	-.7710	-.75
(g)	-.10435	-.10399	-.10293	-.10119	-.9882	-.9588	-.9243	-.8854	-.8430	-.7976	-.75

Table V. (B) is entirely impossible except (a). $r_{12} = r_{13} = 0$. $R = -1$ for all values of selective intensity λ .

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II. *On the Equilibrium of Rotating Liquid Cylinders.*

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INTRODUCTION.

§ 1. As a preliminary to attacking the problem of determining the equilibrium configurations of a rotating mass of liquid, I was led to consider whether some method could not be devised for calculating the potential of a homogeneous mass in a manner more simple than that usually adopted. What was obviously required was a calculus enabling us to write down the potential of such a mass by an algebraical transformation of the equation of its boundary, instead of by an integration extending throughout its volume.

There was found to be no difficulty in reducing the calculation to a problem of algebraical transformation, but in three-dimensional problems the transformations required were, in general, as impracticable as the integrations which they were intended to replace. This was because the transformations depended upon a continued application of the formula which expresses the products or powers of spherical harmonics as the sum of a series of harmonics.

As soon, however, as we pass to the consideration of two-dimensional problems, the spherical harmonics may be replaced by circular functions of a single variable. The transformation now becomes manageable, and for this reason the present paper deals only with two-dimensional problems.

The first part of the paper contains a short sketch of a theory of two-dimensional potentials. I have, however, confined myself strictly to such problems as are required for the solution of the main problem under discussion, namely, that of the rotating liquid; the method does not attempt to be one of general applicability.

THE POTENTIALS OF HOMOGENEOUS CYLINDERS.

General Theory.

§ 2. We shall suppose the cross-section of the cylinder of which the potential is required, to be bounded by a single continuous curve *S* enclosing the origin. Let

V be the potential of this cylinder, supposed to be composed of homogeneous matter of density ρ .

The value of V must be finite and continuous at all points except infinity, and its first differential coefficients must also be finite and continuous at all points. Also V must satisfy $\nabla^2 V = 0$ at all points outside S , and $\nabla^2 V = -4\pi\rho$ at all points inside S . At infinity V must vanish, except for a term proportional to $\log r$.

These conditions suffice to determine V uniquely. For if there were two distinct solutions V and V' , the function $(V - V')$ would satisfy $\nabla^2(V - V') = 0$ at all points of space, would be finite and continuous, together with its first differential coefficients, at all points of space, and would vanish at infinity, except for a term proportional to $\log r$. The only solution satisfying these conditions is known to be $V - V' = 0$, hence any function V satisfying the conditions laid down above must be the potential of which we are in search.

§ 3. Let us use polar co-ordinates r, θ in conjunction with orthogonal co-ordinates x, y , and let us also introduce complex variables ξ, η , defined by

$$\xi = re^{i\theta} = x + iy, \quad \eta = re^{-i\theta} = x - iy.$$

Let us suppose the equation to the curve S to be written in the form

[illegible]

and let us imagine that this equation is solved explicitly for ξ in the form

[illegible]

In general $F(\eta)$ will be a multiple-valued function of η , and the equation (1) may, and probably will, be satisfied for other values of ξ and η than those which occur on the surface S .

Let us, however, suppose that we have succeeded in finding one value of $F(\eta)$ such that this value is a single-valued function of η at every point of S , and is equal to ξ . Let us suppose that we have succeeded in expanding this value of $F(\eta)$ in a series of ascending and descending powers of η , these series each being supposed convergent at every point of S . Let us write

$$\mathbb{F}(\eta) = \phi(\eta) + \psi(\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\phi(\eta) = a_0 + a_1\eta + a_2\eta^2 + \dots \quad (4), \quad \psi(\eta) = \frac{b_1}{\eta} + \frac{b_2}{\eta^2} + \dots \quad (5).$$

We shall consider only the case in which the surface S has the plane $\theta = 0$ as a plane of symmetry. In this case the equation $f(\xi, \eta) = 0$ remains the equation to the curve after the sign of θ is changed, and we therefore have as a second form of this equation,

$$f(\eta, \xi) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

There is therefore a solution of this equation expressing η explicitly as a function of ξ in the form

$$\eta = F(\xi) = \phi(\xi) + \psi(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where F, ϕ, ψ are defined by equations 3, 4 and 5.

Since we have supposed equation (2) to represent the complete solution of equation (1), the equation

$$\eta = F(\xi) \quad , \quad , \quad , \quad , \quad , \quad , \quad , \quad , \quad , \quad , \quad (8)$$

must necessarily represent the complete solution of equation (6), if the meaning of F remains unaltered. Let us suppose that the values of the multiple-valued function $F(\xi)$ (or, what is the same thing, $F(x + iy)$) are exhibited on the appropriate RIEMANN'S surface. The locus of points at which this function is equal to $x - iy$ (also a definite and unique function of position upon this RIEMANN'S surface) will be the complete system of points satisfying equation (6), and will therefore include the curve S . This curve will not, however, be drawn upon a plane, but upon a RIEMANN'S surface.

Now so long as the curve S does not possess a cusp or branch point we have at every point of S ,

$$\left| \frac{d\Gamma(\xi)}{d\xi} \right| = \left| \frac{d(x - iy)}{d(x + iy)} \right| = 1,$$

and hence it follows that no branch point of the RIEMANN'S surface can lie on the curve S. Let the curve S be a single closed curve surrounding the origin, and it follows that we can always so arrange the RIEMANN'S surface that no branch line shall intersect the curve S. In this case it is possible for a solution of the form of equation (6) to satisfy at every point of the curve S, the curve now being regarded as a closed curve lying on one sheet of a RIEMANN'S surface. By simply interchanging the axes of y and $-y$ we can imagine the function $F(\eta)$ represented on the same RIEMANN'S surface, and we see that at every point on the old curve S we shall have $\xi = F(\eta) = \phi(\eta) + \psi(\eta)$.

We have therefore proved that if a particular solution of the form of equation (3) can be found to represent the curve S , then solution (7) will represent the same curve. We have also seen that if a family of curves is described by continuous deformation starting from some curve S_0 , the general solution can be arrived at by continuous variation starting from the solution for the curve S_0 , *so long as no member of the family possesses a cusp or branch point*. For our present purpose it is sufficient to have proved that if we have, at every point of the surface S ,

$$\xi = \phi(\eta) + \psi(\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

then we have also, at every point,

$$\eta = \phi(\xi) + \psi(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Let us now introduce a new function χ defined by

$$\chi = C + \int_0^\xi \phi(\xi) d\xi - \int_\xi^\infty \psi(\xi) d\xi + \int_0^\eta \phi(\eta) d\eta - \int_\eta^\infty \psi(\eta) d\eta - \xi\eta \quad . \quad (11),$$

where C is a constant.

We have by differentiation,

$$\partial\chi/\partial\xi = \phi(\xi) + \psi(\xi) - \eta \quad . \quad . \quad (12), \quad \partial\chi/\partial\eta = \phi(\eta) + \psi(\eta) - \xi \quad . \quad . \quad (13),$$

and therefore at the surface S, by equations (9) and (10), $\partial\chi/\partial\xi = \partial\chi/\partial\eta = 0$.

It follows that χ has a constant value at the surface S , and this, by a suitable choice of the constant C , may be taken to be zero. Also it follows that $\partial\chi/\partial n = 0$ at all points on the surface S , where $\partial/\partial n$ denotes differentiation with respect to the normal. We therefore have at every point of the surface S

$$\chi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14), \quad \partial\chi/\partial n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

§ 4. Let us denote the potential at a point inside the surface S by V_i , and that at a point outside S by V_o . Let us examine, as a trial solution for V ,

$$V_i = \pi\rho \left\{ C + \int_0^\xi \phi(\xi) d\xi + \int_0^\eta \phi(\eta) d\eta - \xi\eta \right\} \quad . \quad . \quad . \quad . \quad . \quad (16),$$

$$V_o = \pi\rho \left\{ \int_\xi^\infty \psi(\xi) d\xi + \int_\eta^\infty \psi(\eta) d\eta \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Since $\eta = re^{-i\theta}$, the greatest value of $|\eta|$ at any point on S is equal to the greatest radius, say R_1 , which can be drawn from the origin to S . Since the series $\phi(\eta)$ is, by hypothesis, convergent for all points on the boundary, it follows that the radius of convergence of the power series $\phi(\eta)$ must be greater than R_1 , and hence that $\phi(\eta)$ is convergent at all points inside the surface S . Hence also $\int_0^\eta \phi(\eta) d\eta$ must be convergent at all points inside S . The same is obviously true if ξ is written instead of η . Hence it follows that if V_i is defined by equation (16), then V_i and its first differential coefficients will be finite and continuous at all points inside S .

In a precisely similar manner it can be shown that if V_o is defined by equation (17), then V_o will be finite and continuous at all points outside S except at infinity, and that the first differential coefficients of V_o will be finite at all points outside S .

From what has been said it follows that V_o and V_i are finite at the boundary. The value of $V_i - V_o$ at the boundary is $\pi\rho\chi$, and this vanishes by equation (14). Hence V is finite and continuous at all points of space (except infinity).

Since the series $\phi(\eta)$ and $\psi(\eta)$ have been supposed to be convergent on the boundary, it follows that the first differential coefficients of V must be convergent on the boundary, and hence that these differential coefficients are finite at all points of space. At the boundary,

$$\partial V_i/\partial n - \partial V_o/\partial n = \pi\rho \partial\chi/\partial n = 0,$$

by equation (15). Hence it follows that the first differential coefficients of V are finite and continuous at all points of space.

At a point inside S ,

$$\nabla^2 V = \nabla^2 V_i = 4\pi\rho \frac{\partial^2}{\partial\xi \partial\eta} \left\{ C + \int_0^\xi \phi(\xi) d\xi + \int_0^\eta \phi(\eta) d\eta - \xi\eta \right\} = -4\pi\rho,$$

and similarly at a point outside S ,

$$\nabla^2 V = \nabla^2 V_o = 4\pi\rho \frac{\partial^2}{\partial\xi \partial\eta} \left\{ \int_\xi^\infty \psi(\xi) d\xi + \int_\eta^\infty \psi(\eta) d\eta \right\} = 0.$$

Lastly, by actual integration, we find as the value of V_0 from equation (17),

$$V_0 = \pi\rho \left\{ -b_1 (\log \xi + \log \eta) + b_2 \left(\frac{1}{\xi} + \frac{1}{\eta} \right) + \frac{1}{2} b_3 \left(\frac{1}{\xi^2} + \frac{1}{\eta^2} \right) + \dots \right\} \quad (18).$$

The term $-b_1 (\log \xi + \log \eta)$ may be replaced by $-2b_1 \log r$; hence V vanishes at infinity, except for a term proportional to $\log r$.

We have now seen that V satisfies all the conditions which must be satisfied by a potential; hence the potential will be given by equations (16) and (17).

§ 5. Calculated by direct integration, the value of V_0 is

$$V_0 = -\rho \iint \log R^2 r' dr' d\theta',$$

where the integration extends throughout the cross-section of the cylinder, and R is the distance of the point r, θ at which the potential is being evaluated, from the point r', θ' of the cylinder. We have

$$R^2 = r^2 + r'^2 - 2rr' \cos (\theta - \theta') = (r - r'e^{i(\theta-\theta')})(r - r'e^{-i(\theta-\theta')}).$$

Now if $|r| > |r'|$, we have

$$\log (r - r'e^{-i(\theta-\theta')}) = \log r + \log \left(1 - \frac{r'e^{i\theta'}}{re^{i\theta}} \right) = \log r - \frac{r'e^{i\theta'}}{re^{i\theta}} - \frac{1}{2} \left(\frac{r'e^{i\theta'}}{re^{i\theta}} \right)^2 - \dots,$$

and hence

$$\log R^2 = 2 \log r - \left(\frac{r'e^{i\theta'}}{re^{i\theta}} + \frac{r'e^{-i\theta'}}{re^{-i\theta}} \right) - \dots$$

Upon integration we obtain

$$V_0 = -\rho \left\{ 2 \log r \iint r' dr' d\theta' - \left(\frac{1}{\xi} \iint r'^2 e^{i\theta'} dr' d\theta' + \frac{1}{\eta} \iint r'^2 e^{-i\theta'} dr' d\theta' \right) - \dots \right\} \quad (19).$$

The cylinder being symmetrical about the plane $\theta = 0$, we have

$$\iint r'^2 e^{i\theta'} dr' d\theta' = \iint r'^2 e^{-i\theta'} dr' d\theta' = \iint r'^2 \cos \theta' dr' d\theta',$$

and hence equation (19) becomes

$$V_0 = -\rho (\log \xi + \log \eta) \iint r' dr' d\theta' + \rho \left(\frac{1}{\xi} + \frac{1}{\eta} \right) \iint r'^2 \cos \theta' dr' d\theta' + \dots \quad (20).$$

Let the area of the cross-section be A , and its centre of gravity be at $x = \alpha$, so that

$$A = \iint r' dr' d\theta' \quad A\alpha = \iint r'^2 \cos \theta' dr' d\theta';$$

then equation (15) becomes

$$V_0 = -\rho A (\log \xi + \log \eta) + \rho A \alpha \left(\frac{1}{\xi} + \frac{1}{\eta} \right) + \dots$$

Comparing this with equation (18), we find that

$$A = \pi b_1 \quad (21), \quad A\alpha = \pi b_2 \quad (22).$$

These last two equations enable us to find, by a process of algebraical transformation, the cross-section and the centre of gravity of the cylinder whose boundary is

At points on the boundary the minimum value of $4ab |\eta^2|$ is $4a$; hence, provided $a^2 > (a-b)^2$, a convergent expansion for ξ is

$$\begin{aligned}\xi &= \frac{1}{a-b} \left\{ -(a+b)\eta + 2\sqrt{ab}\eta \left(1 + \frac{1}{2} \frac{a-b}{ab\eta^2} - \frac{1}{8} \frac{(a-b)^2}{a^2b^2\eta^4} - \dots \right) \right\} \\ &= \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\eta + \frac{1}{\sqrt{ab}\eta} - \frac{1}{4} \frac{a-b}{(ab)^{3/2}\eta^3} - \dots \quad (30).\end{aligned}$$

From this expansion we obtain at once

$$\begin{aligned}V_i &= -\frac{2\pi\rho}{\sqrt{a}+\sqrt{b}} (\sqrt{a}x^2 + \sqrt{b}y^2), \\ A &= \frac{\pi}{\sqrt{ab}}, \quad a=0 \quad \dots \quad (31).\end{aligned}$$

We can obtain V_0 in series at once; if we require its value in finite terms we may proceed as follows:—

From equation (30),

$$\begin{aligned}V_0 &= \pi\rho \int_{\eta}^{\infty} \left(\frac{1}{\sqrt{ab}\eta} - \frac{1}{4} \frac{a-b}{(ab)^{3/2}\eta^3} + \dots \right) d\eta + \text{same function of } \xi, \\ &= \pi\rho \int_{\eta}^{\infty} \left[\frac{1}{a-b} \left\{ -(a+b)\eta + \sqrt{4ab\eta^2 + 4(a-b)} \right\} + \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}\eta \right] d\eta \\ &\quad + \text{same function of } \xi \\ &= \frac{\pi\rho}{a-b} \left\{ -\sqrt{ab}\eta^2 + \frac{\eta}{2} \sqrt{4ab\eta^2 + 4(a-b)} \right. \\ &\quad \left. + \frac{a-b}{\sqrt{ab}} \log (2\sqrt{ab}\eta + \sqrt{4ab\eta^2 + 4(a-b)}) \right\} \\ &\quad + \text{same function of } \xi. \quad (32).\end{aligned}$$

The results obtained agree with the known results if $\sqrt{a}$ and $\sqrt{b}$ are taken of the same sign. There is a second solution, obtained by changing the sign of one of these roots, and this corresponds to a mathematical ellipse of which one axis is negative. The full significance of this will appear later.

Expansion in Powers of a Parameter.

§ 8. When the equation to the surface is of a degree higher than the second, it will not, in general, be possible to obtain a solution in finite terms of the form of equation (29). Suppose, however, that the surface forms one of a family of surfaces, the family being described by the variation of a parameter c , and let this family be chosen so that the surface $c=0$ is one for which the complete solution is known. Then as we know the value of ξ when $c=0$, we shall assume that it will be possible to find the general value of ξ in a series of ascending powers of c , and the equations determining the various coefficients of c will have a unique solution.

least equal to $1/r$. This function is real at every point of space, and is a solution of LAPLACE'S equation. Hence the function is finite at every point of space.

It is therefore clear that if, in equations (16) and (17), we replace $\phi(\eta)$, $\psi(\eta)$ by $\phi(\eta, c)$, $\psi(\eta, c)$, and make the corresponding changes in $\phi(\xi)$, $\psi(\xi)$, we shall have a solution which satisfies all the conditions of the problem, subject to the single condition that c is less than c_1 . At infinity $\psi(\eta, c)$ will be capable of expansion in the form of equation (5),

$$\psi(\eta, c) = b_1/\eta + b_2/\eta^2 + b_3/\eta^3 + \dots,$$

and we now see that equations (21) and (22) can be obtained in the same manner as before.

It is therefore clear that the values obtained for V_i , A , and α will be the true values, even if they have been obtained by the use of divergent series, provided only that the series for V_i remains convergent up to the boundary.

In the case in which $c_1 < c_2$, a similar proposition is true for values of c such that $c_1 < c < c_2$.

§ 10. We now consider the case in which c is greater than either c_1 or c_2 . Let the values of V_i and V_0 which have been found for values of c less than either c_1 or c_2 be extended, by a process of "continuation," to points outside their circles of convergence, and let the values so obtained define the functions V_i and V_0 . These functions will have certain infinities, the position of these infinities depending upon the value of c . When $c = 0$, all the infinities of V_i lie outside S ; all the infinities of V_0 lie inside S . Let us suppose that up to some value of c , say $c = c_3$, no infinity crosses S , but that (if possible) at $c = c_3$ one of these infinities is found on the boundary. The values of V_i and V_0 are functions of c , and $V_i - V_0$ satisfies the requisite algebraical equations from $c = 0$ until c is equal to the smaller of the values $c = c_1$ or c_2 . Hence it must continue to satisfy for all values of c , until the condition found in § 3 is violated, *i.e.*, until the value of c is such that the curve possesses a cusp or branch point. Also V_i and V_0 satisfy the requisite conditions of finiteness, uniqueness, and continuity until c reaches the value c_3 . Now as c approximates to c_3 from the direction in which $c < c_3$, the value of V at same point of the boundary (*viz.*, the point at which the infinity occurs when $c = c_3$) will increase indefinitely, becoming ultimately infinite when $c = c_3$. This value of V will, however, give the true solution for all values of c less than c_3 , and there will be a superior finite limit to the value of V . It therefore follows that there can be no value c_3 at which an infinity crosses the boundary, and the values of V_i and V_0 found by "continuation" of the power series will give the true values of V_i and V_0 until the whole solution is invalidated by the occurrence of a cusp or branch point.

Summing up, it appears that we may neglect the question of convergency of series altogether: so long as the values obtained for either V_i or V_0 are possible values, they must be true values. But care must be taken not to pass through values of the parameter such that S possesses a cusp or branch point for these values.

We are in search of a solution for ξ expanded in powers of c in the form

$$\xi = u_0 + u_1 c + u_2 c^2 + u_3 c^3 + \dots \quad (37),$$

in which $u_0, u_1, \dots$ are functions of η .

Substitute this assumed solution in equation (36), and we obtain

$$\begin{aligned} & \eta (u_0 + u_1 c + u_2 c^2 + u_3 c^3 + \dots) \\ &= \alpha^2 + c \left\{ \eta^n + u_0^n + n u_0^{n-1} u_1 c + \frac{1}{2} n (n-1) u_0^{n-2} u_1^2 c^2 \right. \\ & \quad + n u_0^{n-1} u_2 c^2 + \frac{1}{6} n (n-1) (n-2) u_0^{n-3} u_1^3 c^3 \\ & \quad + n (n-1) u_0^{n-2} u_1 u_2 c^3 + n u_0^{n-1} u_3 c^3 \left. \right\} \\ & \quad + \text{terms of degree 4 and higher in } c. \end{aligned}$$

Equating the coefficients of the various powers of c , we obtain

$$\begin{aligned} \eta u_0 &= \alpha^2, & \eta u_1 &= \eta^n + u_0^n, \\ \eta u_2 &= n u_0^{n-1} u_1, & \eta u_3 &= \frac{1}{2} n (n-1) u_0^{n-2} u_1^2 + n u_0^{n-1} u_2, \\ \eta u_4 &= \frac{1}{6} n (n-1) (n-2) u_0^{n-3} u_1^3 + n (n-1) u_0^{n-2} u_1 u_2 + n u_0^{n-1} u_3, \text{ \&c.} \end{aligned}$$

Solving these equations in succession, we obtain

$$\begin{aligned} u_0 &= \frac{\alpha^2}{\eta}, & u_1 &= \eta^{n-1} + \frac{\alpha^{2n}}{\eta^{n+1}}, & u_2 &= n \alpha^{2n-2} \left(\frac{1}{\eta} + \frac{\alpha^{2n}}{\eta^{n+1}} \right), \\ u_3 &= \frac{1}{2} n (n-1) \alpha^{2n-4} \eta^{n-1} + \frac{n(2n-1)}{\eta^{n+1}} \alpha^{4n-4} + \frac{n(3n-1) \alpha^{6n-4}}{2\eta^{3n+1}}, \\ u_4 &= \frac{1}{6} n (n-1) (n-2) \alpha^{2n-6} \eta^{2n-1} + \frac{3n(n-1)}{2\eta} \alpha^{4n-6}, \\ & \quad + \text{terms of lower degree in } \eta, \text{ \&c.} \end{aligned}$$

Hence we obtain at once

$$\begin{aligned} V_i &= C + \pi \rho \left\{ -\xi \eta + \frac{\xi^n + \eta^n}{n} c + \frac{1}{2} (n-1) \alpha^{2n-4} (\xi^n + \eta^n) c^3 \right. \\ & \quad \left. + \frac{1}{12} (n-1) (n-2) \alpha^{2n-6} (\xi^{2n} + \eta^{2n}) c^4 + \dots \right\} \\ &= C + \pi \rho \left\{ -r^2 + \left(\frac{2c}{n} + (n-1) \alpha^{2n-4} c^3 \right) r^n \cos n\theta \right. \\ & \quad \left. + \frac{1}{6} (n-1) (n-2) \alpha^{2n-6} c^4 r^{2n} \cos 2n\theta + \dots \right\}. \quad (38), \\ A &= \pi \{ \alpha^2 + n \alpha^{2n-2} c^2 + \frac{3}{2} n (n-1) \alpha^{4n-6} c^4 + \dots \}, \end{aligned}$$

and, except in the special case of $n=1, \alpha=0$.

In this way we can, when c is small, write down the potential to any required degree of accuracy.

Deformed Elliptic Cylinder.

§ 13. As a final illustration, we shall find the potential produced by a small deformation of the surface of the elliptic cylinder

$$\xi \eta = \alpha^2 + \alpha_2 (\xi^2 + \eta^2) \quad (39).$$

Let the deformed surface be

$$\xi\eta = a^2 + a_2(\xi^2 + \eta^2) + \sum_1^\infty b_n(\xi^n + \eta^n) \dots \dots \dots (40),$$

and let the solution be, as far as first powers of b 's,

$$\xi = \xi_0 + \sum_1^\infty b_n \xi_n \dots \dots \dots (41),$$

where ξ_n is a function of η , and ξ_0 is the solution when all the b 's vanish.

Substitute solution (41) in equation (40), and equate coefficients of b_n , and we obtain $\xi_n\eta = 2a_2\xi_0\xi_n + \xi_0^n + \eta^n$, or, solving for ξ_n ,

$$\xi_n = (\xi_0^n + \eta^n) / (\eta - 2a_2\xi_0) \dots \dots \dots (42).$$

We can express ξ_0 in the form

$$\xi_0 = \alpha\eta + \beta/\eta + \dots \dots \dots (43).$$

Hence equation (42) becomes

$$\begin{aligned} \xi_n &= \frac{(1 + \alpha^n)\eta^n + n\alpha^{n-1}\beta\eta^{n-2} + \dots}{\eta(1 - 2a_2\alpha) - 2a_2\beta\eta^{-1} + \dots} \\ &= \frac{1 + \alpha^n}{1 - 2a_2\alpha} \eta^{n-1} + \left(\frac{n\alpha^{n-1}\beta}{1 - 2a_2\alpha} + \frac{2a_2\beta(1 + \alpha^n)}{(1 - 2a_2\alpha)^2} \right) \eta^{n-3} + \&c. \end{aligned}$$

Hence we find as the value of V_i ,

$$\begin{aligned} V_i &= \pi\rho \{ C - \xi\eta + \tfrac{1}{2}\alpha(\xi^2 + \eta^2) \} \\ &+ \pi\rho \sum_{n=1}^{n=\infty} (\xi^n + \eta^n) \left\{ \frac{1 + \alpha^n}{1 - 2a_2\alpha} \frac{b_n}{n} + \left(\frac{(n+2)\alpha^{n+1}\beta}{1 - 2a_2\alpha} + \frac{2a_2\beta(1 + \alpha^{n+2})}{(1 - 2a_2\alpha)^2} \right) \frac{b_{n+2}}{n} \right. \\ &\quad \left. + \text{terms in } b_{n+4}, b_{n+6}, \&c. \dots \right\} \dots \dots \dots (44). \end{aligned}$$

The value of ξ_0 given by equation (43) is a solution of equation (39). Substituting this value, and equating the coefficients of the two highest powers of η , we find as the equations determining α and β ,

$$\alpha = a_2(1 + \alpha^2), \quad \beta = a^2 + a_2(2\alpha\beta + 1),$$

equations which will be required later.

ROTATING LIQUID CYLINDER.

General Theory.

§ 14. We now pass to the main problem before us, and consider the equilibrium of a cylinder rotating with angular velocity ω .

The equation to the cylinder for a rotation equal to zero is

$$\xi\eta = a^2 \dots \dots \dots (45).$$

When the rotation ω is different from zero, we shall suppose the equation to the surface referred to its axis of rotation as origin to become

$$\xi\eta = a^2 + a_1(\xi + \eta) + a_2(\xi^2 + \eta^2) + \dots \quad (46),$$

or, in polar co-ordinates,

$$r^2 = a^2 + 2a_1r \cos \theta + 2a_2r^2 \cos 2\theta + \dots \quad (47).$$

This equation is not sufficiently general to represent all cylinders which are symmetrical about the initial line. The value of r^2 being known at the boundary, we shall always be able to find a function v such that v is finite and continuous, together with its first differential coefficients, at all points inside the boundary, and such that $\nabla^2 v = 0$ inside the boundary, and $v = r^2$ at the boundary. The value of v near the origin can be expanded in the form

$$v = a^2 + 2a_1r \cos \theta + 2a_2r^2 \cos 2\theta + \dots \quad (48),$$

and this series will have a circle of convergence, say $r = R$. It is only when the curve lies wholly inside this circle that the cylinder can be represented by an equation of the form of (47).

Let us, however, attach a conventional meaning to equation (47) in the case in which the right hand becomes divergent at the boundary, as follows. Let us suppose that the value of the function v given near the origin by equation (48) is calculated from its known values inside the circle $r = R$, the values outside this circle being obtained by a process of "continuation." Then we shall suppose equation (47) to represent the locus of points at which $r^2 = v$.

Obviously, with this convention, equation (47) is sufficiently general to represent any surface. If this surface is to give an equilibrium configuration under a rotation ω , we must have

$$V_i + \frac{1}{2} \omega^2 r^2 = \text{a constant} \quad (49)$$

at the surface. Now $V_i + \pi\rho r^2$ is a spherical harmonic at all points inside the surface, and equation (49) can be written in the form

$$(V_i + \pi\rho r^2) - \pi\rho \left(1 - \frac{\omega^2}{2\pi\rho}\right) r^2 = \text{a constant},$$

or, what is the same thing,

$$(V_i + \pi\rho r^2) - \pi\rho \left(1 - \frac{\omega^2}{2\pi\rho}\right) v = \text{a constant} \quad (50).$$

This equation is satisfied at the surface S , and each term is a solution of LAPLACE'S equation at every point inside S ; hence the equation must be satisfied at every point inside S .

Now V_i can be calculated by the methods already explained, and we obtain an equation of the form

$$V_i = C - \pi\rho r^2 + \pi\rho \sum_{n=1}^{n=\infty} f_n(a_1, a_2, \dots) (\xi^n + \eta^n),$$

which gives the value of V_i at all points inside a certain circle of convergence. The value of v inside its circle of convergence is, from equation (48),

$$v = a^2 + \sum_{n=1}^{n=\infty} a_n (\xi^n + \eta^n);$$

hence equation (50), at points inside both circles, becomes

$$C = \pi \rho a^2 \left(1 - \frac{\omega^2}{2\pi \rho} \right) + \pi \rho \sum_{n=1}^{n=\infty} \left\{ f_n(a_1, a_2, \dots) - a_n \left(1 - \frac{\omega^2}{2\pi \rho} \right) \right\} (\xi^n + \eta^n) = \text{a constant.} \quad (51).$$

We must therefore have, for all positive integral values of n ,

$$f_n(a_1, a_2, \dots) = a_n \left(1 - \frac{\omega^2}{2\pi \rho} \right) \quad (52).$$

The condition that equation (46) or (47) may represent an equilibrium configuration is therefore

$$f_1/a_1 = f_2/a_2 = f_3/a_3 = \dots \quad (53),$$

and the value of ω^2 is such that each fraction is equal to $1 - \frac{\omega^2}{2\pi \rho}$. Since for any equilibrium configuration the axis of rotation must coincide with the centre of gravity, we see that if the coefficients of (46) satisfy (53), the curve must be referred to its centre of gravity as origin.

The points of bifurcation are given by the Hessian of this system of equations. For our purpose this may be most conveniently written in the form

$$\begin{vmatrix} \frac{\partial f_1}{\partial a_1} - \left(1 - \frac{\omega^2}{2\pi \rho} \right), & \frac{\partial f_2}{\partial a_1}, & \frac{\partial f_3}{\partial a_1}, & \dots \\ \frac{\partial f_1}{\partial a_2}, & \frac{\partial f_2}{\partial a_2} - \left(1 - \frac{\omega^2}{2\pi \rho} \right), & \dots & \dots \\ \frac{\partial f_1}{\partial a_3}, & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0 \quad (54).$$

Our method will, for reasons already explained, break down as soon as a cusp or branch point occurs on any linear series. Any solution will be a true solution provided we can pass from this solution to another, known to be a true solution, over a path through a system of linear series, without passing through a point at which a cusp or branch point occurs. Now the occurrence of a cusp or branch point indicates, in the physical problem, the division of the mass of fluid into two separate masses, and when this occurs the solution breaks down for a second reason also: for equation (49) is only true when the surface of the fluid is continuous; when the surface consists of two distinct parts, the constant on the right-hand side has different values for the two parts of the surface.

This limitation will not cause trouble in the present investigation. For we are only desirous of tracing the changes in the configuration of the fluid up to the separation into two parts, and even if our method had enabled us to proceed beyond this point, it would have been fruitless to do so.

The Series of Circular Cylinders.

§ 15. If it were possible to calculate the f 's and solve equations (53) in the most general case, we should arrive at a complete knowledge of the system of linear series of equilibrium configurations. This being impossible, we shall start from a known series, and calculate successive series by determining the various points of bifurcation.

Now we know (§ 12) that for the small values of the a 's, f_n is of the form $f_n(a_1, a_2, \dots) = \frac{a_n}{n}$. Hence there is a solution of the system of equations (52) given by $a_1 = a_2 = \dots = 0$.

This is the series of circular configurations, and corresponds to the series of Maclaurin spheroids in the three-dimensional problem. When $\omega^2 > 2\pi\rho$ the solution breaks down physically, since the pressure at every point of the liquid becomes negative. In fact, when ω^2 reaches the value $\omega^2 = 2\pi\rho$ the series gives place to a series of annular forms, for each of which ω^2 has the critical value. We can adjust the radius of the annulus so as to give any desired amount of angular momentum greater than the critical value which occurs in the circular configuration when $\omega^2 = 2\pi\rho$.

Points of Bifurcation on Circular Series.

§ 16. To search for points of bifurcation on this linear series, we replace f_n by a_n/n in equation (54). Every term in the determinant on the left hand now vanishes, except the terms of the leading diagonal, and the equation reduces to

$$\left\{1 - \left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\} \left\{\frac{1}{2} - \left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\} \dots \left\{\frac{1}{n} - \left(1 - \frac{\omega^2}{2\pi\rho}\right)\right\} \dots = 0.$$

The different roots correspond to the different integral values of n , and are given by

$$\begin{aligned} n &= 1, \quad 2, \quad 3, \quad 4, \quad 5, \dots \infty, \\ \omega^2/2\pi\rho &= 0, \quad \cdot 5, \quad \cdot 666, \quad \cdot 75, \quad \cdot 8, \dots 1. \end{aligned}$$

The first point of bifurcation ($n = 1$) may be rejected at once, the critical "vibration" being merely a displacement of the entire cylinder as a rigid body.

A displacement in which a_n only occurs for the single value $n = s$ will alter the potential energy only by a term proportional to the square of a_n . Hence the principal vibrations correspond to the different values of n from 2 to ∞ , and are such that a_n only occurs for a single value of n in each. When $\omega^2 = 0$, all these vibrations are stable. When ω^2 reaches the point of bifurcation of order s , the vibration of order s becomes unstable, and, since there is only one point of bifurcation of order s , this vibration remains unstable for all values of ω^2 greater than the value at this point of bifurcation. We therefore see that by the time that ω^2 reaches the limiting value $2\pi\rho$, every vibration is unstable.

The value of ω^2 is given by

$$1 - \omega^2/2\pi\rho = 1/n + \frac{1}{2}(n-1)a_n^{2n-4}a_n^2.$$

It therefore appears that the angular velocity decreases as we recede from the point of bifurcation.

Throughout the series the bounding curve is periodic in θ with a period $2\pi/n$, and it is easily seen that at the remote end of the series is a curve consisting of n equal and symmetrically arranged arms, these arms extending in the limit to infinity, and being of zero breadth at all points except in the immediate neighbourhood of the origin.

In fig. 1 I have drawn the curve for the case of $n = 3$, $a_3 = \frac{1}{4\sqrt{2}a}$. The error in the value of V_i (equation (38)) caused by the neglect of a_3^4 will be seen to be of the order of $0.0003\pi\rho \frac{r^6}{a^4} \cos 4\theta$.

Since the maximum value of r^6/a^4 on this curve is $8a^2$, it appears to be legitimate to neglect this error.

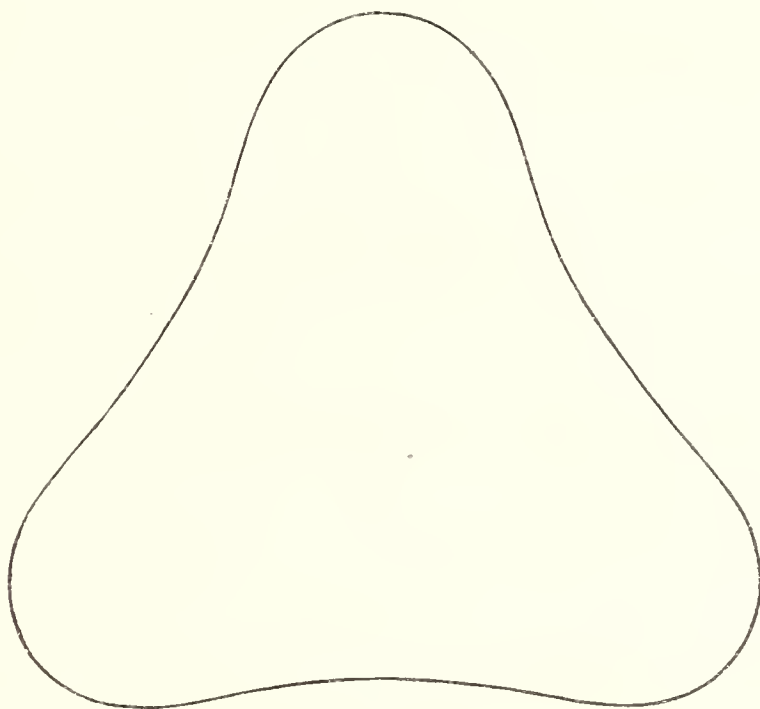


Fig. 1.

Points of Bifurcation on Elliptic Series.

§ 19. Let us now return to the series of elliptic cylinders, and search for points of bifurcation on this series. We have found in § 13 that when a_2 is finite, but when the squares and products of the remaining a 's may be neglected, we have (equation (44)) for values of n different from 2,

$$f_n(a_1, a_2, a_3, \dots) = \frac{1 + a^n}{1 - 2aa_2} \frac{a_n}{n} + \text{terms linear in } a_{n+2}, a_{n+4}, \&c. \dots \quad (61).$$

Let us now examine the form assumed by equation (54) at points upon this series. We have, in the first place, $\partial f_1/\partial a_2 = 0$.

We have also $\partial f_m / \partial f_n = 0$ whenever m is greater than n . Hence we see that the determinant on the left hand of (54) reduces to the products of the terms in its leading diagonal, and that the equation itself is equivalent to the separate equations

$$\partial f_n / \partial a_n = (1 - \omega^2 / 2\pi\rho) \dots \dots \dots (62)$$

taken for all values of n from 1 to ∞ .

Corresponding to a root of (62) there is a point of bifurcation, and the linear series starting from the point must be found from equations (52). From these equations it appears that the linear series corresponding to a root of (62) will be such that, as far as the first order of small quantities, a_s exists only for the values $s = 2, n, n - 2, n - 4, \dots$.

Of these series the series $n = 2$ may be rejected, as corresponding only to a step along the series of elliptic cylinders, and not to a new series at all, and the series $n = 1$ may be rejected, as corresponding merely to a change of origin.

We are left with the values $n = 3, 4, 5, \dots$, and for any one of these values we have, from equation (61),

$$\frac{\partial f_n}{\partial a_n} = \frac{1}{n} \frac{1 + \alpha^n}{1 - 2\alpha a_2}.$$

The points of bifurcation are accordingly given by the equations

$$\frac{1}{n} \frac{1 + \alpha^n}{1 - 2\alpha a_2} = 1 - \frac{\omega^2}{2\pi\rho} \dots \dots \dots (63),$$

where n has the values 3, 4, 5, $\dots$.

These points of bifurcation are points on the series of elliptic cylinders, hence ω^2 , a_2 , and α are connected by equations (57) and (58). If we eliminate ω^2 and a_2 from the three equations (57), (58), and (63), we find, as the equation giving points of bifurcation of order n ,

$$\frac{1 - \alpha^2}{2} = \frac{1 + \alpha^n}{n} \dots \dots \dots (64).$$

This equation must be solved by graphical methods. In fig. 2 the curve which is concave to the axis of α is the parabola

$$y = \frac{1}{2}(1 - \alpha^2) \dots \dots \dots (65).$$

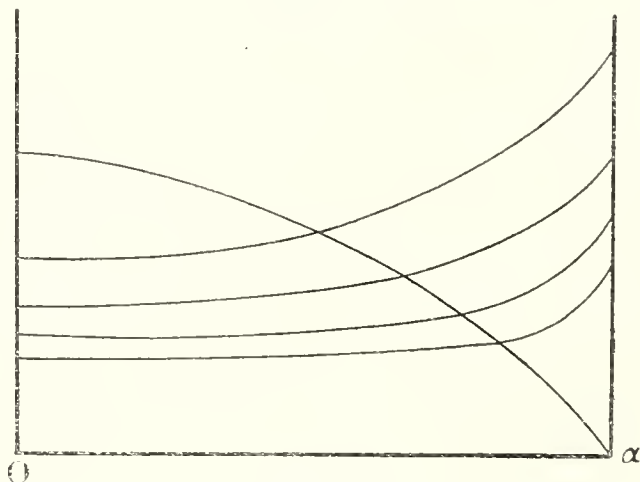


Fig. 2.

The remaining curves are the graphs of

$$y = \frac{1 + \alpha^n}{n} \quad \dots \quad (66)$$

for the values $n = 3, 4, 5, \dots$

The curve (66) cuts $\alpha = 0$ at the point $y = 1/n$, and $\alpha = 1$ at $y = 2/n$. The value of $dy/d\alpha$ is α^{n-1} . It is therefore obvious that the curves are convex to the axis of α , and since for any value of α the value of $dy/d\alpha$ is greatest for that curve for which n is least, it is obvious that the curves can never intersect.

We therefore see that the parabola (65) will meet each of the curves (66) once, and once only, for values of α between 0 and 1. Moreover the smaller n is, the smaller the value of α at the intersection.

As we move along the series of elliptic cylinders, the value of α increases from 0 to 1. Hence there will be an infinite number of points of bifurcation on this series, of orders 3, 4, 5, $\dots$. The point at which we arrive first is that of order $n = 3$; those of orders 4, 5, $\dots$ follow in succession. As before, we find that the configuration at the end of the series of elliptic cylinders ($\alpha = 1$, an infinitely long and thin ellipse) is unstable for every vibration.

The linear series which we expect to be stable is that of order $n = 3$. To find the point of bifurcation of this series we require to solve the equation $\frac{1}{2}(1 - \alpha^2) = \frac{1}{3}(1 + \alpha^3)$ and the solution is found by inspection to be $\alpha = \frac{1}{2}$.

From equations (52) and (53) we find that at this point of bifurcation $\omega^2 = \frac{3}{4}\pi\rho$, and $a_2 = \frac{2}{3}$. The elliptic cylinder at the point of bifurcation is therefore the cylinder

$$\xi\eta = a^2 + \frac{2}{3}(\xi^2 + \eta^2), \quad \dots \quad (67)$$

or, in Cartesian co-ordinates,

$$x^2 + 9y^2 = 5a^2 \quad \dots \quad (68).$$

If we reduce the linear scale of this until the area is a^2 , we find for its equation

$$x^2 + 9y^2 = 3a^2,$$

and for its angular momentum, 1.46 times the greatest angular momentum for which the circular form is stable.

POINCARÉ'S *Series of Pear-shaped Curves.*

§ 20. The configuration of the new linear series of order $n = 3$ is, near the point of bifurcation, of the form

$$\xi\eta = a^2 + \frac{2}{3}(\xi^2 + \eta^2) + b_3(\xi^3 + \eta^3) + b_1(\xi + \eta) \quad \dots \quad (69).$$

This new series is seen to be the series corresponding to POINCARÉ'S series of pear-shaped figures.* Instead of making a separate problem out of the determination of the constants b_1 and b_3 , we shall, in order to avoid repetition at a later stage, pass

* H. POINCARÉ, 'Acta Math.,' vol. 7, p. 347.

at once to the equations determining the general configuration of this series. We therefore replace equation (69) by

$$\xi\eta = a^2 + \frac{2}{5}(\xi^2 + \eta^2) + \sum_{s=1}^{s=\infty} \theta^s \left\{ {}_sC_0 + \sum_{n=1}^{n=\infty} {}_sC_n (\xi^n + \eta^n) \right\} \quad \dots \quad (70).$$

This will be assumed to be the general form of the surface in the linear series now under discussion, the quantity θ being a parameter which vanishes at the point of bifurcation.

The equation expressing explicitly the solution of (70) may be supposed to be

$$\xi = \left(1 - \frac{\omega^2}{2\pi\rho}\right) \left(\frac{8}{5}\xi_0 + \xi_1\theta + \xi_2\theta^2 + \xi_3\theta^3 + \dots\right) \quad \dots \quad (71).$$

in which ξ_0 is the value of ξ when $\theta = 0$, and therefore satisfies

$$\xi_0\eta = a^2 + \frac{2}{5}(\xi_0^2 + \eta^2) \quad \dots \quad (72),$$

and ξ_s is a series of ascending and descending powers of η , say

$$\xi_s = a_0 + a_1\eta + a_2\eta^2 + \dots + a_{-1}\eta^{-1} + a_{-2}\eta^{-2} + \dots \quad \dots \quad (73).$$

If we calculate the value of V_i from (71), we find

$$V_i = -\pi\rho\xi\eta + \frac{8}{5}\pi\rho \left(1 - \frac{\omega^2}{2\pi\rho}\right) U_0 + \pi\rho \left(1 - \frac{\omega^2}{2\pi\rho}\right) \sum \theta^s U_s \quad \dots \quad (74),$$

where, if ξ_s is given by (73), the value of U_s is

$$U_s = C + a_0(\xi + \eta) + \frac{1}{2}a_1(\xi^2 + \eta^2) + \dots \quad \dots \quad (75).$$

Using the value of V_i given by (74), the equation to be satisfied at the surface is

$$-\xi\eta \left(\pi\rho - \frac{1}{2}\omega^2\right) + \frac{8}{5}\pi\rho \left(1 - \frac{\omega^2}{2\pi\rho}\right) U_0 + \pi\rho \left(1 - \frac{\omega^2}{2\pi\rho}\right) \sum \theta^s U_s = \text{const.},$$

or, dividing throughout by $\pi\rho - \frac{1}{2}\omega^2$,

$$\xi\eta = \sum \theta^s U_s + \text{terms independent of } \theta \quad \dots \quad (76).$$

Equation (76) must be identical with (70), the right-hand members of both being spherical harmonics, and hence we must have

$$U_s = {}_sC_0 + \sum_{n=1}^{n=\infty} {}_sC_n (\xi^n + \eta^n),$$

and therefore, by equation (75), ${}_sC_n = a_{n-1}/n$ for all positive values of n .

Instead of being given by equation (73), the value of ξ_s may now be supposed to be given by

$$\xi_s = {}_sC_1 + 2{}_sC_2\eta + 3{}_sC_3\eta^2 + \dots + a_{-1}\eta^{-1} + a_{-2}\eta^{-2} + \dots \quad \dots \quad (77).$$

If we introduce the limitation that the curve is to remain of constant area, we must put $a_{-1} = 0$. If we now replace $a_{-2}, a_{-3}, \dots$ by new unknowns ${}_sC_{-1}, {}_sC_{-2}, \dots$, we can write equation (77) in the symmetrical form

$$\xi_s = \sum_{n=-\infty}^{n=+\infty} n {}_sC_n \eta^{n-1} \quad \dots \quad (78),$$

in which we know that ${}_sC_{-1}$ must ultimately be equal to zero, in order that the centre of gravity may coincide with the origin (*cf.* equation 22).

Now ω^2 will be a function of θ , and the relation between ω^2 and θ is at our disposal, this relation being virtually the definition of θ . We have, however, already assumed that at the point of bifurcation $\theta = 0$ and $\partial\omega^2/\partial\theta = 0$. We therefore take as the relation between ω^2 and θ ,

$$1 - \omega^2/2\pi\rho = \delta_0 + \delta_2\theta^2 + \delta_3\theta^3 + \dots \quad (79),$$

in which $\delta_0 = \frac{5}{8}$, and $\delta_2, \delta_3, \dots$ are as yet undetermined.

The value of ξ given by equation (71) is now

$$\xi = \xi_0 + \theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) + \theta^3(\delta_0\xi_3 + \delta_2\xi_1 + \frac{8}{5}\delta_3\xi_0) + \dots \quad (80).$$

If we substitute this value in equation (70), of which we are supposing it to be a solution, we obtain

$$\begin{aligned} \eta(\xi_0 + \theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) + \dots) \\ = \alpha^2 + \frac{2}{5}\{\eta^2 + \xi_0^2 + 2\xi_0(\theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) + \dots) \\ + (\theta\delta_0\xi_1 + \theta^2(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) + \dots)^2\} \\ + \sum_{s=1}^{s=\infty} \theta^s({}_sC_0 + \sum_{n=1}^{n=\infty} {}_sC_n(\xi^n + \eta^n)) \quad (81), \end{aligned}$$

and if we equate the coefficients of successive powers of θ in this, we obtain

$$\xi_0\eta = \alpha^2 + \frac{2}{5}(\xi_0^2 + \eta^2) \quad (82),$$

$$\delta_0\xi_1(\eta - \frac{4}{5}\xi_0) = {}_1C_0 + \sum_1^\infty {}_1C_n(\xi_0^n + \eta^n) \quad (83),$$

$$\begin{aligned} (\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0)(\eta - \frac{4}{5}\xi_0) = \frac{2}{5}(\delta_0\xi_1)^2 + \delta_0\xi_1 \sum_1^\infty n {}_1C_n \xi_0^{n-1} \\ + {}_2C_0 + \sum_1^\infty {}_2C_n(\xi_0^n + \eta^n) \quad (84), \end{aligned}$$

$$\begin{aligned} (\delta_0\xi_3 + \delta_2\xi_1 + \frac{8}{5}\delta_3\xi_0)(\eta - \frac{4}{5}\xi_0) = \frac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) \\ + (\delta_0\xi_1)^2 \sum_1^\infty \frac{1}{2}n(n-1) {}_1C_n \xi_0^{n-2} + (\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) \sum_1^\infty n {}_1C_n \xi_0^{n-1} \\ + \delta_0\xi_1 \sum_1^\infty n {}_2C_n \xi_0^{n-1} + {}_3C_0 + \sum_1^\infty {}_3C_n(\xi_0^n + \eta^n). \quad (85). \end{aligned}$$

Equation (82) is, as it ought to be, identical with (72). Equation (83) enables us to determine the constants which occur in ξ_1 . These having been found, equation (84) enables us to determine ξ_2 , and so on in succession.

§ 21. As far as first powers of θ , we know that the configuration is of the form given by equation (69). We therefore assume at once for ξ_1 the form

$$\xi_1 = 3c_3\eta^2 + c_1 - \frac{c_{-1}}{\eta^2} - \frac{3c_{-3}}{\eta^4} - \dots \quad (86),$$

in which c_n is temporarily written for ${}_1C_n$.

Since $\delta_0 = \frac{5}{8}$, and since the c 's higher than c_3 must vanish, we find that equation (83) takes the form

$$\frac{5}{8}(\eta - \frac{4}{5}\xi_0)\xi_1 = c_3(\xi_0^3 + \eta^3) + c_1(\xi_0 + \eta) \quad (87).$$

Since ξ_0 satisfies equation (82), we have

$$\xi_0 = \frac{5}{4}\eta \pm \frac{1}{2} \sqrt{\left(\frac{9}{4}\eta^2 - 10a^2\right)}. \quad (88),$$

or, expanding in the appropriate form, and taking $a = 1$,

$$\xi_0 = \frac{1}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \frac{1000}{3^5\eta^5} + \frac{25,000}{3^7\eta^7} + \dots \quad (89).$$

From equations (82) and (89) we have

$$\xi_0^2 = \frac{5}{2}\xi_0\eta - \eta^2 - \frac{5}{2} = \frac{1}{4}\eta^2 + \frac{5}{3} + \frac{125}{27\eta^2} + \frac{2500}{3^5\eta^4} + \dots \quad (90).$$

By a similar process we obtain

$$\xi_0^3 = \left(\frac{21}{4}\eta^2 - \frac{5}{2}\right)\xi_0 - \frac{5}{2}\eta^3 - \frac{25}{4}\eta = \frac{1}{8}\eta^3 + \frac{5}{4}\eta + \frac{50}{9\eta} + \frac{1375}{81\eta^3} + \dots \quad (91).$$

Equation (87) can now be put into the form

$$\begin{aligned} & \left[\frac{3}{8}\eta - \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{243\eta^5} - \dots \right] \left[3c_3\eta^2 + c_1 - \frac{c_{-1}}{\eta^2} - \frac{3c_{-3}}{\eta^4} - \dots \right] \\ & = c_3 \left[\frac{9}{8}\eta^3 + \frac{5}{4}\eta + \frac{50}{9\eta} + \frac{1375}{81\eta^3} + \dots \right] \\ & + c_1 \left[\frac{3}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \dots \right] \quad (92). \end{aligned}$$

Equating the coefficients of the various powers of η we obtain

$$\begin{aligned} \frac{9}{8}c_3 &= \frac{9}{8}c_3 \\ -\frac{5}{2}c_3 + \frac{3}{8}c_1 &= \frac{5}{4}c_3 + \frac{3}{2}c_1 \\ -\frac{25}{9}c_3 - \frac{5}{6}c_1 - \frac{3}{8}c_{-1} &= \frac{50}{9}c_3 + \frac{5}{3}c_1 \\ -\frac{500}{81}c_3 - \frac{25}{27}c_1 + \frac{5}{6}c_{-1} - \frac{9}{8}c_{-3} &= \frac{1375}{81}c_3 + \frac{50}{27}c_1. \end{aligned}$$

The first equation is, as it ought to be, an identity. We may assign to c_3 any value, and therefore take $c_3 = 1$, this being equivalent to fixing the linear scale of measurement of θ . Solving the remaining equations in succession, we obtain the following scheme of values:—

$$c_3 = 1, \quad c_1 = -\frac{10}{3}, \quad c_{-1} = 0, \quad c_{-3} = -\frac{1000}{81}.$$

The vanishing of c_{-1} shows that the centre of gravity of the curve is, as it ought to be, at the origin. We now have as the value of ξ_1 , equation (86),

$$\xi_1 = 3\eta^2 - \frac{10}{3} + \frac{1000}{27\eta^4} + \dots \quad (93).$$

§ 22. We now proceed to the determination of ξ_2 . Equation (84) takes the form

$$\begin{aligned} (\eta - \frac{1}{3}\xi_0) (\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) &= \frac{2}{5}\delta_0^2\xi_1^2 + \delta_0\xi_1 (3\xi_0^2 - \frac{10}{3}) \\ &+ {}_2C_0 + \sum_{n=1}^{n=\infty} {}_2C_n (\xi_0^n + \eta^n) \quad (94). \end{aligned}$$

The value of ξ_2 is of the form (equation (78)),

$$\xi_2 = \sum_{n=-\infty}^{+\infty} n_2 C_n \eta^{n-1}. \quad (95).$$

Suppose this value substituted for ξ_2 in equation (94), and in the equation so obtained equate the coefficients of the various powers of η . For any power of η greater than the fourth it will be seen that equation (94) may be replaced by

$$\delta_0 \left(\eta - \frac{4}{5} \xi_0 \right) \sum_{n=-\infty}^{n=+\infty} n {}_2C_n \eta^{n-1} - \sum_{n=1}^{n=\infty} {}_2C_n (\xi_0^n + \eta^n) = 0.$$

The equation obtained by equating to zero the coefficient of any power of η greater than the fourth will therefore be of the form

$$\text{a linear homogeneous function of } {}_2C_n, {}_2C_{n+1}, \dots = 0,$$

and there is an equation of this form for every value of n greater than 4. This system of equations can only be satisfied by taking

$${}_2C_5 = {}_2C_6 = {}_2C_7 = \dots = 0.$$

We may therefore assume for ξ_2 (equation (95)) an expansion of the form

$$\xi_2 = 4d_4\eta^3 + 3d_3\eta^2 + 2d_2\eta + d_1 - \frac{d_{-1}}{\eta^2} - \frac{2d_{-2}}{\eta^3} \dots \dots \dots (96),$$

in which d_n is written for ${}_2C_n$.

From equation (82) we have

$$\delta_0 \xi_1 (3\xi_0^2 - \frac{10}{3}) = \xi_1 (\frac{7.5}{16} \xi_0 \eta - \frac{1.5}{8} \eta^2 - \frac{3.25}{48}),$$

and from equation (87),

$$(\eta - \frac{4}{5} \xi_0) \xi_1 = \frac{8}{5} (\xi_0^3 + \eta^3) - \frac{16}{3} (\xi_0 + \eta) \dots \dots \dots (97).$$

From these last two equations we obtain

$$\delta_0 \xi_1 (3\xi_0^2 - \frac{10}{3}) = \xi_1 (\frac{2.55}{64} \eta^2 - \frac{3.25}{48}) - \frac{7.5}{8} \eta (\xi_0^3 + \eta^3) + \frac{1.25}{4} \eta (\xi_0 + \eta) \dots (98)$$

With the help of equations (97) and (98) we can write equation (94) in the form

$$\begin{aligned} (\eta - \frac{4}{5} \xi_0) (\delta_0 \xi_2 + \frac{8}{5} \delta_2 \xi_0) &= \frac{5}{32} \xi_1^2 + \xi_1 (\frac{2.55}{64} \eta^2 - \frac{3.25}{48}) \\ &\quad - \frac{7.5}{8} \eta (\xi_0^3 + \eta^3) + \frac{1.25}{4} \eta (\xi_0 + \eta) \\ &\quad + d_4 (\xi_0^4 + \eta^4) + d_3 (\xi_0^3 + \eta^3) + d_2 (\xi_0^2 + \eta^2) + d_1 (\xi_0 + \eta) + d_0 \dots (99). \end{aligned}$$

It is clear upon examination of this equation that the equations found upon equating the coefficients of η^3 , η , η^{-1} , &c., will contain only terms multiplied by d_3 , d_0 , d_{-1} , &c., without constant terms. We therefore take

$$d_3 = d_1 = d_{-1} = \dots = 0.$$

Equation (99) now contains only even powers of η . Before we can calculate the coefficients of these powers we must obtain series for ξ_1^2 and ξ_0^4 . By squaring equation (93) we get

$$\xi_1^2 = 9\eta^4 - 20\eta^2 + \frac{100}{9} + \frac{2000}{9\eta^2} + \dots \dots \dots (100).$$

Next we have from equation (82)

$$\xi_0^2 + \eta^2 = \frac{5}{2} (\xi \eta - 1).$$

Squaring this, and subtracting $2\xi_0^2\eta^2$ from each side,

$$\xi_0^4 + \eta^4 = \frac{25}{4} - \frac{25}{2}\xi_0\eta + \frac{17}{4}\xi_0^2\eta^2.$$

Using the series which have already been obtained for ξ_0^2 and ξ_0 (equations (89) and (90)), we obtain

$$\xi_0^4 = \frac{1}{16}\eta^4 + \frac{5}{6}\eta^2 + \frac{275}{54} + \frac{5000}{243\eta^2} + \dots$$

We can now evaluate that part of the right-hand side of equation (99) which does not contain d_4 , d_2 , or d_0 . We have

$$\begin{aligned} \frac{5}{32}\xi_1^2 &= \frac{45}{32}\eta^4 - \frac{25}{8}\eta^2 + \frac{125}{72} + \frac{625}{18\eta^2} + \dots, \\ \xi_1\left(\frac{255}{64}\eta^2 - \frac{325}{48}\right) &= \frac{765}{64}\eta^4 - \frac{1125}{32}\eta^2 + \frac{1625}{72} + \frac{10625}{72\eta^2} + \dots, \\ -\frac{75}{8}\eta(\xi_0^3 + \eta^3) &= -\frac{675}{64}\eta^4 - \frac{375}{32}\eta^2 - \frac{625}{12} - \frac{34375}{216\eta^2} + \dots, \\ \frac{125}{4}\eta(\xi_0 + \eta) &= \frac{375}{8}\eta^2 + \frac{625}{12} + \frac{3125}{54\eta^2} + \dots \end{aligned}$$

By addition the sum of the terms in question is found to be

$$\frac{45}{16}\eta^4 - \frac{25}{16}\eta^2 + \frac{875}{36} + \frac{4375}{54\eta^2} + \dots$$

Lastly, we have

$$\begin{aligned} \frac{8}{5}(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) &= 4d_4\eta^3 + (2d_2 + \frac{32}{5}\delta_2)\eta + \frac{64}{15}\delta_2\eta^{-1} \\ &\quad + (-2d_{-2} + \frac{128}{27}\delta_2)\eta^{-3} + \dots \end{aligned}$$

Collecting the various series, we find as the form assumed by equation (99),

$$\begin{aligned} &\left(\frac{3}{8}\eta - \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{3^5\eta^5} - \dots\right) \\ &\left(4d_4\eta^3 + (2d_2 + \frac{32}{5}\delta_2)\eta + \frac{64}{15}\delta_2 + (-2d_{-2} + \frac{128}{27}\delta_2)\eta^{-3} + \dots\right) \\ &= \frac{45}{16}\eta^4 - \frac{25}{16}\eta^2 + \frac{875}{36} + \frac{4375}{54\eta^2} + \dots \\ &+ d_4\left(\frac{17}{16}\eta^4 + \frac{5}{6}\eta^2 + \frac{275}{54} + \frac{5000}{243\eta^2} + \dots\right) \\ &+ d_2\left(\frac{5}{4}\eta^2 + \frac{5}{3} + \frac{125}{27\eta^2} + \dots\right) \\ &+ d_0. \end{aligned}$$

Equating the coefficients of the various powers of η , we obtain

$$\begin{aligned} \frac{3}{4}d_4 &= \frac{45}{16} + \frac{17}{16}d_4, \\ -\frac{10}{3}d_4 + \frac{3}{4}d_2 + \frac{12}{5}\delta_2 &= -\frac{25}{16} + \frac{5}{6}d_4 + \frac{5}{4}d_2, \\ -\frac{100}{27}d_4 - \frac{5}{3}d_2 + \frac{8}{15}\delta_2 &= \frac{875}{36} + \frac{275}{54}d_4 + \frac{5}{3}d_2 + d_0, \\ -\frac{2000}{243}d_4 - \frac{50}{27}d_2 - \frac{80}{27}\delta_2 - \frac{3}{4}d_{-2} &= \frac{4375}{54} + \frac{5000}{243}d_4 + \frac{125}{27}d_2. \end{aligned}$$

Solving, we obtain in succession,

$$\begin{aligned} d_4 &= \frac{45}{7}, & d_2 &= -\frac{2825}{56} + \frac{24}{5}\delta_2. \\ d_0 &= \frac{5500}{63} - \frac{8}{3}\delta_2, & d_{-2} &= \frac{4375}{54} - \frac{992}{81}\delta_2. \end{aligned}$$

We therefore find as the value of ξ_2

$$\xi_2 = \frac{180}{7}\eta^3 - \frac{2825}{28}\eta - \frac{4375}{27\eta^3} + \dots + \delta_2 \left\{ \frac{48}{5}\eta + \frac{1984}{81\eta^3} + \dots \right\} \quad (101).$$

§ 23. We now proceed to the determination of ξ_3 . Equation (85) takes the form

$$\begin{aligned} (\delta_0\xi_3 + \delta_2\xi_1 + \frac{8}{5}\delta_3\xi_0)(\eta - \frac{4}{5}\xi_0) \\ &= \frac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) \\ &\quad + (\delta_0\xi_1)^2 3\xi_0 + (\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0)(3\xi_0^2 - \frac{10}{3}) \\ &\quad + \delta_0\xi_1(\frac{180}{7}\xi_0^3 - \frac{2825}{28}\xi_0) + \delta_0\delta_2\frac{48}{5}\xi_0\xi_1 \\ &\quad + {}_3C_0 + \sum_1^{\infty} {}_3C_n(\xi_0^n + \eta^n) \quad \dots \quad (102). \end{aligned}$$

All the terms on the right-hand side, except those in the last line, are of odd degree in η and of degree 5 at most. The same is true of the terms on the left-hand which are multiplied by δ_2 . The terms multiplied by δ_3 are of even degree, two at most. It is therefore clear that we may at once take

$$\begin{aligned} {}_3C_6 &= {}_3C_7 = {}_3C_8 = {}_3C_9 = \dots = 0, \\ \delta_3 &= 0; \quad {}_3C_4 = {}_3C_2 = {}_3C_0 = \dots = 0, \end{aligned}$$

and assume for ξ_3 an expansion of the form

$$\xi_3 = 5e_5\eta^4 + 3e_3\eta^2 + e_1 - \frac{e_{-1}}{\eta^2} - \dots \quad (103).$$

We now calculate the various series which occur in equation (102). We have

$$\delta_0(\eta - \frac{4}{5}\xi_0)\xi_3 = \left[\frac{3}{8}\eta - \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{243\eta^5} - \dots \right] \xi_3,$$

and from equation (97),

$$\begin{aligned} \delta_2(\eta - \frac{4}{5}\xi_0)\xi_1 &= \delta_2 \left[\frac{8}{5}(\xi_0^3 + \eta^3) - \frac{16}{3}(\xi_0 + \eta) \right] \\ &= -\delta_2 \frac{64}{375\eta} \left[-\frac{75}{8}\eta(\xi_0^3 + \eta^3) + \frac{125}{4}\eta(\xi_0 + \eta) \right]. \end{aligned}$$

This last bracket can be at once calculated from the series of the last page; we have

$$\delta_2(\eta - \frac{4}{5}\xi_0)\xi_1 = \delta_2 \left(\frac{9}{5}\eta^3 - 6\eta + 0 \cdot \eta^{-1} + \dots \right) \quad (104).$$

Next

$$\frac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) + (\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0)(3\xi_0^2 - \frac{10}{3}) = (\frac{5}{8}\xi_2 + \frac{8}{5}\delta_2\xi_0)(\frac{1}{2}\xi_1 + 3\xi_0^2 - \frac{10}{3})$$

From equations (93) and (90) we have

$$\begin{aligned} \frac{1}{2}\xi_1 + 3\xi_0^2 - \frac{10}{3} &= \left(\frac{3}{2}\eta^2 - \frac{5}{3} + \frac{500}{27\eta^4} \right) + \left(\frac{3}{4}\eta^2 + 5 + \frac{125}{9\eta^2} + \frac{2500}{81\eta^4} \right) - \frac{10}{3} \\ &= \frac{9}{4}\eta^2 + \frac{125}{9\eta^2} + \frac{4000}{81\eta^4} \quad \dots \quad (105). \end{aligned}$$

From equations (101) and (89) we have

$$\begin{aligned}
\frac{5}{8}\xi_2 + \frac{8}{5}\delta_2\xi_0 &= \frac{5}{8}\left\{\frac{180}{7}\eta^3 - \frac{2825}{28}\eta - \frac{4375}{27\eta^3} - \dots\right\} + \frac{5}{8}\delta_2\left\{\frac{48}{5}\eta + \frac{1984}{81\eta^3} + \dots\right\} \\
&\quad + \delta_2\left\{\frac{4}{3}\eta + \frac{8}{3\eta} + \frac{80}{27\eta^3} + \dots\right\} \\
&= \frac{225}{14}\eta^3 - \frac{14125}{224}\eta - \frac{21875}{216\eta^3} + \dots + \delta_2\left\{2\eta + \frac{8}{3\eta} + \frac{1480}{81\eta^3} + \dots\right\},
\end{aligned}$$

and hence, by multiplication with (105),

$$\begin{aligned}
\frac{4}{5}\delta_0\xi_1(\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0) + (\delta_0\xi_2 + \frac{8}{5}\delta_2\xi_0)(3\xi_0^2 - \frac{10}{3}) \\
= \frac{2025}{56}\eta^5 - \frac{127125}{896}\eta^3 + \frac{3125}{14}\eta - \frac{78125}{352\eta} + \dots \\
+ \delta_2\left\{\frac{9}{2}\eta^3 + 6\eta + \frac{620}{9\eta} + \dots\right\}.
\end{aligned}$$

We have from equations (89) and (100)

$$\xi_0 = \frac{1}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \frac{1000}{243\eta^5} + \dots$$

$$\xi_1^2 = 9\eta^4 - 20\eta^2 + \frac{100}{9} + \frac{2000}{9\eta^2} + \dots$$

and hence, by multiplication, we find

$$\xi_0\xi_1^2 = \frac{9}{2}\eta^5 + 5\eta^3 - \frac{100}{9}\eta + \frac{3500}{81\eta} + \dots$$

$$\text{Therefore } (\delta_0\xi_1)^2(3\xi_0) = \frac{675}{128}\eta^5 + \frac{375}{64}\eta^3 - \frac{625}{48}\eta + \frac{21875}{144\eta} + \dots$$

From equations (91) and (89) we have

$$\frac{180}{7}\xi_0^3 - \frac{2825}{28}\xi_0 = \frac{45}{14}\eta^3 - \frac{1025}{56}\eta - \frac{2125}{84\eta} + \frac{154375}{784\eta^3} + \dots$$

and from equation (93),

$$\delta_0\xi_1 = \frac{15}{8}\eta^2 - \frac{25}{12} + \frac{625}{27\eta^4} + \dots$$

By multiplication of these last two series,

$$\delta_0\xi_1\left(\frac{180}{7}\xi_0^2 - \frac{2825}{28}\xi_0\right) = \frac{675}{112}\eta^5 - \frac{2625}{64}\eta^3 - \frac{3125}{336}\eta + \frac{6250}{7\eta} + \dots$$

We have also

$$\begin{aligned}
\delta_0\delta_2\frac{48}{5}\xi_0\xi_1 &= \frac{6}{5}\delta_2\left\{\frac{1}{2}\eta + \frac{5}{3\eta} + \frac{50}{27\eta^3} + \dots\right\}\left\{3\eta^2 - \frac{10}{3} + 0\cdot\eta^{-2} + \dots\right\} \\
&= \delta_2\left\{\frac{9}{5}\eta^3 + 4\eta + 0\cdot\eta^{-1} + \dots\right\}.
\end{aligned}$$

The last series we require is ξ_0^5 . By multiplication of the series (91) and (92), we find

$$\xi_0^5 = \frac{1}{32}\eta^5 + \frac{25}{48}\eta^3 + \frac{875}{216}\eta + \frac{5000}{243\eta} + \dots$$

If we now collect the various series which have been obtained and substitute them in equation (102), we find, as the equivalent of this equation,

$$\begin{aligned}
& \left[\frac{3}{8}\eta + \frac{5}{6\eta} - \frac{25}{27\eta^3} - \frac{500}{243\eta^5} - \dots \right] \left[5e_5\eta^4 + 3e_3\eta^2 + e_1 - \frac{e_{-1}}{\eta^2} - \dots \right] \\
& + \delta_2 \left[\frac{9}{5}\eta^3 - 6\eta + 0 \cdot \eta^{-1} + \dots \right] \\
& = \delta_2 \left[\frac{9}{5}\eta^3 + 6\eta + \frac{620}{9\eta} \right] + \delta_2 \left[\frac{9}{5}\eta^3 + 4\eta + 0 \cdot \eta^{-1} + \dots \right] \\
& + \frac{2 \cdot 0 \cdot 2 \cdot 5}{5 \cdot 6} \eta^5 - \frac{1 \cdot 2 \cdot 7 \cdot 1 \cdot 2 \cdot 5}{8 \cdot 9 \cdot 6} \eta^3 + \frac{3 \cdot 1 \cdot 2 \cdot 5}{1 \cdot 4} \eta - \frac{78,125}{352\eta} + \dots \\
& + \frac{6 \cdot 7 \cdot 5}{1 \cdot 2 \cdot 8} \eta^5 + \frac{3 \cdot 7 \cdot 5}{6 \cdot 4} \eta^3 - \frac{6 \cdot 2 \cdot 5}{4 \cdot 8} \eta + \frac{21,875}{144\eta} + \dots \\
& + \frac{6 \cdot 7 \cdot 5}{1 \cdot 1 \cdot 2} \eta^5 - \frac{2 \cdot 6 \cdot 2 \cdot 5}{6 \cdot 4} \eta^3 - \frac{3 \cdot 1 \cdot 2 \cdot 5}{3 \cdot 3 \cdot 6} \eta + \frac{6250}{7\eta} + \dots \\
& + e_5 \left(\frac{3 \cdot 3}{3 \cdot 2} \eta^5 + \frac{2 \cdot 5}{4 \cdot 8} \eta^3 + \frac{8 \cdot 7 \cdot 5}{2 \cdot 1 \cdot 6} \eta + \frac{5000}{243\eta} \right) \\
& + e_3 \left(\frac{9}{8} \eta^3 + \frac{5}{4} \eta + \frac{50}{9\eta} + \dots \right) \\
& + e_1 \left(\frac{3}{2} \eta + \frac{5}{3\eta} + \dots \right)
\end{aligned}$$

Equating the coefficients of the various powers of η , we obtain

$$\frac{1 \cdot 5}{8} e_5 = \frac{6 \cdot 0 \cdot 7 \cdot 5}{1 \cdot 2 \cdot 8} + \frac{3 \cdot 3}{3 \cdot 2} e_5 \quad . \quad . \quad . \quad . \quad . \quad . \quad (106),$$

$$- \frac{2 \cdot 5}{6} e_5 + \frac{9}{8} e_3 - \frac{9}{2} \delta_2 = - \frac{1 \cdot 5 \cdot 8 \cdot 6 \cdot 2 \cdot 5}{8 \cdot 9 \cdot 6} + \frac{2 \cdot 5}{4 \cdot 8} e_5 + \frac{9}{8} e_3 \quad . \quad . \quad . \quad . \quad (107),$$

$$- \frac{1 \cdot 2 \cdot 7 \cdot 5}{2 \cdot 7} e_5 - \frac{5}{2} e_3 + \frac{3}{8} e_1 - 16 \delta_2 = \frac{5 \cdot 6 \cdot 2 \cdot 5}{2 \cdot 8} + \frac{8 \cdot 7 \cdot 5}{2 \cdot 1 \cdot 6} e_5 + \frac{5}{4} e_3 + \frac{3}{2} e_1 \quad . \quad . \quad (108),$$

$$- \frac{2 \cdot 5 \cdot 0 \cdot 0}{2 \cdot 4 \cdot 3} e_5 - \frac{2 \cdot 5}{9} e_3 - \frac{5}{6} e_1 - \frac{6 \cdot 2 \cdot 0}{9} \delta_2 - \frac{3}{8} e_{-1} = \frac{4 \cdot 4 \cdot 0 \cdot 6 \cdot 2 \cdot 5}{1 \cdot 0 \cdot 0 \cdot 8} + \frac{5 \cdot 0 \cdot 0 \cdot 0}{2 \cdot 4 \cdot 3} e_5 + \frac{5 \cdot 0}{9} e_3 + \frac{5}{3} e_1 \quad (109).$$

Solving the first two of these equations, we obtain

$$e_5 = \frac{2 \cdot 2 \cdot 5}{4} \quad . \quad . \quad . \quad . \quad (110), \quad \delta_2 = - \frac{8 \cdot 6 \cdot 2 \cdot 5}{4 \cdot 4 \cdot 8} \quad . \quad . \quad . \quad . \quad (111).$$

Equations (108) and (109) may be written

$$- \frac{6 \cdot 2 \cdot 5}{7 \cdot 2} e_5 - \frac{1 \cdot 5}{4} (e_3 + \frac{3}{10} e_1) - 16 \delta_2 = \frac{5 \cdot 6 \cdot 2 \cdot 5}{2 \cdot 8} \quad . \quad . \quad . \quad . \quad (112),$$

$$- \frac{2 \cdot 5 \cdot 0 \cdot 0}{8 \cdot 1} e_5 - \frac{2 \cdot 5}{3} (e_3 + \frac{3}{10} e_1) - \frac{6 \cdot 2 \cdot 0}{9} \delta_2 - \frac{3}{8} e_1 = \frac{4 \cdot 4 \cdot 0 \cdot 6 \cdot 2 \cdot 5}{1 \cdot 0 \cdot 0 \cdot 8} \quad . \quad . \quad . \quad (113).$$

Multiply (112) and (113) by 20 and 9 respectively, and subtract, and we obtain

$$\frac{6 \cdot 2 \cdot 5}{6} e_5 + 300 \delta_2 - \frac{1 \cdot 5}{2} e_{-1} = \frac{9 \cdot 3 \cdot 7 \cdot 5}{1 \cdot 1 \cdot 2} \quad . \quad . \quad . \quad . \quad . \quad (114).$$

From equations (110) and (111) we obtain

$$\frac{6 \cdot 2 \cdot 5}{6} e_5 + 300 \delta_2 = \frac{9 \cdot 3 \cdot 7 \cdot 5}{1 \cdot 1 \cdot 2},$$

and hence, by comparison with (114), $e_{-1} = 0$.

This vanishing of e_{-1} supplies a searching test of the accuracy of the work. Equation (112) becomes, after simplification,

$$e_3 + \frac{3}{10} e_1 = - \frac{1 \cdot 7 \cdot 0 \cdot 7 \cdot 5}{1 \cdot 6 \cdot 8} \quad . \quad . \quad . \quad . \quad . \quad (115).$$

Our equations do not enable us to determine e_3 and e_1 separately; they are, however, all satisfied by taking

$$e_3 = - \frac{1 \cdot 7 \cdot 0 \cdot 7 \cdot 5}{1 \cdot 6 \cdot 8} + \lambda \quad . \quad . \quad . \quad (116), \quad e_1 = - \frac{1 \cdot 0}{3} \lambda \quad . \quad . \quad . \quad (117)$$

where λ is unknown. We therefore have (equation 103)

$$\xi_3 = \frac{11.25}{4}\eta^4 - \frac{17.075}{56}\eta^2 + \lambda(3\eta^2 - \frac{1.0}{3}) + \text{terms in } \eta^{-4}, \eta^{-6}, \&c. \quad (118).$$

Using the value of δ_2 given by equation (111), we find as the values of d_4, d_2 , and d_0 (p. 91),

$$d_4 = \frac{4.5}{7}, \quad d_2 = -\frac{9.65}{14}, \quad d_0 = \frac{6.9875}{504}.$$

§ 24. Collecting the values of the various constants, we find as the equation to the surface (equation 70),

$$\begin{aligned} \xi\eta = & 1 + \frac{2}{3}(\xi^2 + \eta^2) + \{(\xi^3 + \eta^3) - \frac{1.0}{3}(\xi + \eta)\}(\theta + \lambda\theta^3) \\ & + \theta^2\{\frac{4.5}{7}(\xi^4 + \eta^4) - \frac{9.65}{14}(\xi^2 + \eta^2) + \frac{6.9875}{504}\} \\ & + \theta^3\{\frac{2.25}{4}(\xi^5 + \eta^5) - \frac{17.075}{168}(\xi^3 + \eta^3)\} + \text{terms in } \theta^4, \theta^5, \&c. \quad (119). \end{aligned}$$

The occurrence of the indeterminate quantity λ can easily be accounted for. For if we have a solution

$$\xi\eta = a^2 + \theta f_1 + \theta^2 f_2 + \theta^3 f_3 + \dots \quad (120),$$

corresponding to a parameter θ which is connected with the rotation by the relation

$$1 - \omega^2/2\pi\rho = \delta_0 + \delta_2\theta^2 + \delta_3\theta^3 + \dots \quad (121).$$

then we can obtain this same solution in another form by replacing the parameter θ by a new parameter $\theta + \lambda\theta^3$. As far as θ^3 this leaves the relation (121) between ω^2 and θ unaltered, whilst the equation to the surfaces as far as θ^3 becomes

$$\xi\eta = a^2 + (\theta + \lambda\theta^3)f_1 + \theta^2 f_2 + \theta^3 f_3 + \dots \quad (122).$$

It accordingly appears that in equation (119) the value of λ is entirely at our disposal. We shall therefore take $\lambda = 0$.

Investigation of Stability.

§ 25. There is a large *a priori* probability that the linear series we are now considering will be stable for small values of θ , but it will be well to rigorously examine the question.

It appears from POINCARÉ'S researches that the whole investigation reduces to determining whether the angular momentum is a maximum or a minimum at $\theta = 0$.* We therefore require to calculate the angular momentum as far as θ^2 , and the answer to our problem will depend upon the sign of the term containing θ^2 .

As far as θ^2 , the equation to the surface in polar co-ordinates (r, ϕ) is (equation (119))

$$\begin{aligned} r^2 = & 1 + \frac{4}{3}r^2 \cos 2\phi + 2\theta(r^3 \cos 3\phi - \frac{1.0}{3}r \cos \phi) \\ & + \theta^2(\frac{9.0}{7}r^4 \cos 4\phi - \frac{9.65}{7}r^2 \cos 2\phi + \frac{6.9875}{504}). \quad (123). \end{aligned}$$

The moment of inertia is given by

$$I = \iint r^2 dr r d\phi$$

* H. POINCARÉ, "Sur la Stabilité de l'Equilibre des Figures Pyriformes" 'Phil. Trans.' A. vol. 198, p. 333.

taken over the surface, and therefore by

$$I = \frac{1}{2} \int_0^\pi r^4 d\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124),$$

where r is given by equation (123).

Let us assume a solution as far as θ^2 of the form

$$r^4 = \alpha^4 (1 + \beta\theta + \gamma\theta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125),$$

where α , β , γ are functions of ϕ only, α being given by

$$\alpha^2 (1 - \frac{4}{5} \cos 2\phi) = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126).$$

The value of r^2 corresponding to solution (125) is

$$r^2 = \alpha^2 (1 + \frac{1}{2}\beta\theta + \theta^2 (\frac{1}{2}\gamma - \frac{1}{8}\beta^2)),$$

and if we substitute this value for r^2 , and the similar values for r and r^3 in equation (123), and equate the coefficients of θ and θ^2 , we obtain

$$\frac{1}{2}\beta = 2 (\alpha^3 \cos 3\phi - \frac{10}{3}\alpha \cos \phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127),$$

$$\frac{1}{2}\gamma - \frac{1}{8}\beta^2 = \frac{3}{2}\alpha^3\beta \cos 3\phi - \frac{5}{3}\alpha\beta \cos \phi + \frac{90}{7}\alpha^4 \cos 4\phi - \frac{965}{7}\alpha^2 \cos 2\phi + \frac{69875}{504},$$

whence, by elimination of β ,

$$\begin{aligned} \frac{1}{2}\gamma = 8 (\alpha^3 \cos 3\phi - \frac{10}{3}\alpha \cos \phi) (\alpha^3 \cos 3\phi - \frac{5}{3}\alpha \cos \phi) \\ + \frac{90}{7}\alpha^4 \cos 4\phi - \frac{965}{7}\alpha^2 \cos 2\phi + \frac{69875}{504}. \end{aligned}$$

We can eliminate ϕ from this equation by the help of equation (126). The resulting value for $\frac{1}{2}\gamma$ contains only even powers of α ; if we simplify this, and transform the numerical coefficients to decimals, we find

$$\frac{1}{2}\gamma = 68.1\alpha^6 - 319.8\alpha^4 + 264.4\alpha^2 + 159.4.$$

Now we require to find the coefficient of θ^2 in I (equation (124)), and this is $\frac{1}{2} \int_0^\pi \alpha^4 \gamma d\phi$. We therefore require to know the value of $\int_0^\pi \alpha^{2n} d\phi$ for $n = 2, 3, 4, 5$. This integral can easily be evaluated for all positive integral values of n ; the values which we require at present are as follows:—

$n = 2,$	$3,$	$4,$	$5,$
$\frac{1}{\pi} \int_0^\pi \alpha^{2n} d\phi = 4.6$	17.0	70.1	305.3

Substituting these values, we find at once that $\frac{1}{2} \int_0^\pi \alpha^4 \gamma d\phi$ is a positive quantity.

Thus the moment of inertia and the angular velocity both increase as θ^2 increases from the value $\theta = 0$. The moment of momentum is therefore a *minimum* for the value $\theta = 0$, and this proves the stability of the series of pear-shaped figures.

The Series of Pear-Shaped Curves.

§ 26. The equation to the curves of this linear series has already been calculated as far as θ^3 . In order to obtain a still better idea of the shape of the curves I have carried the calculation two degrees further. The calculation of these last two degrees is extremely heavy, and I have omitted all details in order to save space. The method is precisely similar to that which was followed in the calculations of §§ 20–23.

It was found that the coefficients multiplying terms in θ^4 and θ^5 were inconveniently large, and to obviate this, the parameter has been changed from θ to $10^{3.2}\theta$. After making this change we find, as far as θ^5 , for the equation to the surface expressed in polar co-ordinates, and for the equation determining ω^2 ,

$$\begin{aligned} r^2 = & (1 + \cdot 139\theta^2 + \cdot 023\theta^4 + \dots) - \cdot 211\theta r \cos \phi \\ & + (\cdot 8 - \cdot 138\theta^2 - \cdot 069\theta^4 + \dots) r^2 \cos 2\phi \\ & + (\cdot 063\theta - \cdot 0064\theta^3 - \cdot 0031\theta^5 \dots) r^3 \cos 3\phi \\ & + (\cdot 013\theta^2 + \cdot 0008\theta^4 + \dots) r^4 \cos 4\phi \\ & + (\cdot 0036\theta^3 + \cdot 00093\theta^5 \dots) r^5 \cos 5\phi \\ & + (\cdot 0011\theta^4 + \dots) r^6 \cos 6\phi \\ & + (\cdot 00043\theta^5 + \dots) r^7 \cos 7\phi + \&c. \dots \dots (128). \end{aligned}$$

$$1 - \omega^2/2\pi\rho = \cdot 625 - \cdot 0196^2 - \cdot 016\theta^4. \dots \dots (129).$$

§ 27. We must next consider within what limits the calculated terms of equation (128) will give a good approximation to the complete equation. It is clear that for given values of r and θ the worst approximation may be expected when $\phi = 0$. Let us therefore consider the function $\Phi(r, \theta)$, defined by

$$\begin{aligned} \Phi(r, \theta) = & (1 + \cdot 139\theta^2 + \cdot 023\theta^4 + \dots) - \cdot 211\theta r \\ & - (\cdot 2 + \cdot 138\theta^2 + \cdot 069\theta^4 + \dots) r^2 \\ & + (\cdot 063\theta - \cdot 0064\theta^3 - \cdot 0031\theta^5) r^3 \\ & + (\cdot 013\theta^2 + \cdot 0008\theta^4 + \dots) r^4 + (\cdot 0036\theta^3 + \cdot 00093\theta^5) r^5 \\ & + (\cdot 0011\theta^4 + \dots) r^6 + (\cdot 00043\theta^5 + \dots) r^7 + \&c. \dots (130). \end{aligned}$$

The value of $\Phi(r, \theta)$ is expressed by a doubly infinite series, of which only a few terms are known. When $r = 0$, $\theta = 0$, the value of Φ is known to be accurately equal to unity. For small values of r and θ , equation (130) will give Φ with considerable accuracy, but for larger values of r and θ , the terms calculated will be inadequate to give a good approximation to the value of Φ . What then, we

inquire, are the values of r and θ over which this approximation may be regarded as good?

The coefficient of each power of r is an infinite series of powers of θ , of which all terms up to θ^5 have been calculated. A glance at these series will show that the approximation is tolerably good so long as $\theta^2 < 1$, but begins to break down as soon as θ exceeds this unit value.

Supposing that we have assigned to θ some definite value less than unity, the value of $\Phi(r, \theta)$ will be given by an infinite series of powers of r , of which only the first seven are known. For small values of r these first few terms will give a sufficiently good approximation, for larger values the approximation will be bad.

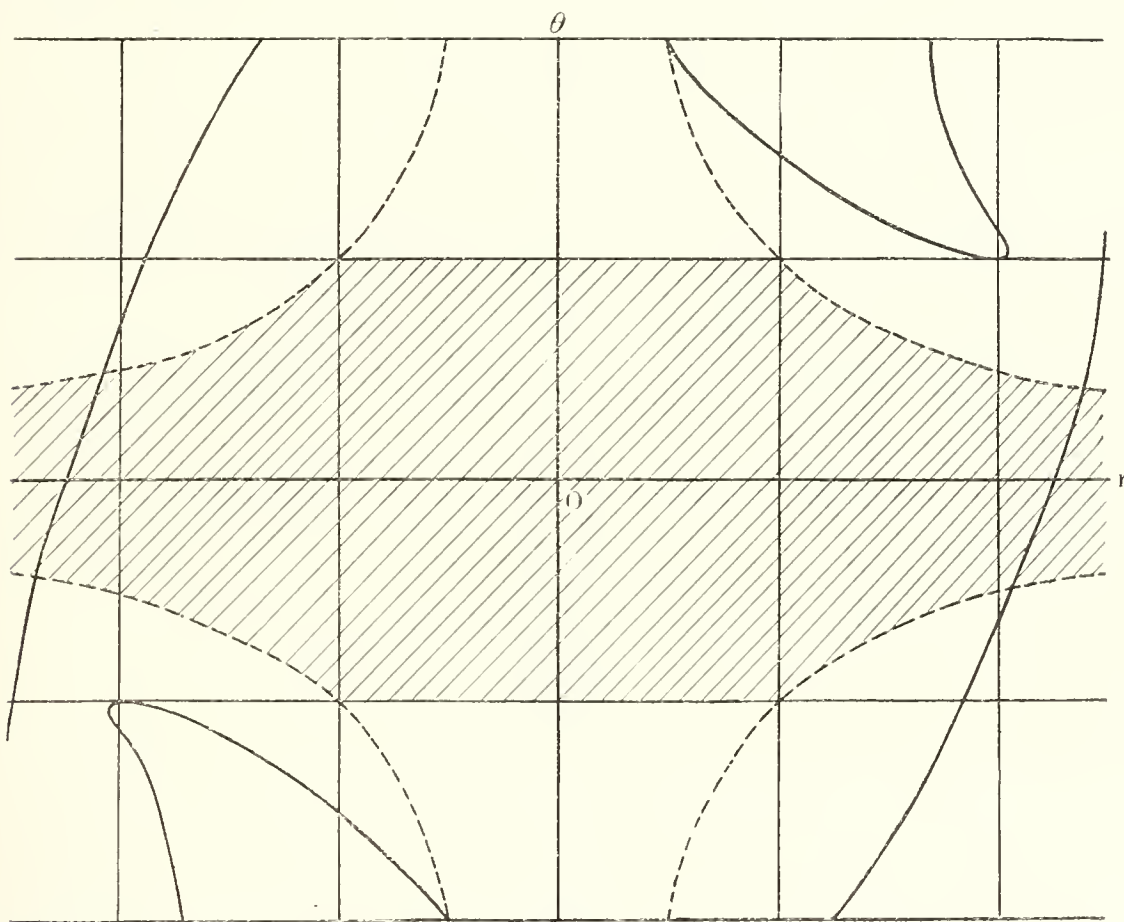


Fig. 3.

and for still greater values the series will become divergent, so that the first few terms give no approximation at all. It will be seen from inspection of equation (128) that the approximation will be tolerably good so long as $r^2 < 1/\theta^2$.

The conditions under which the calculated terms will give a good approximation may accordingly be supposed to be that $\theta^2 < 1$, and $r^2 < 1/\theta^2$. In fig. 3 is represented the plane of r, θ . The part of this plane over which the approximation is good is that bounded by the four curves

$$\theta = 1, \quad \theta = -1, \quad r\theta = 1, \quad r\theta = -1.$$

This is the portion which is shaded in the figure.

In this same figure the thick curves represent the locus,

$$\Phi(r, \theta) = 0,$$

calculated upon the supposition that the calculated terms of $\Phi(r, \theta)$ give a sufficiently good approximation to the whole. For the greater part of the curve this assumption is not justifiable, so that the curve requires adjustment, the amount of this adjustment increasing as we recede from the shaded portions of the plane. The most important points on the curve are those at which $d\theta/dr = 0$. These may with sufficient accuracy for our present purpose be taken to be $r = 2$, $\theta = 1$, and $r = -2$, $\theta = -1$.

§ 28. The points at which the axis $\phi = 0$ meets the curve of which the equation is (128) are given by

$$\Phi(r, \theta) = 0.$$

Fig. 3 accordingly enables us to trace the motion of these points as we move along the linear series, *i.e.*, as θ increases from zero upwards. At $\theta = 0$ we have, of course, two equal and opposite roots— $r = \pm\sqrt{5}$. As θ increases the positive root increases,

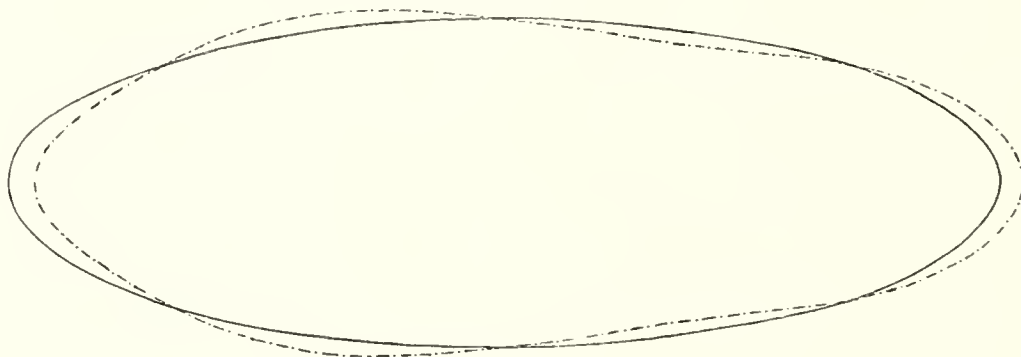


Fig. 4.

while the negative root numerically decreases. Remembering that the centre of gravity of the curve must remain at the origin, we see that this indicates a general thickening of the half of the curve in which $\phi > \pi/2$, with a diminution in the thickness of the forward half, and consequent lengthening of this half. These features become more marked as θ increases, until we reach the value $\theta = 1$, at which a new feature presents itself. For here there are two new roots occurring at the point $r = 2$. This indicates that the fluid separates into two portions when the value $\theta = 1$ is reached, the point of separation being $r = 2$ (approximately). We are at once struck by the great inequality in size between the primary and satellite: the former extending approximately from $r = -2$ to $r = +2$, and the latter only from $r = 2$ to $r = 2\frac{1}{2}$. The ratio of the linear dimensions will therefore be something like 8 to 1, but it must be remembered that our results require considerable correction on account of the imperfections in our approximations.

§ 29. In fig. 4 the thick curve is the elliptic cylinder corresponding to $\theta = 0$, and the dotted curve is the adjacent curve corresponding to a small value of θ ($\theta = \frac{1}{5}$).

In figs. 5 and 6 the curves are those corresponding to $\theta^2 = \frac{1}{2}$ and $\theta^2 = 1$. A glance at fig. 3 will show that there are difficulties in the way of drawing these latter curves with much accuracy. I have given in detail some of the calculations used in drawing the curve $\theta^2 = 1$, in order that the reader may judge for himself as to the closeness or otherwise of the approximations. The curve $\theta^2 = \frac{1}{2}$ is of course much easier. Before passing on to the calculations, two points ought to be noticed.

(i.) It will be noticed that in the various θ -series (the coefficients of powers of r in equation (128)) the terms last calculated are without exception of the same sign as those previously calculated. There is therefore some justification for hoping that the remainders in these series will be of the same sign as these last terms. If this is so, the error introduced by the neglect of these remainders could, to an appreciable extent, be reduced by an adjustment in the value of θ . Thus we shall be attempting to calculate the curve for (say) $\theta = 1$, and shall obtain a curve which is much more like the curve for some smaller value of θ (say $\theta = .98$) than it is like the curve $\theta = 1$. Regarded as an attempt at tracing a surface of equilibrium the error will be much smaller than if regarded as an attempt at tracing the particular curve $\theta = 1$.

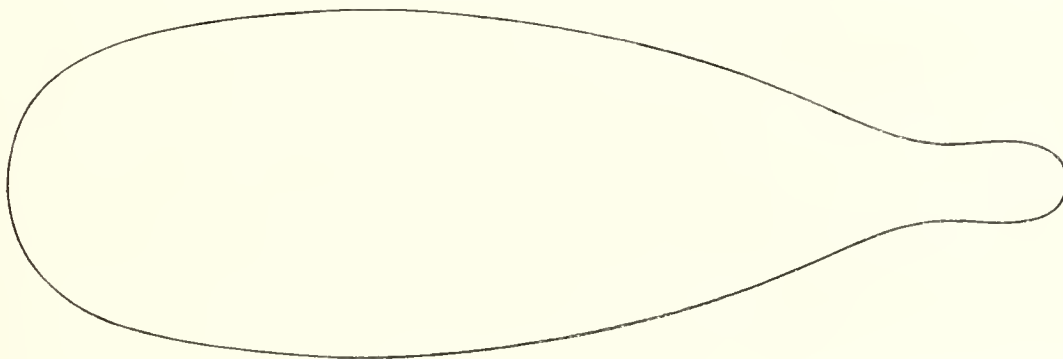


Fig. 5.

(ii.) It will be noticed that the sign of the leading terms in each of the series multiplying r^4, r^5, r^6, r^7 , is positive. An examination of the method by which these leading terms are calculated will show that this is a general law: all the leading terms after r^3 are of positive sign. Thus the error will be reduced by supposing the series (128) continued to higher powers of r by a suitably chosen series of terms. I have accordingly done this in the calculations, and the conjectural terms are, throughout, enclosed in square brackets.*

The Curve $\theta^2 = \frac{1}{2}$. (Fig. 5.)

§ 30. In tracing this curve I started from $\phi = \pi$, and calculated a series of points on the curve for decreasing values of ϕ . The approximation at $\phi = \pi$ was found to

* The effect of these corrections must, of course, be small; but, at the same time, it seems as well to make use of any definite knowledge that we possess.

be good, the root being 1.98, and the error occurring only in the third decimal place. As ϕ decreases the approximation improves, and at $\phi = \pi/2$ the error occurs only in the fifth decimal place. At $\phi = 7\frac{1}{2}^\circ$ the error again appears in the third place, and after this the approximation is bad. My plan was to calculate for smaller values of ϕ as well as I could, taking care to keep the values of r in defect rather than excess of their true values. The curve was then plotted out on paper ruled with squares of 1 millim., the unit of length being taken to be 50 millims.*

The area of the elliptic cylinder of fig. 4 is known to be

$$13,090 \text{ sq. millims.,}$$

and this would also have been the area of the present curve had it been accurately drawn. The area of the curve (obtained by counting squares) was, however, found to be

$$12,776 \text{ sq. millims.}$$

The moments about the axis $\phi = \pi/2$ of the two parts of the curve ($\phi > \pi/2$ and

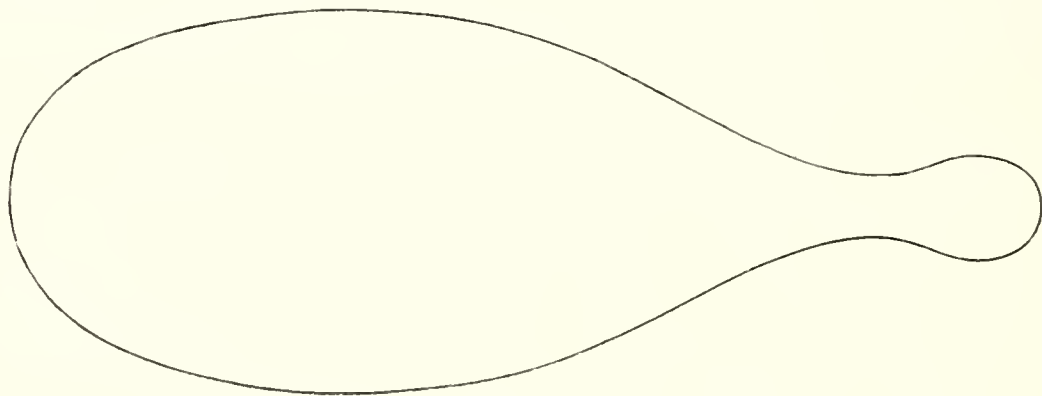


Fig. 6.

$\phi < \pi/2$) ought of course to be equal: these were found to be respectively

$$298,290 \text{ cub. millims. and } 302,850 \text{ cub. millims.}$$

It was therefore obvious that the curve had been too much shortened in the region in which $\phi < 7\frac{1}{2}^\circ$.

Readjusting the curve in this region so as to divide the error as equally as possible, I arrived at a curve with outstanding errors in area and moment of

$$139 \text{ sq. millims. and } -139 \text{ sq. millims. at } r = 125 \text{ millims., } \phi = 0.$$

This is the curve given in fig. 5. It will be seen that the error is one of about 1 per cent.

Calculation of the Curve $\theta = 1$. (Fig. 6.)

§ 31. The equation of the curve is found to be

$$\begin{aligned} r^2 = & 1.162 - .211r \cos \phi + .593r^2 \cos 2\phi + .053r^3 \cos 3\phi + .014r^4 \cos 4\phi \\ & + .0045r^5 \cos 5\phi + .0011r^6 \cos 6\phi + .00043r^7 \cos 7\phi \\ & + [.0002r^8 \cos 8\phi + .0001r^9 \cos 9\phi + \dots] \end{aligned}$$

the terms in square brackets being those mentioned at the end of § 29.

* This has been photographically reduced to 25 millims. before printing.

We calculate in succession the value of the radius vector when $\phi = 180^\circ, 150^\circ, 120^\circ, 90^\circ, 60^\circ, 45^\circ$, &c. The value of r is a root of $\Phi(r) = 0$, where

$$\Phi(r) = -r^2 + 1.162 - .211r \cos \phi + \&c.$$

Our method will be to find a value of r for which $\Phi(r)$ is small, and calculate $\Phi(r)$ and $d\Phi/dr$ at this point. The true root R is then given by

$$R = r - \frac{\Phi(r)}{d\Phi(r)/dr}.$$

$$\text{When } \phi = \pi, \quad \Phi = 1.162 + .211r - .407r^2 - .053r^3 + .014r^4 - .0045r^5 + .0011r^6 \\ - .00043r^7 + [.0002r^8 - .0001r^9 + \dots].$$

$$\text{When } r = 1.8, \quad \Phi = 1.162 + .380 - 1.319 - .310 + .147 - .085 + .038 - .026 \\ + [.020 - .018 + \dots] = -.003 + [.01],$$

$$d\Phi/dr = .21 - 1.4 - .5 + .3 - .2 + .1 - \dots = -1.4,$$

$$\text{therefore} \quad R = 1.8 - .002 + [.007], \quad \text{or,} \quad R = 1.805.$$

$$\text{When } \phi = 5\pi/6, \quad \Phi = 1.162 + .182r - .704r^2 - .007r^4 + .0039r^5 - .0011r^6 \\ + .00036r^7 - [.0001r^8 + .0000r^{10} + \dots].$$

$$\text{When } r = \sqrt{2} = 1.414, \quad \Phi = 1.162 + .257 - 1.408 - .028 + .022 - .009 + .004 - [.001] \\ = .000 - [.001],$$

$$\text{therefore} \quad R = 1.414.$$

$$\text{When } \phi = \pi/2, \quad \Phi = 1.162 - 1.593r^2 + .014r^4 - .0011r^6 + [.0002r^8] + \dots$$

$$\text{When } r^2 = .74, \quad r = .859;$$

$$\Phi = 1.162 - 1.179 + .007 - .0003 + [.0000] + \dots = -.010;$$

$$d\Phi/dr = -2.5; \text{ therefore } R = .859 - .004 = .855.$$

$$\text{When } \phi = 7\frac{1}{2}^\circ, \quad \Phi = 1.162 - .209r - .427r^2 + .046r^3 + .012r^4 + .0036r^5 + .0008r^6 \\ + .0003r^7 + [.0001r^8 + .00004r^9 + \dots].$$

$$\text{When } r^2 = 3, \quad r = 1.732, \quad \Phi = 1.162 - .348 - 1.281 + .241 + .108 + .056 + .021 \\ + .014 + [.008 + .005 + \dots] = -.027 + [.01].$$

$$d\Phi/dr = -.209 - 1.44 + .42 + .23 + .17 + .08 + \dots = -.75; \text{ therefore} \\ R = 1.732 - .036 + [.01] = 1.71.$$

$$\text{When } \phi = 4^\circ, \quad \Phi = 1.162 - .211r - .410r^2 + .052r^3 + .014r^4 + .004r^5 + .001r^6 \\ + .0004r^7 + [.00017r^8 + .00008r^9 + \dots].$$

$$\text{When } r = 2, \quad \Phi = 1.162 - .422 - 1.640 + .416 + .224 + .128 + .064 + .051 \\ + [.05 + .04 + \dots] = -.02 + [.1?].$$

$$d\Phi/dr = -.211 - 1.640 + .624 + .448 + .320 + .192 + .178 + [.18 + .18 + \dots] \\ = .01 + [?].$$

$$\text{When } r = 1.8, \quad \Phi = 1.162 - .390 - 1.328 + .297 + .147 + .076 + .034 + .023 \\ + [.017 + .014 + \dots] = .02 + [.05].$$

$$d\Phi/dr = -.211 - 1.47 + .49 + .33 + .21 + .11 + .09 + [.08 + .07 + \dots] \\ = -.45 + [.2].$$

There is therefore a root at about $r = 1.95$, but it is clear that we are already in the region at which the approximation ceases to be satisfactory, and that we are close upon the region in which the series become actually divergent.

The smallness of $d\Phi/dr$ indicates that somewhere in the neighbourhood of the point just calculated we come to a point at which there is a pair of equal roots for r , and therefore a “minimum” in the value of ϕ . This is the “neck” of which the first signs are apparent in fig. 5. Let us refer to all the matter to the left of this “neck” as the “primary,” to all that to the right as the “satellite.” Let the exact line of division be a vertical line at a distance 2 from the origin.

The primary has been drawn with fair accuracy ; the satellite must be drawn in the manner adopted in the difficult region of the former curve.

The area of the primary was found to be

11778 sq. millims.,

and this leaves 1312 sq. millims. to be accounted for by the satellite and the error in drawing. The centre of gravity of the primary was found to be 4.7 millims. to the left of the origin. Distributing the error as equally as possible, I have arrived at the curve of fig. 6. The area of primary and satellite are respectively

11778 and 881 sq. millims.,

the error in area is a defect of 431 sq. millims., and that in the moment about $\phi = \pi/2$ is that of an excess of 431 sq. millims. at the point $r = 125$ millims., $\phi = 0$. The error in the whole curve is therefore about 3 per cent.; that in the satellite is unfortunately of the same order of magnitude as the satellite itself.

§ 32. The following table sums up the results which have been obtained, and also contains some new results. The moments of momentum of the last curves were obtained by a process of counting on squared paper, and are not carried to any great accuracy.

	Curve.		Area.	$\frac{\omega^2}{2\pi\rho}$.	Angular momentum (omitting factor $\sqrt{\frac{2\rho^3}{\pi}}$).
(1)	Circle	—	1	0	0
(2)	„	—	1	.43	.33
(3)	„	Point of bifurcation . .	1	.5	.35
(4)	Ellipse	Ratio of axes $\sqrt{5} : 1$. .	1	.43	.44
(5)	„ fig. 4 . .	Point of bifurcation : ratio of axes 3 : 1	1	.375	.51
(6)	Fig. 5	$\theta^2 = \frac{1}{2}$	1	.39	.53
(7)	Fig. 6	$\theta^2 = 1$	1	.42 ?	.55 ?
(8)	„	Separation of fluid into primary and satellite	1	.43 ?	.57 ?
(9)	} The two parts { of curve (8) {	Primary	.93 ?	.43 ?	.40 ?
(10)		Satellite	.07 ??	.43 ?	.17 ??

In this table the quantities of which the numerical values are doubtful are marked with a query. A single query indicates a probable error of 1 or 2 per cent.; a double query indicates that the error may be comparable with the quantity itself.

Let us examine the state of things just after separation has taken place. The satellite is describing an orbit about the primary, both bodies rotating with the same angular velocity. This angular velocity is, to within a few per cent., given by

$$\omega^2/2\pi\rho = \cdot43 \dots \dots \dots (131).$$

If the satellite exerted no attraction upon the primary, the figure of the primary would be a figure of equilibrium under the influence of a rotation given by (131). The force exerted by the satellite may be divided into two parts, a uniform force in the direction $\theta = 0$, and a tide-generating force of the usual kind. If the former of these existed alone, the configuration of the primary would still be one of equilibrium under a rotation of amount given by (131). We therefore see that the actual configuration of the primary may be regarded as a configuration of equilibrium under rotation given by (131), disturbed by the tide-generating potential which is caused by the satellite.

Since this tide-generating potential is small, except in the immediate neighbourhood of the satellite, it ought to be possible to remove the tides from the surface of the primary, and form a pretty good idea of the configuration which would be the configuration of the primary except for tidal disturbance. If this is done with the primary of fig. 6, it will be found that the remaining curve is a very good ellipse. We may therefore conjecture that curve (9) is an ellipse deformed by the tidal influence of its satellite.

Now the ellipse corresponding to the amount of rotation given by (131) is curve 4 of the preceding table. We see that the axes are in the ratio $\sqrt{5}:1$, and this is in good agreement with the ellipse obtained by removing the tides in fig. 6. The momentum of the ellipse of unit area of which the axes are in the ratio $\sqrt{5}:1$ (curve 4) is $\cdot44$. If we reduce this so as to apply to an ellipse of area $\cdot93$ instead of to one of unit area, we find an angular momentum of $\cdot38$. Since this ellipse must be supposed to rotate not about its centre, but about the centre of gravity of itself and a satellite about one-fifteenth of its mass, situated at the end of its axis, this angular momentum must be increased to about $\cdot39$. The small discrepancy between this and the value $\cdot40$ obtained for curve 9 may be accounted for partly by errors of approximation, and partly by the increase of momentum caused by the tidal deformation of the ellipse.

We can check our result in another way. The equation of the ellipse being

$$ax^2 + by^2 = 1 \dots \dots \dots (132),$$

the force at $x, 0$, a point near the extremity of the major axis and outside the ellipse,

may, to a good approximation, be taken to be $\frac{2\omega\rho}{\sqrt{a} + \sqrt{b}} \frac{2}{\sqrt{a \cdot x}}$, and if $x, 0$ is the centre of gravity of the satellite, this must be equal to

$$\cdot 93\omega^2 x,$$

the factor $\cdot 93$ being introduced to allow for the displacement of the centre of gravity of the primary.

Taking $\sqrt{b} = \sqrt{(5a)}$, we find the equation

$$\frac{2}{1 + \sqrt{5}} \frac{1}{ax^2} = \cdot 93 \frac{\omega^2}{2\pi\rho},$$

and putting $\omega^2/2\pi\rho = \cdot 43$, this gives the values

$$ax^2 = 1\cdot 54, \quad \sqrt{a \cdot x} = 1\cdot 24.$$

Now $\sqrt{a \cdot x}$ is the sum of the semi-axes of primary and satellite divided by the semi-axes of the primary. The equation just found is therefore about as true as could be expected, the linear diameters of primary and satellite being approximately in the ratio of 4 to 1.

The ellipse given as curve 4 is stable, and, since the mass of the satellite is small compared with that of the primary, we may suppose the combination of primary and satellite to be stable. Thus, if our conjecture as to the interpretation of curve 8 is sound, it appears that the linear series which commences with curve 5 remains stable until the mass separates into two masses.

The motion of a gradually-cooling mass will therefore be through the following cycle of changes. Firstly, increase of the ratio [angular momentum $\div$ (area)²] until we reach curve 5. Then motion along POINCARÉ'S linear series until we reach curve 9. At this point separation takes place, and the primary is left as a tidally-distorted form of curve 4. As the satellite recedes the tidal distortion decreases, and as the value of the ratio [angular momentum $\div$ (area)²] again increases, the configuration moves along the Jacobian series of elliptic cylinders until curve 5 is again reached. This completes the cycle, and the continual repetition of the cycle can only be ended by solidification, or some similar cause which is outside our present considerations.

I have not attempted to give any discussion of results from the point of view of dynamical astronomy. The complications introduced by the heterogeneity and compressibility of natural substances, as well as by the difference between the two-dimensional and three-dimensional problems, are so great that any discussion with reference to the actual conditions of astronomy would be impossible in the present paper.

I have had the advantage of frequent conversations with Professor DARWIN on the subject of this paper; my thanks are also due to Professor FORSYTH for advice in connection with the earlier sections.

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constant Volume and at different initial pressures.

Hydrogen and Helium gases
rises below its boiling point
Part II.—On the Vapor Pressure of Liquid Oxygen and Nitrogen

Hydrogen and Helium Series
pertains below its B-branch on the ground-state
Part III—On the vapor pressure of liquid hydrogen at low

the first is a σ -algebra on Σ and the second is a σ -algebra on $\mathcal{A}$.

Temperature.

Part I.—On the Pressure Coefficient of Hydrogen and Helium at Constant Volume and at Different Initial Pressures.

Hydrogen and Helium gases
 rises below its Boiling-Point in the presence of Liquid
 Part II.—On the Vapor Pressure of Liquid Helium and Liquid

Part III.—On the Vapor Pressure of Liquid Water at Temperatures below its Boiling Point on the Celsius Scale.
Hydrogen and Helium Gases.

021 601 97, 2001 02 10, A. J. Smith, 1999

Temperature.

Part I.—On the Pressure Coefficients of Hydrogen and Helium at Constant Volume and at different Initial Pressures.

Hydrogen and Helium gases.

part III.—On the Vapour Pressure of Liquid Hydrogen at Temperatures below its Boiling-Point on the Celsius and Fahrenheit Scales.

$$\log(-\tilde{G})^T = \log(-G)^T + \log(1 - \frac{1}{2} \log 2) = \log(-G)^T - 0.17329$$

...the effect of temperature.

Blüh. Zeit: A. vol. 200. 1906. p. 127. 130.
LEWERS, M. H. S. *Blüh. Zeit: A. vol. 200. 1906. p. 127. 130.*

Point; Thermometry.
Helmholtz Pressure Coefficient of: Expansion of Gases; Heat of and Boiling

Phil Brown / 1-800-755-2500

Point, Vapour Pressure; Thermometer
Hydrogen Pressure Coefficient of Diffusion of Hydrogen Gas

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FRANKLIN, M. H., "EFFECT OF TEMPERATURE ON THE VISCOSITY OF SOLID POLYMER SOLUTIONS."

III. *On the Measurement of Temperature. Part I.—On the Pressure Coefficients of Hydrogen and Helium at Constant Volume and at different Initial Pressures. Part II.—On the Vapour Pressures of Liquid Oxygen at Temperatures below its Boiling-Point on the Constant-Volume Hydrogen and Helium Scales. Part III.—On the Vapour Pressures of Liquid Hydrogen at Temperatures below its Boiling-Point on the Constant-Volume Hydrogen and Helium Scales.*

By MORRIS W. TRAVERS, *D.Sc., Fellow of University College, London,*
GEORGE SENTER, *B.Sc.,* and ADRIEN JAQUEROD, *D.Sc.*

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[PLATE 1.]

PART I.

On the Pressure Coefficients of Hydrogen and Helium at Constant Volume and at different Initial Pressures.

By MORRIS W. TRAVERS, *D.Sc., Fellow of University College, London,*
and ADRIEN JAQUEROD, *D.Sc.*

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1. *Introduction.*

CONSIDERING how much depends upon our knowledge of the relationship between the thermodynamic scale of temperature and the various scales of temperature as determined by means of gas thermometers, it is surprising that so little attention has been paid to the matter in recent years. Indeed, instead of attempting to arrive at the necessary experimental data for the solution of the problem, it appears as if chemists and physicists were prepared to accept arbitrary scales of temperature, which differ considerably one from another.

Since the problem was first attacked by JOULE and Lord KELVIN, more than half a century ago, it has been made the subject of investigation by more than one mathematician. Recently ('Phil. Mag.,' 1901, vol. 2, p. 130) J. ROSE-INNES has succeeded in developing some fairly simple equations by means of which the difference between the thermo-dynamic scale of temperature and the temperature determined by means of gas thermometers can be calculated. The data necessary in the case of each gas are the following:—

- (a.) The coefficient of increase of pressure at constant volume.
- (b.) The coefficient of increase of volume at constant pressure.
- (c.) The variation of "p.v." with pressure at different temperature.
- (d.) The Joule-Thomson effect and its variation with temperature.

Applying existing data to his equations, ROSE-INNES arrived at results which, as the following table shows, are anything but satisfactory:—

Correction of the Constant-Volume Thermometer.

Gas.	Pressure on gas at ice-point.	Temperature of ice- point (reciprocal of pressure coefficient).	Correction.	Absolute ice- point.
	millims.			
Hydrogen	1000	273·04	0·12	273·16
Nitrogen	995	272·13	1·23	273·36

Correction of the Constant-Pressure Thermometer.

Gas.	Temperature of ice-point (reciprocal of volume coefficient).	Correction.	Absolute ice-point.
Hydrogen	273·13	- 0·13	273·00
Nitrogen	273·44	0·72	273·16

Of the data employed in making the foregoing calculation only the pressure coefficients of hydrogen and nitrogen, which have been determined by P. CHAPPUIS, of the "Bureau International des Poids et Mesures," and the variation of "p.v." with pressure at the normal temperature, which has been recently investigated by ONNES ('Leiden Laboratory Reports,' 1901) in the case of hydrogen, can be considered as being known with sufficient certainty. AMAGAT has determined the variations of "p.v." for nitrogen, but the experiment was made at high pressure and not at the pressure under which thermometric measurements are usually made. In view of the fact that both air and oxygen appear to obey BOYLE'S law at pressures below 150 millims. of mercury (RAYLEIGH, 'Phil. Trans.,' A, vol. 198, p. 417), extrapolation of AMAGAT'S results at high pressure does not seem justifiable; and to obtain the data necessary for the correction of the gas thermometer the variations of "p.v." should be carefully investigated at pressures below two atmospheres. REGNAULT'S values for the volume coefficients of nitrogen and hydrogen are not to be relied upon.

The Joule-Thomson effect for air and hydrogen was determined by JOULE and Lord KELVIN more than half a century ago. Without discussing their results, it may be pointed out that by employing electrical methods for the measurement of temperature, it should now be possible to measure the effect with a much higher degree of accuracy, and over a wider range of temperature than was then possible. An investigation in this direction has been commenced by Dr. R. A. LEHFELDT conjointly with one of us.

2. *Suitable Thermometric Substances.*

For the reason that the composition of atmospheric air varies with the prevailing condition, air is not a suitable substance for use in thermometry. Atmospheric nitrogen appears, however, to contain a constant quantity of argon and its companions, and may be employed in thermometry over a range, limited, perhaps, by the temperatures at which it condenses on the one hand and dissociates on the other.

Until recently, hydrogen appeared to be the most important thermometric substance.

The constant-volume hydrogen scale appeared to approximate closely to the absolute scale of temperature, and the gas could be employed in any measurements to which thermometers could be applied. For this reason the Comité International adopted as the normal scale of temperature the scale of a constant-volume hydrogen thermometer, in which the pressure at the ice-point was 1000 millims. of mercury.

Recently, however, we have succeeded in reaching temperatures at which the vapour pressure of hydrogen itself is extremely small, and which cannot be investigated by means of the hydrogen thermometer with any degree of accuracy (see p. 179). Further, it has now been ascertained that at high temperatures hydrogen will reduce glass, porcelain, and even silica; and as the gas diffuses readily through the walls of platinum vessels, its application in this direction must be considered limited.

It has been pointed out by one of us ("Experimental Study of Gases," p. 156) that pure helium appears to be a much more perfect thermometric substance than hydrogen. It is chemically inactive; and, so far as we can predict, is incapable of undergoing dissociation at high temperatures. Hence it is probable that measurements of high temperatures, made by means of quartz thermometers filled with helium, are alone to be relied upon. For the measurement of low temperatures it presents similar advantages; for as it appears probable that its critical point lies below 12° abs., it may be considered as remaining a very perfect gas down to the lowest temperatures which it has hitherto been possible to reach.

Further, as the following results show, the hydrogen and helium scales agree closely between 0° and 100° C., and consequently it may be assumed that at temperatures above 0° C. the difference between the thermo-dynamic scale of temperature and the scale of a constant-volume thermometer filled with one of these gases is small. At lower temperatures the divergence between the two scales is small, but the contraction of the helium being always less than that of the hydrogen indicates that the former remains the more perfect gas. Unless some hitherto unknown objection is attached to the use of the helium thermometer, the helium scale should replace the hydrogen scale as the normal scale of temperature.

3. *Previous Measurements of the Pressure Coefficient of Hydrogen and Helium.*

The first step in the series of researches on which we have embarked consisted in the determination of the pressure coefficients at constant volume of hydrogen and helium between the melting-point of ice and the boiling-point of water under standard pressure, which, in the basement of University College, is equivalent to 759.56 millims. of mercury at 0° C. The coefficient for hydrogen, at a pressure of 100 millims. of mercury at the ice-point, is given by CHAPPUIS ("Travaux et Mémoires du Bureau International des Poids et Mesures") as 0.00366254, the mean of seven determinations, of which the highest is 0.00366271 and the lowest 0.00366231, and on this deter-

mination the normal scale of temperature of the Comité International is based. Kammerlingh ONNES ('Leiden Communications,' 1901, No. 60) gives 0.0036627 as the value of the coefficient, the actual numbers being 0.0036624, 0.0036628, 0.0036628. The result obtained by REGNAULT, 0.0036613, need not now be considered.

It has hitherto been assumed that the pressure coefficient for hydrogen is independent of the initial pressure, and does not, as in the case of nitrogen, decrease and at length attain a limiting value. As important deductions have been based on this statement, we decided to measure the coefficient for our standard gases at initial pressures corresponding to 350, 500, and 700 millims. of mercury.

An approximate measurement of the pressure coefficient of helium was made by KUENEN and RANDALL in 1895 ('Phil. Mag.'). In employing the helium thermometer for the measurement of low temperatures, OLSZEWSKI assumed that the coefficient is the same as that of hydrogen.

4. Accuracy of the Results.

In order to determine the fifth significant figure in the pressure coefficient, the total error must not exceed 1 part in 40,000, a degree of accuracy which it is practically impossible to attain in individual experiments. As will be seen later, the greatest error in our observations may be attributed to inaccuracy in the scale which, over the whole length, may be considered accurate to at least 0.01 millim., or to $1/25,000$ of the increase of pressure on the gas between 0° and 100° C. at an initial pressure of 700 millims.

5. Apparatus employed in the Research. The Barometer.

Since at the boiling-point of water a change of pressure of 0.1 millim. of mercury corresponds to a change of temperature of 0.0036° C., or $1/20,000$ of the difference between the boiling and freezing-points of water, it was sufficient to observe the atmospheric pressure with this degree of accuracy. The barometer employed is shown in fig. 1; it was constructed by us specially for this research. The upper chamber A was 20 millims. in diameter and 200 millims. in length. It was sealed to the stem B, which was drawn out at the lower end C to a capillary tube of an internal diameter of about 1 millim. In filling the barometer, the capillary end of the tube B was sealed to a T tube (fig. 2), of which the branch D led to the mercury pump, and the branch C projected

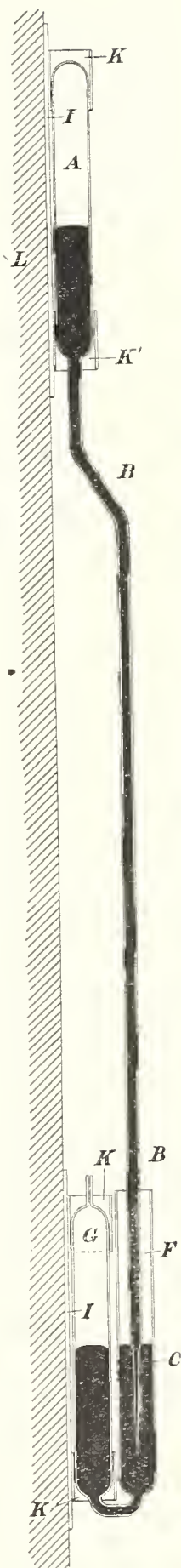


Fig. 1.

downward and was drawn to a fine point, which was sealed. The portions A and B of the apparatus were heated, and, when the apparatus was thoroughly exhausted, the point of the tube E was broken below the surface of some warm mercury in a basin. The mercury entered slowly, and, passing through the constriction C, ran

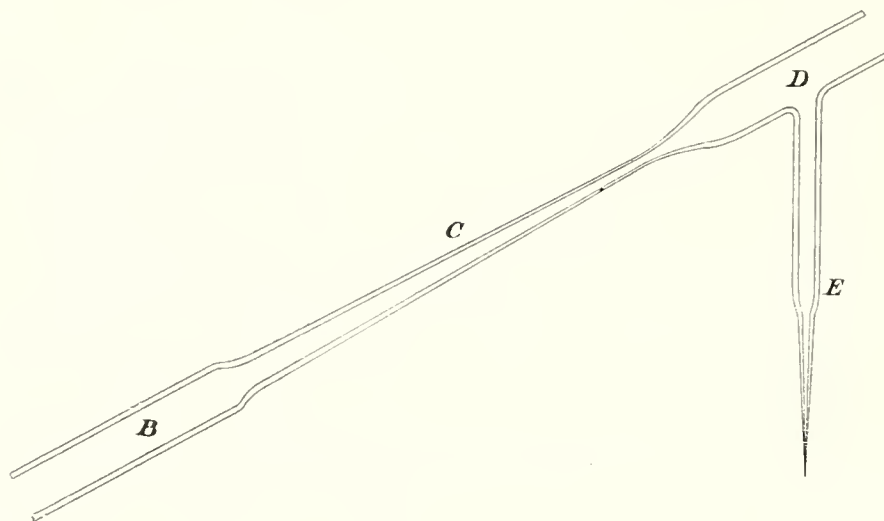


Fig. 2.

down into A, which was heated sufficiently to boil the liquid. When the apparatus was completely filled, the tube was cut through the constriction C, and the small bubble of air which remained in B was replaced by mercury.

Both the upper chamber and the limb of the U tube which formed the lower chamber were set by means of plaster of Paris into brass mounts KK', which were soldered to plates II', shown in section in the drawing. In the case of the upper chamber this was done after it had been filled with mercury. In setting up the barometer the U tube FG was first fixed to the piece of wood L which supported the instrument, and was then filled with mercury. The upper portion was then inverted, the open end plunged into the limb F of the U tube, and the brass plate I screwed to wooden support L. A dish was placed below the apparatus to catch the mercury overflowing from F; finally some of the mercury in the U tube was removed by means of a syphon, so that the surface of the mercury in the upper and lower chambers appeared as in the figure.

The scale, which was placed in front of the barometer, consisted of a piece of plate-glass to which were joined, by means of Canada balsam, two short scales 100 millims. in length and divided in millimetres. To prevent them shifting, through the flowing of the cement, they were further fixed by a binding of copper-wire covered with plaster of Paris at the top and bottom. The scale was compared with a standard scale from the Physikalische Reichsanstalt at Charlottenburg, and afterwards at the National Physical Laboratory with a standard metre (p. 119). The temperature of the column of mercury was measured by means of a thermometer divided to $\frac{1}{20}^{\circ}$ C. placed half-way between the top and bottom.

Readings of the barometer were made by means of two telescopes placed on a stand opposite the instrument. The barometer itself was attached to a pillar so that a card placed behind (fig. 3) it at an angle of 45° to the line of sight gave a good illumination from a window at the side. The upper half of the card was black, the lower half white, and before taking a reading the card was adjusted so that the dividing line appeared about 1 millim. above the surface of the mercury. By this arrangement the meniscus appeared perfectly sharp, as all surface reflections were completely eliminated.

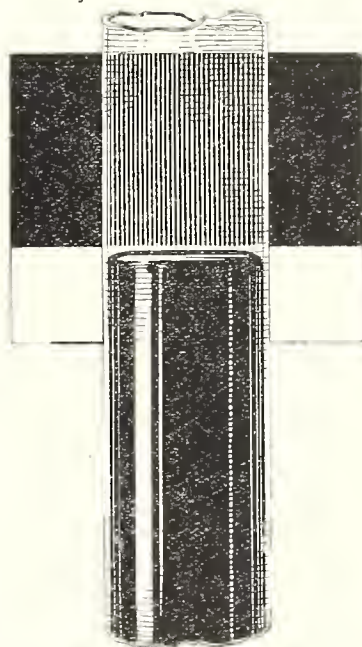


Fig. 3.

Mercury Thermometers.—The mercury thermometers employed in these observations were compared with the thermometer employed to measure the temperature of the dead-space. This thermometer was itself compared with the gas thermometer and the error over the small range of temperature required, viz., 8° to 12° C., was known to within 0.02° . The weight of such an error will be discussed later.

6. *The Constant-Volume Thermometer.*

The apparatus employed in the determination of the pressure coefficients of the gases differs in many respects from that employed by other observers. CHAPPUIS employed thermometers of hard glass and of platino-iridium, and ONNES bulbs of hard glass; in both cases the connection between the thermometer bulb and the dead-space was made by means of a steel tube connected with the glass by means of cement.

Beyond the fact that hard glass has a slightly lower coefficient of expansion than soda glass, there is no advantage in employing it; on the other hand, soda glass is more easily worked, and as by using it the bulb, stem, and dead-space can all be made in one piece, considerable advantage is to be gained. The bulb of our instrument (fig. 4) was made by sealing a piece of glass tube 30 millims. in diameter to a capillary tube 0.75 millim. in internal diameter, which for convenience in calibration was divided in millimetres for a portion of its length. The stem was bent at right angles at C (fig. 4), and ultimately sealed at D to the portion of the stem leading to the dead-space.

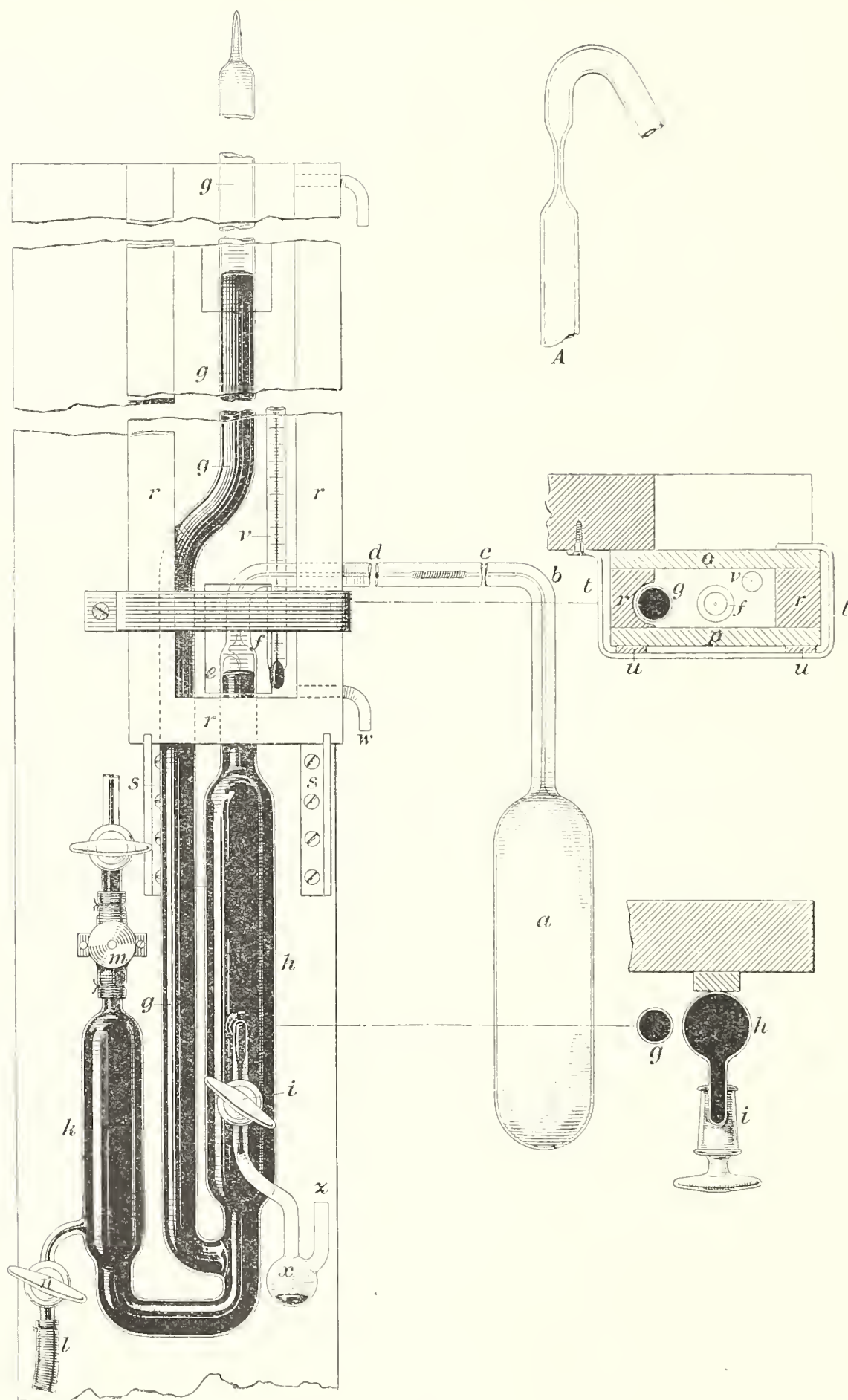


Fig. 4.

7. *The Calibration of the Bulb.*

The volume of the bulb was determined by weighing it empty, and afterwards filled with water at the melting-point of ice. A glass tube was connected with the stem by a piece of rubber tube to prevent loss of water when the temperature rose. The following are the results :—

	(a.)	(b.)
Weight of bulb empty	71·9551	71·9568
„ „ „ filled with water at 0° C. to division 126 on stem	170·6657	170·6657
Weight of catch water	4·5856	4·5860
„ „ water	94·1230	
„ „ „ corrected for weight of air in bulb, &c.	94·1975	
Volume of bulb to division 70 at 0° C.	94·2096	

The coefficient of expansion of the glass of which the thermometer bulb was made was determined by the weight-thermometer method. The weight-thermometer was made from the same glass tube as the thermometer bulb. The value for the coefficient was found to be 0·0000285.

The composition of the glass employed, which was obtained from Messrs. C. E. MÜLLER & Co., of High Holborn, appears to be very constant; for the coefficients of expansion of samples of it determined by KUENEN and RANDALL (*loc. cit.*, p. 109) in 1896, and by one of us in 1901, have the same value as that last determined.

The change of volume of the bulb due to change of the pressure on the gas inside it was also measured. It was determined by filling the bulb with water to a point on the stem and measuring the change in the position of the meniscus when the pressure was reduced by means of a water pump. The change of volume amounted to 0·0017 cub. centim. per atmosphere.

8. *The Dead-Space.*

Both CHAPPUIS and ONNES employed a glass tube with a steel cap attached to it by cement, and enclosing a steel point for the adjustment of the mercury meniscus. In the construction of our instrument we have entirely dispensed with steel and cement connection, and have employed an opaque glass point *e* sealed into the tube *f*, which formed the dead-space at its point of junction with the stem. The tube for the dead-space was cut from a piece of glass tube of 9 millims. internal diameter, selected for its straightness and regularity of bore; the manometer tube *g* was cut from the same piece. In making the junction with the capillary tube great care was

taken to heat only about 3 millims. of the wider tube, so as not to disturb the regularity of its surface at the level at which the mercury meniscus was afterwards observed. The operation of sealing the glass point into the tube was not an easy one.

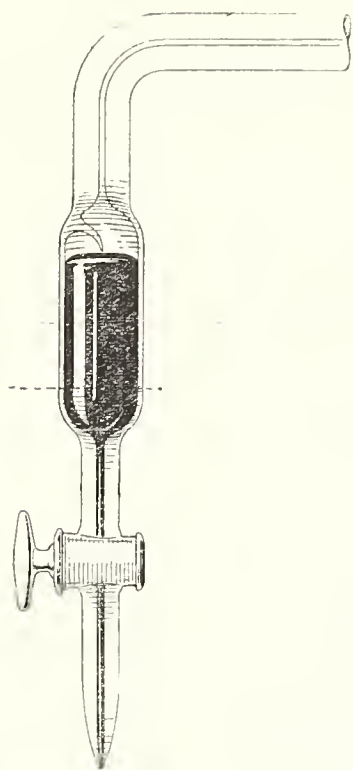


Fig. 5.

After bending the capillary tube as in the figure, the volume of the dead-space was determined in the following manner. The wide tube was cut about 30 millims. below its junction with the capillary tube and sealed to a capillary stop-cock drawn out to a point at its other end (fig. 5). The apparatus was fixed in a vertical position, a tap-funnel and rubber tube were attached to the stop-cock, and mercury was run into the tube till its surface came exactly into contact with the point; the final adjustment was made by closing the tap on the funnel and compressing the rubber tube with a screw-clip. The stop-cock on the dead-space was then closed, the rubber tube removed, and the mercury in the dead-space run out into a porcelain crucible and weighed. The rubber tube was then again attached to the apparatus, which was next filled with mercury to a mark on the stem. This mercury was weighed, and from the difference of the two weighings the volume of gas in the dead-space, between a

mercury meniscus in contact with the point and a mark on the stem, was calculated.

The experimental results are as follows :—

Weight of mercury from stop-cock to point.	Temperature.	Volume.
	°C.	
a. 34·1522	20·5	2·5213
b. 34·1684	20·1	2·5223
c. 34·1640	20·2	2·5220
		} 2·5219
Weight of mercury from stop-cock to mark on stem.		
a. 39·3633	} 20·0	2·9123
39·3638		
Volume of dead-space to mark on stem . . .		$\left\{ \begin{array}{l} 2·9123 \\ 2·5219 \\ \hline 0·3904 \end{array} \right.$

In making an actual experiment, the mercury in the dead-space was brought to within a short distance of the point; this distance was measured by means of the telescope and micrometer eye-piece used in measuring the pressure (p. 120). As the

volume of the dead-space was thus somewhat greater than that determined by the method just described, it was necessary to apply a correction amounting to $\pi r^2 \Delta$, where Δ is the distance between the surface of the mercury and the point, and r is the radius of the tube (see p. 121); $\pi r^2 = 0.636$.

As will be shown later, it was necessary in measuring the pressure to apply a correction for the heights of the mercury meniscus in the dead-space and manometer tube of the thermometer, which varied slightly in every experiment. If the meniscus in the dead-space be considered to have the form of a semi-ellipsoid of height h and of radius r , the change of volume of the dead-space with change of height of the meniscus will be given by $\frac{1}{3}\pi r^2 (h - h')$. The mean height of the meniscus was found to be 1.25 millims., and the extreme variation for any observation never exceeded 0.2 millim., which corresponds to a change of volume of 0.004 cub. centim. in the dead-space. As such an error had a considerable effect on the value of P_0 and P_{100} , we were obliged to take it into consideration.

9. The Stem.

In our earlier experiment we had found that, if by any accident the mercury reached the top of the dead-space, minute globules of it adhered to the glass and were in time carried over into the bulb. As the vapour pressure of mercury is 0.27 millim. at 100° C., this gave rise to serious errors in our measurements.

To obviate this difficulty we introduced into a portion of the stem between c and d a closely-wound coil of silver wire 15 millims. long and slightly over 2 millims. in diameter. The tube was made slightly larger at this point. The volume of the air space was easily determined by weighing the tubes empty and full of water.

	(a.)	(b.)
Weight of stem filled with water	10.4304	10.4306
" " empty	10.3636	10.3632
" water	0.0668	0.0674
Volume of stem	0.0671 cub. centim.	

10. Volumes of the Bulb, Stem, and Dead-Space.

When the instrument was set up permanently and the mercury brought close to the point in the dead-space, the whole of the internal space was considered as divided into three regions.

I. The bulb and stem to within 40 millims of the horizontal portion.

Volume at ice-point = 94.2096 cub. centims.

Volume at boiling-point = $\left\{ 94.2096 (1 + 0.000285t) + 0.0017 \frac{P_{100} - P_0}{760} \right\}$ cub. centims.

II. The stem from the point already referred to to the middle of the wall of the water space enclosing the barometer tube. The temperature of the stem was determined with sufficient accuracy by means of a thermometer with the bulb placed between *c* and *d*.

$$\text{Volume} = 0.1031 \text{ cub. centim.}$$

In some of our earlier experiments, about 3 centims. of the stem outside the hypsometer was enclosed between sheets of asbestos, and the temperature was determined by means of a mercury thermometer. The mean temperature of this portion of the stem was usually about 90°C ., and, as its volume was only 0.0182 cub. centim., its temperature could be taken as 100°C . without affecting the value of P_{100} .

III. The dead-space and remainder of the stem. The dead-space and barometer were enclosed in a water jacket, as will be described presently; the temperature was determined by means of the thermometer *v*.

$$\text{Volume} = \{0.3904 + 0.636\Delta + 0.212(0.125 - h)\},$$

where Δ is the distance in centimetres between the point and the surface of the mercury, and h is the height of the meniscus.

11. *The Manometer and its Connection with the Dead-Space, &c.*

The pressure on the gas in the thermometer was determined directly by observing the difference of the level of this mercury meniscus in the dead-space *f* and in a manometer *g*. By employing this method it was necessary to take only two readings, and the observed pressures were entirely independent of the atmospheric pressure.

The tube *f* which formed the dead-space after removing the stop-cock was sealed at its lower end to a wider tube *h*, which was connected with the manometer tube *g* and with a tube *k*, which served to catch any air bubbles which might enter the apparatus through the stop-cock *n* and the rubber tube *l* leading to a mercury reservoir. A tube 5 millims. in diameter, sealed into the side of the tube *h*, was connected to a stop-cock *i* of capillary bore, through which the gas could be introduced into the apparatus. The tube *k* terminated above in a narrow glass tube, connected by a piece of india-rubber pressure tubing with one limb of a glass stop-cock, through which traces of air entering through the tube *l* could be expelled. By closing the stop-cock *n* and compressing the rubber tube by the pinch-cock *m* the final adjustment of the level of the mercury in the dead-space could be effected.

The manometer tube was carried vertically upwards from the bottom of *h* and bent above the level of the dead-space, so that its axis coincided with that of the latter. In observing the pressure on the gas in the thermometer, it was only necessary to read the difference of level between surfaces of the mercury in the dead-space and in the manometer tube which lay in the same vertical line. The top of the

manometer tube was 1500 millims. above the level of the dead-space, so that when the pressure on the gas in the thermometer was 1000 millims. there was still a considerable vacuous space above the mercury. The manometer tube was boiled out and the apparatus filled with mercury before the thermometer was fixed to its stand, or the bulb sealed to the capillary tube leading to the dead-space. For this purpose the apparatus was set up temporarily in clamps, and the top of the manometer tube, which was drawn out to a fine capillary (fig. 4. A), was sealed to a tube leading to the mercury pump. The capillary tube connected with the dead-space was also closed.

After thoroughly exhausting the apparatus, and heating it to remove trace of water condensed on the glass, mercury was admitted through the stop-cock *n*. As the mercury entered the manometer tube it was boiled by heating the tube with a Bunsen burner. By this method, which we have frequently employed, the rising surface of the mercury remains in contact with mercury vapour only, and as the exhaustion is continued throughout the operation a very perfect barometric vacuum is finally obtained. By admitting air to the tube *h* through the stop-cock *i*, and lifting the reservoir attached to the tube *l*, the mercury in the manometer tube was raised nearly to the top. By heating, mercury vapour was made to pass through the constriction, and condense in the descending tube. The constriction was then heated in the blow-pipe, and the manometer was sealed. The mercury was then allowed to fall in the manometer tube.

The bulb was not sealed to the rest of the apparatus till this operation had been completed, as it was found that it was impossible to prevent mercury vapour condensing in it. The connection was now made by sealing the stem with a mouth blow-pipe. It was unnecessary to make any correction for the distortion of the stem, as the volume of 1 millim. of its length amounted only to 0.000444 cub. centim.

The instrument was now fixed to its permanent support, as in fig. 4. This consisted of a stout wooden plank, 3 centims. thick and 15 centims. wide at the bottom, cut away for 6 centims. of its width from the level of the dead-space upwards to allow of the illumination of the scale from behind. The support was stayed rigidly at the top to the wall and to a beam running transversely across the room, so as to prevent all lateral movement.

We do not intend to give an account of the experiments which we carried out before we commenced the series of measurements on which we base our results. It will suffice to state that about four months were spent detecting and eliminating sources of error such as that due to lack of rigidity in the apparatus.

12. *Introduction of the Gas into the Apparatus.*

The gases employed in these experiments were hydrogen and helium, and it will be convenient to consider the method by which they were prepared and purified at a later stage. The gas was introduced into the thermometer through the stop-cock *i*.

which for this purpose was connected through a bulb x and a tube z to the apparatus containing the gas, and to a mercury pump. The tube x was intended to receive the mercury contained in the portion of the side tube above the stop-cock.

By lowering the reservoir connected by the rubber tube l with the stop-cock n , the level of the mercury in h could be made to fall below the point of entry of the side tube connected with i . Then by carefully opening the cocks i and n alternately and manipulating the mercury pump, the gas in the apparatus could be completely removed. When this was effected the mercury in the tubes h , g , and k stood at the same level. Before introducing the gas the bulb was heated in an air-bath for about an hour to 150° C. to move all traces of moisture, and was washed out several times with the dry gas.

The operation of filling the apparatus was the reverse of that of emptying it. The gas was admitted slowly by opening the stop-cock i , and at the same time mercury was admitted through the stop-cock n . The quantity of gas required to fill the apparatus to the desired pressure was at first found by trial, but after the first experiment it was known that when the level of the mercury in the tube h lay just below the tube leading to i , the mercury rose to a certain height in the manometer tube g . When sufficient gas had been admitted the stop-cock i was closed and more mercury was allowed to enter the apparatus through h . The gas in the portion of the side tube above i was easily displaced by mercury. (See also Part III., p. 171. *et seq.*)

13. *The Scale.*

The measurement of the pressure in the thermometer bulb could have been easily and effectively made by employing a well constructed cathetometer with a standard scale attached to it. As, however, we had no means of obtaining such an instrument, we were obliged to employ the following method.

On the surface of a piece of plate-glass short scales ruled in millimetres by ZEISS of Jena, were cemented by means of Canada balsam. The scale so constructed was fixed in front of the mercury column and dead-space, so that when the lowest scale, which was 50 millims. in length, was in front of the dead-space, the four remaining scales were 100 millims. in length. With this arrangement it was possible to fill the thermometer at three initial pressures, viz., at 350, at 500, and at 700 millims. of mercury, and to determine the pressure coefficient of the gas at each pressure. The method of observation will be described later.

The distances between the points on the scale at which observations were made were first of all determined by us by means of a micrometer apparatus, designed by Mr. HILGER for measuring spectro-photographs, with a screw of 1-millimetre pitch and a drum divided into 100 parts. By means of this instrument it was possible to determine lengths of 150 millims. to 0.002 millim. At the end of our research the

scale was sent to the National Physical Laboratory, and was there compared with a standard brass metre scale (Société Genevoise, No. 59); the comparison was guaranteed to 0.01 millim. For this we wish to express our thanks to the Director, Mr. R. T. GLAZEBROOK.

The results of the two sets of comparison are very concordant, except so far as the distance between scale I. and scale III. is concerned. With regard to the discrepancy of 0.015 millim. which exists here, we wish to state that the values of the pressure-coefficients first obtained for hydrogen at initial pressures of 350 and 500 millims., indicated that there was probably a small error in the scale at this point. This has been corrected.

In the following table column A gives the distances between the line 9 on the lowest scale (I.) and the middle points on the four remaining scales, as determined by us; column C gives the value stated in the certificate furnished by the National Physical Laboratory, while the numbers in column B are calculated from those in column C by multiplying the latter by the factor $962.395/962.28$.

	A.	B.	C.
Line 9, scale I., to line 50, scale II. . . .	369.895	369.895	369.85
„ „ „ „ III. . . .	521.505	521.52	521.46
„ „ „ „ IV. . . .	684.644	684.64	684.56
„ „ „ „ V. . . .	962.395	962.395	962.28

It need not be pointed out here, that in the measurements of the pressure-coefficient it is only necessary to know the absolute length of the scale approximately. We have accordingly taken the figures in the second column (B) as representing the distances between the standard points on our scale.

14. *The Temperature of the Mercury Column.*

Since in these experiments we were attempting to measure the pressure with a degree of accuracy approaching 0.01 millim., it was necessary to take special precaution with regard to the temperature of the mercury column. In our first experiments we attempted to measure the temperature of the column by means of four thermometers, the bulbs of which were enclosed in glass tubes, of the same diameter as the barometer column, filled with mercury. We found, however, that the temperature could not be read with certainty to 0.2 C., nor even with that degree of accuracy when the steam point was being determined. As an error of 0.01 in the temperature of the column involves a corresponding error of 0.016 millim. in P_{100} (950 millims.), we enclosed the dead-space and mercury columns in a water jacket.

The manner in which this was effected is shown in fig. 4. The dead-space and manometer column were enclosed between two pieces of plate-glass, p and o , the pieces of plate-glass p in front forming the scale; strips of wood, r and r' , formed the side and bottom of the water jacket, the whole being made water-tight by means of a mixture of red-lead and gold size. The arrangement was carried on brass brackets ss screwed to the face of the main support. The two plates of glass were clamped to the wooden strips rr' by brass bands tt' and wooden wedges uu' ; the brass bands were screwed at one end to the main support. Water entered the jacket at the bottom, and escaping at the top by a rubber tube, flowed over the tubes h and k , and the portion of g below the water jacket. This part of the apparatus was swathed in cotton cloth, and was maintained at a constant, though indefinite temperature, by means of the current of water. The temperature of the water in the jacket remained constant within $0^{\circ}02$ for a sufficient time for the measurement of the pressure, and rarely varied by $0^{\circ}1$ during the period of one hour, necessary for four consecutive observations. The error due to temperature on P_{100} (950 millims.) is thus reduced to 0.002 millim., which is considerably less than the error of observation. It may also be noted here that, as the temperature of the dead-space was made with the same degree of accuracy, and an error of 0.1 in determining it makes a difference of 0.002 millim. on P_{100} , this source of error also disappears.

The height of the column of which the temperature varied between 8° and 17° C. was reduced in every case to the height at 10° C.

It is obvious that as the manometer tube and the glass scale in front of it are not parallel, and the space between them is filled with water, a refraction error may be introduced which will have a constant influence on the value of P_0 and P_{100} . If the deviation of the manometer tube amounts to 2 millims. in its total length, the error in reading the height of the column would be roughly 0.01 millim. By observing the height of the column when the jacket contained water, and then allowing the water to escape rapidly, we could, however, detect no change in the position of the meniscus. Further, since the manometer tube was sufficiently straight for the error to affect P_0 and P_{100} equally, the error due to refraction can be neglected.

15. *The Measurement of the Pressure on the Gas in the Thermometer.*

The observations were made by means of a telescope placed at a distance of one metre from the scale. The telescope was fitted with a Hilger ocular micrometer, with a screw, divided into 100 parts, each corresponding to a movement of the cross-wire of 0.005 millim., or to 0.007 millim. on the scale. Each meniscus was illuminated from behind by means of an electric glow-lamp covered with white tissue paper; cards placed behind the glass plate O , cut off the light at a height of 1 or 2 millims. above the level of the mercury (*cf.* p. 109).

When the temperature of the bulb and column were steady, the mercury in the dead-space was adjusted first of all by means of the mercury reservoir and stop-cock *n*, and finally by means of the pinch-cock *m*. The mercury was always brought to within about 0.1 millim. of the point. After tapping the support with the hand, and waiting a short time for the mercury to settle down, the observations were made.

The spider-line of the micrometer was first brought into coincidence with the line on the scales next above the meniscus, and then with the top of the meniscus itself, readings being taken in both positions. A light was then held in front of the instrument, and the height of the meniscus was determined by bringing the spider line into coincidence with the point at which the vertical bright line, caused by the reflection of the light from the curved surface of the mercury column, came to an end. The height of the mercury meniscus was applied to the calculation of the correction to be applied for capillarity (KOHLRAUSCH). The determination of the position of the point *c* with regard to the scale was also necessary for the calculation of the distance of the mercury from it, and from this the volume of the dead-space. Its value, and the value of one division of the micrometer screw in terms of a division of the scale, were measured separately.

Observations of the pressure were invariably made by each one of us alternately, the other reading the thermometer enclosed in the water jacket, and the thermometer placed close to the horizontal portion of the stem. The position of the lower meniscus was observed, then that of the upper meniscus, and finally the lower meniscus again. The level of the mercury in the apparatus was usually readjusted between each pair of measurements.

16. *The Ice-Point.*

The bulb was immersed to within 40 millims. of the horizontal portion of the stem in an inverted glass bell-jar, and surrounded with broken ice packed tightly round it. The bell-jar was filled nearly to the top of the vessel with distilled water. Samples of the ice were on two occasions melted and the liquid evaporated; in neither case was it found to contain more than a minute trace of solid matter. A piece of wood, with a slit cut in it for the stem of the thermometer, was placed on top of the bell-jar containing the ice.

17. *The Boiling-Point.*

The bulb of the thermometer was surrounded with dry steam by means of a large double-walled copper hypsometer of the usual form. The stem of the thermometer passed through a hole in the cover of the hypsometer, and the aperture was made fairly steam-tight by means of a small strip of wet rag wound round the stem. No correction of the difference of pressure inside and outside the hypsometer was necessary.

To shield the apparatus against radiation from the heated vessel a piece of wood, kept wet by a stream of water, was used as a screen. A similar piece of wood, with a slit cut in it for the stem, was held in a clamp horizontally about 1 centim. above the top of the hypsometer to screen the horizontal portion of the stem. The temperature of the short length of stem between the hypsometer and the screen was usually about 96° , a difference from 100° too small to be considered. During each experiment the barometer was always read three times; at the beginning, at the end, and between the two pairs of observations.

18. *Calculation of the Results.*

In calculating the value of the pressure coefficient from the results of our experiments, we employed the usual methods. The pressure P_0 and P_{100} which the gas would exert at the temperature of melting ice, and at that of saturated water vapour at the normal pressure, if confined in a space corresponding to the volume of the thermometer at 0° C., were first calculated. The pressure coefficient was then obtained from the equation

$$\alpha = (P_{100} - P_0)/100 P_{100}.$$

In calculating P_0 and P_{100} it is of course necessary to assume some value for the pressure coefficient in order to apply the necessary correction for the gas contained in the stem and dead-space; and in reducing the temperature of the bulb to 100° C. in calculating P_{100} . These corrections are, however, small; and no appreciable error is introduced by taking the coefficient as $1/273$.

The following equations were employed in calculating P_0 and P_{100} :—

$$P_0 = P \left\{ V_b + \frac{V_s \times 273}{273 + T_s} + \frac{V_{ds} \times 273}{273 + T_{ds}} \right\} \frac{1}{V}$$

$$P_{100} = P' \left\{ \frac{V_b' \times 373}{273 + T} + \frac{V_s \times 373}{273 + T_s'} + \frac{V_{ds} \times 373}{273 + T_{ds}'} \right\} \frac{1}{V}.$$

Where P is the observed pressure at the ice-point.

P' is the observed pressure at the steam-point.

T is the temperature of steam corresponding to the barometric pressure.

T_s and T_s' are the temperatures of the stem.

T_{ds} and T_{ds}' are the temperatures of the dead-space.

V_b is the volume of the bulb at the ice-point.

V_b' is the volume of the bulb at the steam-point.

V_s is the volume of the stem.

V_{ds} is the volume of the dead-space.

$V = (V_b + V_s + V_{ds})$, is the total volume of the thermometer at 0° C.

The method by which the temperature, pressure, and volume were calculated and corrected has already been dealt with.

19. *Details of the Results of One Experiment.*

As we have already stated, each determination of the ice-point or boiling-point consisted of four separate observations of the height and temperature of the barometer column, of the temperature of the stem, &c. The readings were taken by each of us alternately. In determining the boiling-point the barometer was also read three times, at the beginning, at the end, and in the middle of each set of experiments, and the temperature of the bulb was calculated from the mean corrected pressure. It would be useless to set down the whole of the measurements involved in every experiment, but in order to give an idea of the accuracy of the observations, we give the full details of one complete set of measurements.

Ice-Point, February 24th, 1902.

	I.		II.		III.		IV.	
	Micrometer readings.	Mean in millims.	Micrometer readings.	Mean in millims.	Micrometer readings.	Mean in millims.	Micrometer readings.	Mean in millims.
Lower meniscus { Line 9 to point Line 9 to meniscus Height of meniscus	67, 67, 68	0.469	73, 73, 70	0.504	77, 78, 77	—	69	0.483
	80, 80, 80	0.560	80, 80, 78	0.563	181, 181, 184	0.539	74, 75, 78, 79	0.539
	177, 178	1.25	180, 180	1.26	100, 100	1.27	185, 180	1.27
Upper " { Line 60 to meniscus Height of meniscus	102, 102	0.714	98, 99	0.689	144, 146	0.700	99, 99	0.693
	129, 132	0.98	140, 139	0.98	—	1.01	146, 144	1.01
Temperature of dead-space	8.78° C.		8.76° C.		8.64° C.		8.60° C.	
" " stem	19.1° C.		19.6° C.		19.6° C.		18.6° C.	

	I.	II.	III.	IV.
Line 9 (lower scale) to line 60 (upper scale)	694.644	694.644	694.644	694.644
Δ bottom	+0.560	+0.553	+0.539	+0.539
Δ top	-0.714	-0.689	-0.700	-0.693
Temperature correction to 10° C.	+0.141	+0.143	+0.158	+0.162
Correction for capillarity	-0.081	-0.084	-0.078	-0.078
Pressure in millims. of mercury at 10° C.	694.550	694.567	694.563	694.574
Volume of dead-space	0.3958	0.3959	0.3945	0.3945

Mean pressure 694.564 millims.
Mean volume of dead-space 0.3950 cub. centim.
Mean temperature of dead-space 8.70° C.
" " stem 19.3° C.
P₀ 694.458.

	I.		II.		III.		IV.	
	Micrometer readings.	Mean in millims.	Micrometer readings.	Mean in millims.	Micrometer readings.	Mean in millims.	Micrometer readings.	Mean in millims.
Lower meniscus { Line 9 to meniscus .	78, 79, 79	0·553	78, 79, 79	0·553	80, 78, 79	0·553	78, 79, 77	0·546
{ Height of meniscus .	185, 182, 177	1·27	185, 186	1·29	175, 178, 180	1·25	180, 180, 186	1·27
Upper " { Line 31 to meniscus .	38, 39	0·269	36, 39	0·262	39, 39	0·273	35, 35, 35	0·247
{ Height of meniscus .	158, 155	1·10	157, 154	1·08	150, 152	1·06	157, 159	1·11
Temperature of dead-space	8·38° C.		8·31° C.		8·31° C.		8·30° C.	
" " stem (a)	27·0° C.		27·1° C.		27·1° C.		27·0° C.	
" " (b)	92·5° C.		92·0° C.		92·0° C.		92·0° C.	

	I.	II.	III.	IV.
Line 9 (lower scale) to line 31 (upper scale)	943·394	943·394	943·394	943·394
Δ bottom	+0·553	+0·553	+0·553	+0·546
Δ top	-0·269	-0·262	-0·273	-0·246
Temperature correction to 10° C. . . .	+0·253	+0·265	+0·265	+0·266
Correction for capillarity. . . .	-0·051	-0·063	-0·057	-0·051
Pressure corrected to 10° C. . . .	943·880	943·887	943·882	943·910
Volume of dead-space	0·3944	0·3944	0·3944	0·3940
Observed height of barometer	I. 754·11 millims.	II. 754·06 millims.	III. 754·06 millims.	
Temperature of barometer	16·8° C.	17·1° C.	17·1° C.	17·8° C.

Mean pressure	943·889 millims.	Mean volume of dead-space	0·3943
Volume of bulb	94·4776 cub. centims.	Mean temperature of dead-space	8·33° C.
Mean barometric pressure	752·11	" " stem	(a) 27·0°, (b) 92° C.
Temperature of steam	99·709	P ₁₀₀	948·824

20. The Pressure Coefficient of Hydrogen.

Series I. and II. at initial pressures of about 700 millims.
Series III. " " " 500 " "
Series IV. " " " 350 " "
Series V. " " " 700 " "

I.) Hydrogen.

Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
Feb. 24th	694·550, ·567, ·563, ·574	694·564	0·3949	° C. 8·70	° C. 19·3	694·458
Feb. 25th	694·524, ·542, ·563, ·581	694·552	0·3950	7·74	20·0	694·452

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem (a).	Mean temperature of stem (b).	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
Feb. 24th	943·755, ·731, ·765, ·773	943·756	0·3949	° C. 8·17	° C. 26·3	° C. 94	751·18	° C. 99·675	948·789
Feb. 25th	943·880, ·887, ·882, ·910	943·889	0·3944	8·33	27·0	92	752·11	99·709	948·824
Feb. 26th	943·640, ·674, ·662, ·671	943·665	0·3948	8·54	26·3	94	749·96	99·630	948·809

Mean P₀, 694·455. Mean P₁₀₀, 948·807. $z = 0·00366261$.

(II.) Hydrogen.

Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
Feb. 26th	696·178, ·186, ·200, ·225	696·198	0·3956	° C. 8·14	° C. 19·2	696·103
Feb. 27th	696·176, ·216, ·207, ·183	696·196	0·3958	8·37	17·7	696·102

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem (a).	Mean temperature of stem (b).	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
Feb. 27th	945·157, ·193, ·151, ·156	945·160	0·3975	° C. 8·15	° C. 26·1	° C. 87	742·37	° C. 99·346	951·059
Feb. 28th	945·687, ·716, ·713, ·736	945·713	0·3959	9·03	21·5	93	748·24	99·565	951·044

Mean P₀, 696·103. Mean P₁₀₀, 951·052. $z = 0·00366252$.

In these experiments the short length of the stem (b) outside the hypsometer was considered separately. No error is however, introduced by considering it to be heated to 100° C. The thermometer was completely exhausted and was refilled with hydrogen between the first and second series of experiments.

(III.) Hydrogen.

Ice-Point

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
March 4th	520·384, ·411, ·406, ·430	520·408	0·3950	° C. 9·94	° C. 16·4	520·326
March 5th	520·388, ·403, ·381, ·380	520·388	0·3952	9·53	15·3	520·311

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem (a).	Mean temperature of stem (b).	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
March 3rd	707·808, ·794, ·783, ·794	707·795	0·3965	° C. 9·06	° C. 25·0	° C. 92	761·27	° C. 100·046	710·897
March 4th	707·831, ·832, ·837, ·839	707·835	0·3966	9·46	25·2	68	761·71	100·062	710·882
March 5th	707·914, ·915, ·907, ·896	707·908	0·3954	9·42	24·7	85	762·41	100·087	710·907

Mean P₀ = 520·319. Mean P₁₀₀ = 710·895. $\alpha = 0·00366268$.

(IV.) Hydrogen.

Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
March 6th	348·036, ·041, ·006, ·010	348·023	0·3952	° C. 9·55	° C. 18·7	347·968
March 11th	348·056, ·054, ·060, ·070	348·060	0·3947	9·74	17·6	348·002
March 11th	348·086, ·049, ·072	348·069	0·3949	9·17	19·0	348·009

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem (a).	Mean temperature of stem (b).	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
March 6th	473·403, ·417, ·404, ·397	473·405	0·3950	° C. 9·65	° C. 27·0	° C. 86	761·59	° C. 100·057	475·428
March 7th	473·294, ·295, ·286, ·288	473·291	0·3946	9·59	29·0	68	759·05	99·964	475·439
March 10th	473·342, ·333, ·334, ·331	473·335	0·3944	9·49	23·2	91	760·14	100·004	475·436

Mean P₀ = 347·993. Mean P₁₀₀ = 475·434. $\alpha = 0·00366133$.

(V.) Before determining the pressure coefficient for helium, as a considerable time had elapsed since the measurements of the coefficients for hydrogen had been made, the thermometer was refilled with hydrogen, and an ice-point and a boiling-point were determined.

Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space. ° C.	Mean temperature of stem. ° C.	P ₀ .
May 8th	706·604, ·622, ·649, ·623	706·663	0·3970	11·14	14·4	706·528

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space. ° C.	Mean temperature of stem. ° C.	Barometric pressure.	Temperature of steam. ° C.	P ₁₀₀ .
May 7th	961·504, ·516, ·435, ·427	961·431	0·3986	11·26	21·3	765·18	100·189	965·291

$P_0 = 706.528, \quad P_{100} = 965.291, \quad \alpha = .00366246.$

Since this value for the coefficient is the result of one observation of the ice-point and one of the boiling-point, it can only be taken as confirming our previous determination, and as indicating that our apparatus was still in thorough working order.

21. *The Pressure Coefficient of Helium.*

Series I. and II. at initial pressures of about 700 millims.

Series III. and IV. " " " 500 "

(I.) Helium. (For preparation of the gas see Part III., p. 171.)
Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
May 29th	690·385, ·346, ·398, ·376	690·376	0·3976	°C. 15·11	°C. 17·6	690·232
May 30th	690·391, ·385, ·377, ·391	690·386	0·3950	15·48	18·7	690·238

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
May 29th	938·548, ·507, ·524, ·509	938·522	0·3996	°C. 15·13	°C. 22·8	755·54	°C. 99·863	943·044
May 30th	938·174, ·154, ·164, ·133	938·154	0·3964	15·40	22·0	751·86	99·700	943·044
May 31st	938·208, ·200, ·220, ·196	938·206	0·3987	15·33	24·1	753·21	99·750	942·992

Mean P₀ = 690·235. Mean P = 943·027. $\alpha = 0·00366241$.

(II.) Helium. (A small quantity of gas was removed from the bulb between Experiment I. and II.)
Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
June 2nd	671·582, ·559, ·589, ·565	671·574	0·3977	°C. 15·98	°C. 19·1	671·422
June 3rd	671·580, ·563, ·587, ·566	671·574	0·3957	16·10	19·2	671·408

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
June 2nd	913·602, ·588, ·573, ·601	913·591	0·3974	°C. 15·94	°C. 20·8	763·22	°C. 100·118	917·352
June 3rd	913·511, ·510, ·511, ·503	913·509	0·3953	16·10	22·2	762·41	100·089	917·322
June 4th	913·428, ·476, ·485, ·515	913·476	0·3977	16·49	22·5	761·96	100·072	917·328

Mean P₀ = 671·415. Mean P₁₀₀ = 917·334. $\alpha = 0·00366270$.

(III.) Helium at lower pressure.

Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
June 5th	523·108, ·098, ·117, ·118	523·108	0·3964	° C. 16·15	° C. 17·85	522·984
June 6th	523·081, ·098, ·111, ·119	523·108	0·3962	16·02	18·6	522·984

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
June 5th	711·598, ·581, ·595, ·611	711·599	0·3967	° C. 17·43	° C. 24·1	761·96	100·072	714·576
June 6th	711·244, ·242, ·234, ·231	711·238	0·3969	16·09	23·5	757·73	99·917	714·529
June 7th	710·632, ·602, ·620, ·581	710·594	0·3947	15·62	22·8	747·76	99·548	714·577

Mean P₀ = 522·984. Mean P₁₀₀ = 714·560. $\alpha = 0·00366313.$

(IV.)

Ice-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	P ₀ .
June 16th	523·120, ·122, ·116	523·119	0·3957	° C. 13·40	° C. 15·1	523·016
June 17th	523·147, ·130, ·129, ·121	523·132	0·3939	13·40	15·2	523·020

Boiling-Point.

Date.	Observed pressure, corrected.	Mean pressure.	Mean volume of dead-space.	Mean temperature of dead-space.	Mean temperature of stem.	Barometric pressure.	Temperature of steam.	P ₁₀₀ .
June 16th	710·987, ·967, ·973, ·985	710·978	0·3929	° C. 13·48	° C. 20·5	753·46	99·759	714,568
June 17th	711·436, ·423, ·453, ·438	711·438	0·3940	13·38	19·7	760·12	100·005	714,583

Mean P₀ = 523·018. Mean P₁₀₀ = 714·576. $\alpha = 0·00366255.$

22. *Accuracy of the Results.*

In describing the construction of the apparatus and the manner in which the experiments were carried out, we have already pointed to the errors which may occur and their effect on the accuracy of the final result. It will have been noticed that with the exception of the errors in the barometer reading, and in the temperature of the steam in the hypsometer, the errors of observation affect P_0 and P_{100} almost equally, and hence have little effect on the value of the coefficient. Further, since an error of 0.1 millim. in the barometer reading only changes the value of P_{100} (950 millims.) by 0.003 millim., the effect of such an error on the value of the coefficient is less than five units in the sixth significant figure.

With regard to the measurement of the pressure, as we have already stated, the errors in the scale were certainly less than 0.01 millim. An error in the difference $P_{100}-P_0$ of this magnitude would involve an error of one unit in the fifth significant figure. The actual extreme difference between the value of P_{100} in the set of three observations which constitute each series usually amounts to 0.02 to 0.03 millim., but these differences appear to be unbiassed and do not influence the final result in one direction or in the other. The value of the temperature correction practically disappears, for, as we have pointed out, the temperature of the column could be determined to 0.02°C. , and in successive determinations of P_0 and P_{100} the temperatures were practically identical. The capillarity correction involving the height of the meniscus we considered at first to have a doubtful character, but as the height of the meniscus varied by 0.04 millim. in different observations, and the application of the correction brought the observed pressures to the same value, we have come to the conclusion that it is sound.

The errors in the measurement of the volume and temperature of the dead-space are small. Their effect on the value of the coefficient has already been sufficiently discussed.

23. *The Final Values for the Pressure Coefficients of Hydrogen and Helium.*

As the tables in Sections 20 and 21 show, we have two series of determinations of the pressure coefficient both in the case of hydrogen and of helium at initial pressure of about 700 millims. of mercury. The results are as follows:—

Gas.	Observed values of the coefficient, α .	Mean value.
Hydrogen	0.00366261, 0.00366252	0.00366255
Helium.	0.00366241, 0.00366270	0.00366255

Each of the observed values is based, as the tables show, on eight measurements of the pressure on the gas, the temperature of the dead-space, &c., when the bulb of the thermometer was immersed in ice, and twelve similar measurements where the bulb was surrounded with steam. The agreement between the observations is satisfactory.

As has already been pointed out at the commencement of this paper, the best determinations of the pressure coefficient for hydrogen are those of CHAPPUIS and ONNES. CHAPPUIS' work is to be found in the 'Travaux et Mémoires du Bureau International des Poids et Mesures,' vol. 6 (1888), and in the subsequent volumes; an account of his work is also to be found in the 'Rapports du Congrès International de Physique' (Paris, 1901). The results obtained in 1887 with his large thermometer, which has a platinum-iridium bulb of nearly 1 litre capacity filled at an initial pressure of 1000 millims. of mercury, are as follows :—

Value of the coefficient.	Mean values.
0·00366271	} 0·00366254
0·00366248	
0·00366225	
0·00366231	
0·00366256	
0·00366270	
0·00366269	

The mean value of four determinations carried out in 1899 with the same thermometer was 0·00366296, and of five determinations made in 1895 with a bulb of "verre dur" was 0·00366217. The mean value of the seven determinations made in 1887 is, however, retained as the probable value of the coefficient, which forms the basis for the definition of the so-called normal scale of temperature.

The three determinations of the coefficient by ONNES ('Communications from the Physical Laboratory of the University of Leiden,' No. 60) are as follows :—

$$0\cdot0036628, \qquad 0\cdot0036624, \qquad 0\cdot0036628.$$

As in the case of CHAPPUIS' experiments, the initial pressure in the thermometer was about 1000 millims. of mercury.

Though the highest initial pressure at which our measurements were made was 700 millims. of mercury, there is, as we shall presently show, no reason for assuming that the coefficient varies with the pressure, and the agreement between CHAPPUIS' results and our own may be considered as confirmatory of the latter. The combined results may probably be considered correct to the fifth significant figure.

24. *The Pressure Coefficients of Hydrogen and Helium at Lower Initial Pressures.*

Though the accuracy of the determinations of the pressure coefficients at lower pressures is reduced in proportion to the initial pressure, the results we have obtained indicate that the pressure coefficient for the two gases is practically independent of the pressure. The value of the coefficient found for hydrogen at an initial pressure of 500 millims. is 0·0036627, and for helium 0·0036625 and 0·0036631; if the coefficient attained a limiting value corresponding to 0·003660 at zero pressure, the value at 500 millims. should be 0·0036612.

The determination of the pressure coefficient for hydrogen at an initial pressure of 350 millims. is practically valueless; only one set of measurements was made, and we found that our method was not sufficiently accurate to investigate the coefficient at such a low pressure.

25. *General Conclusions.*

CHAPPUIS has determined the pressure coefficients of nitrogen at different pressures with the following results :—

Initial pressure.	Coefficient.
millims.	
1000	0·00367445
793·5	0·0036718
530·8	0·0036683

On plotting the values of the coefficient against the initial pressure, the value of the coefficient at zero initial pressure is found on linear extrapolation to lie between 0·003662 and 0·003663.

As DANIEL BERTHELOT ('Comptes Rendus,' 1898) has pointed out, the specific volumes of the common gases at zero pressure, calculated from their densities under normal conditions, and the variation of "p.v." with pressure, are the same. Their densities under the same conditions are proportional to their chemical atomic weights. If this is the case, they should, at zero pressure, behave as perfect gases, and the temperature scale on a thermometer filled with any gas at very low pressure should be coincident with the absolute scale of temperature.

If our conclusion is correct, that the pressure-coefficient for hydrogen and for helium has the value 0.00366255 , and is independent of the pressure below 1000 millims. of mercury, it is probable that the melting-point of ice on the absolute scale does not lie very far from $273^{\circ}.03$. Before we can apply ROSE-INNES' equations (*loc. cit.*, p. 106) to the investigation of the problem, it will be necessary, however, to obtain a more complete knowledge of the thermo-dynamic properties of these gases.

PART II.

On the Vapour Pressures of Liquid Oxygen at Temperatures below its Boiling-Point on the Constant-Volume Hydrogen and Helium Scales.

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1. *Introduction.*

AFTER determining the pressure coefficients of hydrogen and helium between the melting-point of ice and the boiling-point of water, and so obtaining the fundamental intervals of our thermometer scales, we proceeded to apply our results to the determination of the boiling-point and vapour pressures of liquid oxygen. In these experiments we have employed three different thermometers; the coefficient of expansion of the glass at low temperatures has been specially determined; every precaution has been taken in preparing the different samples of hydrogen and helium employed; and the errors due to impurity in the oxygen and to the superheating of the liquid have, we believe, been eliminated. The following account contains full details of our experimental method and results.

2. Previous Measurements of the Vapour Pressures of Liquid Oxygen.

The earliest measurement to which any importance can be attached is that of WROBLEWSKI in 1888 ('Wiener Berichte,' 97. Abth. 2A. p. 1321). The temperature of the boiling-point of oxygen was measured on the constant-volume hydrogen scale, but no details are given in the paper, which is entitled "Die Zusammendrückbarkeit des Wasserstoffs," and was published after his death. The number given by him is 182.4°C .

OLSZEWSKI appears to have measured the boiling-point of oxygen on several occasions. In 1896 he published in 'Nature' (vol. 54, p. 377) an account of his comparison of the hydrogen and helium thermometers at temperatures corresponding to vapour pressures below 741 millims. mercury. By extrapolation the temperature of the boiling-point on either thermometer appears to be -182.36°C . In his paper on the liquefaction of gases ('Phil. Mag.,' 1895, vol. 39, p. 188) he gives -181.4° as the boiling-point. The capacity of the bulb of his thermometer was only 2 cub. centims.

ESTREICHER ('Phil. Mag.,' (5), vol. 40, p. 454) also made a large number of measurements. He made twenty-five measurements of the temperature corresponding to 744.8 millims., and gives -182.56°C . as the mean of the results. The boiling-point calculated by extrapolation is -182.4°C . (ΔT being 0.0122° per millimetre). ESTREICHER worked in OLSZEWSKI's laboratory, and employed the same instrument as the latter.

WITKOWSKI ('Phil. Mag.,' 1896, vol. 42) gives the boiling-point of oxygen as -182.446°C ., employing the value 0.00366 for the coefficient of expansion of hydrogen.

HOLBORN and WIEN ('Wied. Ann.,' 1896, vol. 59, p. 213) compared the hydrogen and air thermometers at the temperature of liquid air, and found that the readings of the air thermometer were 0.65° lower than those of the hydrogen thermometer. Later, HOLBORN ('Ann. der Physik,' (4), 1901, vol. 6, p. 242) standardised a platinum resistance thermometer by means of an air thermometer, and by means of it measured the boiling-point of pure oxygen. The temperature referred to the constant-volume hydrogen scale is -182.7°C . Full details of the work, which was carried out in the Physikalische Reichsanstalt at Charlottenburg, are given in the paper.

LADENBURG and KRUGEL ('Ber.,' 1900, vol. 32, p. 1818, vol. 33, p. 637) found -182.2° for the temperature corresponding to a pressure of 745 millims. (boiling-point -182.05°C .). The measurement was made by means of a thermo-electric junction, standardised at the temperature of liquid air, which was supposed to be -191.25° .

DEWAR ('Proc. Roy. Soc.,' 1901, vol. 68, p. 44), employing constant volume thermometers filled with hydrogen and oxygen, obtained the following values. The numbers in third value of the following table are calculated from the mean value of dp/dt .

Hydrogen Thermometer.

Pressure.	Temperature.	Boiling-point ; calculated.
	° C.	° C.
760·6	- 182·20	- 182·21
764·4	- 182·67	- 182·72
759·5	- 181·62	- 181·52

The coefficient of expansion for hydrogen is taken as 0·0036625. Taking the same coefficient for oxygen, a proceeding which does not appear to be justified, he obtained the following results :—

Oxygen Thermometer.

Pressure.	Temperature.	Boiling-point ; calculated.
	° C.	° C.
772·5	- 182·29	- 182·43
756·0	- 183·46	- 183·42
753·5	- 182·95	- 182·90

BALY ('Phil. Mag.,' 1900, vol. 49, p. 517) measured the vapour pressures of oxygen on the constant-pressure hydrogen scale, employing a Callendar thermometer. The results appear to be affected by a constant error; they will be discussed elsewhere.

So far as the evidence contained in the original papers is concerned, it appears to us that greatest weight is to be attached to the work of HOLBORN and WIEN. It is true that the results of WROBLEWSKI, OLSZEWSKI, and ESTREICHER show a close agreement, but none of these experimentalists furnish sufficient data to make criticism possible.

In all the experiments we have referred to, the boiling-point or vapour pressure of oxygen has been measured by immersing the thermometer in the liquid and measuring the pressure at which it was boiling. ESTREICHER and OLSZEWSKI measured the vapour pressures at lower temperatures by connecting the vacuum vessel containing the liquid with an exhaust pump, and making the liquid boil under reduced pressure.

The difficulty of making accurate measurements by this method is twofold. In the first place, it is not easy to liquify a sufficient quantity of pure oxygen; and in the second place, the liquid oxygen tends to become superheated, and does not boil steadily. The impurity in the oxygen would most probably be air, which would lower the boiling-point; other impurities would, however, have the opposite effect.

In every case it appears to have been assumed that the coefficient of expansion for glass remains constant over the whole range of temperature.

3. *The Pressure Coefficients of Hydrogen and Helium between 0° and 100° C.*

As has already been pointed out, there is no appreciable difference between the pressure coefficient of the two gases, which may be considered as approximating very closely to 0.00366255. The reciprocal of this number is 273.03, and though an error of 0.01° only is incurred by taking 273 as the temperature of melting ice on the scale of either thermometer, we have in calculating our results employed the true coefficient.

4. *The Coefficient of Expansion of Glass between 16° and — 190° C.*

The coefficient of expansion of the glass was determined in the following manner. The inner tube of a cylindrical vacuum vessel, which was 30 millims. wide and 1000 millims. long, was graduated for short distances close to its two ends. The vacuum vessel was enclosed in a water-jacket, and was set up vertically in front of two telescopes, of short focus, fitted with micrometer eye-pieces. The cross wires of the micrometer eye-pieces were first brought into coincidence with marks at the two ends of the vacuum vessel. The vessel was then filled with liquid air and the distances through which the marks on the vacuum vessel moved were observed. The following result for the volume coefficient is calculated from the linear contraction of the glass :—

Coefficient of expansion 0° to 100° C. (by weight thermometer)	. 0.0000285.
„ „ 0° to — 190° C. (from linear contraction)	. 0.0000218.

5. *Method of Experiment.*

In our experiments we have overcome the difficulty last referred to by immersing our thermometer, together with a bulb in which pure oxygen could be liquefied, and which was connected with a manometer, in liquid air or oxygen contained in a vacuum vessel. The liquid in the vacuum vessel could, if necessary, be made to boil steadily, by passing through it a current of air or hydrogen, and could be maintained at a constant temperature by enclosing the vacuum vessel in another vacuum vessel of larger dimensions also containing liquid air. Readings of the thermometer and of the manometer connected with the bulb containing the pure liquid oxygen were taken simultaneously by two observers.

6. *The Large Constant-Volume Thermometer.*

The thermometer employed in determining the pressure coefficients of hydrogen and helium between 0° and 100° C. has been fully described in the previous memoir (Part I.). The same instrument, and three others of smaller dimensions, were

employed in these researches, the larger thermometer being only used to determine four points on the vapour-pressure curve for liquid oxygen on the hydrogen scale as a check on the measurements made by means of the smaller instruments. It will be remembered that in the large instrument the mercury column and dead-space were enclosed in a water-jacket, and the pressure was measured by means of a scale of special construction, which formed the first surface of the water-jacket.

In these measurements the bulb of the thermometer and about 20 millims. of the stem were immersed in a liquid, consisting mainly of oxygen, contained in a vacuum vessel holding about 450 cub. centims., which was enclosed in a larger vacuum vessel containing a little liquid air. Beside the thermometer was placed, as in fig. 1, a glass tube *m*, in which pure oxygen, obtained by heating potassium permanganate, could be liquefied. This tube communicated with the pump, with the apparatus for generating the oxygen, and with a manometer of the type shown in fig. 2. During an experiment a stream of hydrogen was passed through the liquid in the inner vacuum vessel, to prevent superheating, and to stir it thoroughly.

In calculating the results it was possible to consider the whole of the stem from the top of the bulb to the level of the top of the vacuum vessel as at the temperature of liquid air. The portion not immersed in the liquid was only about 30 millims. long, and the error so introduced would be considerably less than 0.01° C. The coefficient of expansion of the glass between the freezing-point of water and the temperature of liquid air was taken as 0.0000218, a number which we obtained by actual experiment (p. 138). The thermometer was filled with pure dry hydrogen by the method described in the second appendix to Part III. of this paper.

The freezing-point of water on the constant-volume hydrogen scale is taken as 273.03, the reciprocal of the pressure coefficient 0.00366255.

The formula employed in calculating the results is as follows :—

$$P_0 \left\{ \frac{V}{273.03} + \frac{V_s}{273 + T_s} + \frac{V_{ds}}{273 + T_{ds}} \right\} = P \left\{ \frac{V(1 - \alpha(273 - T))}{T} + \frac{V_s}{273 + T'_s} + \frac{V_{ds}}{273 + T'_{ds}} \right\}$$

where P_0 is the pressure on the gas when the bulb is surrounded with melting ice ;
 P is the pressure on the gas when the bulb is at the temperature to be measured ;

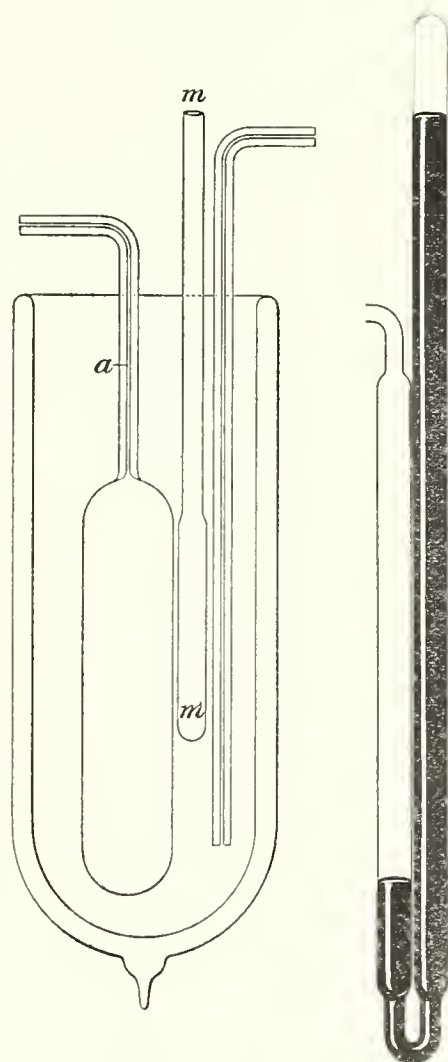


Fig. 1.

Fig. 2.

V is the volume of the bulb to a (fig. 1), at 0°C. ; V_{ds} is the volume of the dead-space and the portion of the stem within the water-jacket; V_s is the volume of the remaining portion of the stem; α is the coefficient of expansion of glass; T is the temperature to be measured; $T_s T'_s$ and $T_{ds} T'_{ds}$ are the temperatures of the stem and dead-space respectively in degrees Centigrade. The left-hand side of the equation gives the value of the constant for the thermometer.

The Ice-Point.

Volume of bulb at 0°C.	94.2096 cub. centims.
„ „ dead-space	0.3904 „ „
„ „ stem	0.1031 „ „
Pressure on gas corrected to 0°C.	987.00 millims.
Temperature of dead-space	11.29°C.
„ „ stem	17.4°C.
Constant	342.235.

The Vapour Pressures of Liquid Oxygen.

Volume of bulb at 86.5 abs.	93.8336.
-------------------------------------	----------

Pressure on gas in thermometer, corrected to 0°C.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	From curve (p. 151).
	$^\circ \text{C.}$	$^\circ \text{C.}$	millims.	abs.	abs.
315.17	11.1	17.1	514.5	85.63	86.54
315.26	11.1	17.1	516	86.66	86.56
315.40	11.1	17.1	518.5	86.70	86.61
315.59	11.1	17.1	519.5	86.75	86.64

The figures in the last columns are obtained from the smoothed results of the measurements of the vapour pressure of oxygen obtained by means of the small thermometer. There appears to be a constant difference of 0.1°C.

In discussing the accuracy of the result it is necessary to point out first of all that it is extremely difficult to maintain a large mass of liquid air or oxygen in a steady state of ebullition; and for this reason it is probable that the results obtained by means of the smaller thermometers are the more accurate. In the case of liquid air we have to deal not only with the difficulty which arises from the superheating of the liquid, but, as the liquid evaporates, its temperature rises rapidly. Liquid oxygen is not easy to obtain in large quantities, and it is extremely difficult to make it boil steadily: it may, in fact, become superheated to the extent of a whole degree Centigrade.

In the determination of the ice-point the pressure could be measured without difficulty to 0.02 millim., the temperature of the dead-space to $0.01^{\circ}\text{C}.$; as has been pointed out in Part I. of this paper, the accuracy of this measurement was of the order of 1 part in 20,000. The errors are thus confined to the measurement of the temperature of the liquid oxygen. In this part of the experiment the pressure could not be measured with certainty to 0.05 millim., though the error certainly never exceeded 0.1 millim., which is equivalent to $0.03^{\circ}\text{C}.$ Such errors as exist must be attributed to the unequal heating of thermometer bulb, and of the bulb containing the liquid oxygen.

7. *The Smaller Constant-Volume Thermometers.*

As has already been stated, the use of a thermometer with a large bulb for the measurement of low temperatures is open to serious objections. In investigating the whole range of vapour pressure of liquid oxygen and of liquid hydrogen, three instruments were used with bulbs of capacities corresponding to 12, 26 and 27 cub. centims. These instruments (fig. 3) did not differ in any important particular from the large thermometer already described, but as it was not necessary to determine the pressure with a degree of accuracy greater than 0.1 millim., the manometer column g , which was 7 millims. in diameter, was not jacketed with water, and the distance between the surface of the mercury in g and in the dead-space d was read directly on a glass scale h , placed behind the instrument, by means of two telescopes; the readings were corrected to $0^{\circ}\text{C}.$ The temperature of the dead-space d , and of the stem as far as c , was measured by means of a thermometer placed close beside d . The temperature of the manometer column was taken at the mean between the temperature of the dead-space and the temperature indicated by a thermometer placed at the level of the surface of the mercury in the manometer tube g .

The thermometer bulb d , which had a capacity of 12, 26 or 27 cub. centims. in the three thermometers, was sealed to the stem as indicated in the figure, so that if any mercury were accidentally introduced into the bulb, it could be completely removed from it by simply opening the stop-cock f and lowering the reservoir connected with it.

In making a measurement of temperature by means of a constant-volume gas thermometer, the temperature of the bulb a and of the dead-space d are supposed to be constant; and when the latter is measured by means of a mercury thermometer placed beside it, the former can be calculated from the pressure exerted by the gas. The temperature of the vertical portion of the stem c is, however, uncertain, and is usually taken to be the mean of the extreme temperatures. In measuring very low temperatures, the density of the gas in the dead-space becomes so low that its mass becomes nearly negligible, while the amount of gas in the stem becomes sufficiently

large to make it important to determine its temperature to within a few degrees, particularly when, as in our experiments, it is necessary to employ a small thermometer bulb.

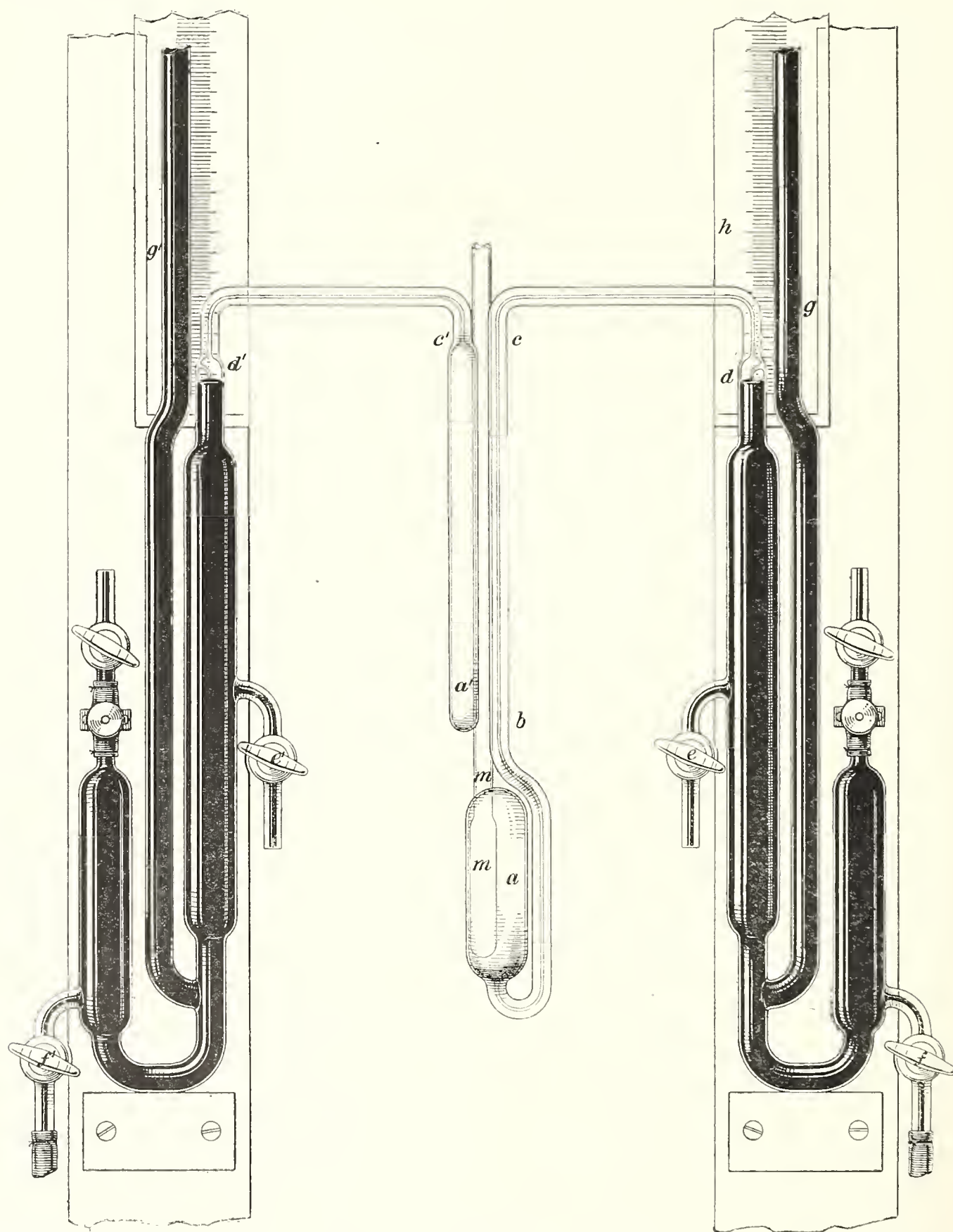


Fig. 3.

To measure the temperature of this portion of the stem we employed the device which is shown in the figure. A second thermometer similar to the one already described, with a cylindrical bulb a' of the same length as the portion of the stem bc , was mounted opposite the main thermometer. From the readings of this auxiliary thermometer, taking the temperature of the dead-space and mercury column to be the same in the case of the main thermometer, the mean temperature of the stem could be calculated with sufficient accuracy for our purpose. This device has already been applied by other investigators to similar measurements. The stem of the main and auxiliary thermometers passed at c through a rubber stopper not shown in the figure. A tube mm , terminating in a bulb at the lower end, also passed through the rubber stopper, and communicated with a manometer of the type shown in fig. 2, and with an apparatus for generating pure oxygen.

The volume of the bulb and stem to the point b , of the stem from b to c , and of the remainder of the stem and dead-space, were determined by the methods described in Part I. of this paper. As in the previous case, the stem was divided in millimetres between b and c .

8. Calibration of the Thermometers.

Calibration of the stem.—The volume of one division of the stem was determined before the bulb was sealed to it, by introducing into it a thread of mercury which was measured and subsequently weighed.

Calibration of the dead-space.—The volume of the dead-space to the zero point C on the stem was determined by the method described in Part I. of this paper p. (114). A capillary stop-cock was sealed to the tube which joined the lower part of d . Mercury was introduced through the stop-cock by means of a rubber tube and mercury reservoir, till it was brought into contact with the point. The stop-cock was then closed, the rubber tube removed, and the mercury run out of the apparatus and weighed. The rubber tube was then re-attached, and, by raising the reservoir, mercury was introduced so as to fill the dead-space and stem to the zero point c on the scale. This quantity of mercury was weighed as before.

Volumes of the Bulbs.

	A.	B.		C.	
		I.	II.	I.	II.
Weight of bulb filled with water to mark on stem . .	47·5376 (10°·2)	56·3542 (21°·8)	56·3804 (15°·6)	50·0494 (19°·3)	50·0570 (17°·8)
Weight of bulb filled with water to mark on stem . .	35·3322	30·4265	30·4253	23·3048	23·3048
Weight of air in bulb . . .	0·0147	0·0380	0·0280	0·0289	0·0289
„ water in bulb . . .	12·2183	25·9557	25·9959	26·7157	26·7233
Volume of bulb	12·2217 (10°·2)	26·0110 (21°·8)	26·0075 (15°·6)		
„ „ at 0° C. (Coef. of exp. of glass = 0·000028)	12·218	25·9948	25·9959	26·7444	26·7455

Volumes of the Stems.

Bulb A. 92.5 millims. of stem, 0.2865 gramme of mercury ; 1 millim. = 0.00319 gramme
103 „ „ 0.3300 „ „ 1 „ = 0.00320 „
1 millim. „ = 0.000232 cub. centim.
Bulb B. 168.8 millims. „ = 0.65529 millim.
1 millim. „ = 0.000286 cub. centim.
Bulb C. The stem was of the same diameter as that of bulb B.

Volume of the Dead-Space.

Mercury from stop-cock to mark on stem.	Mercury from stop-cock to glass point.	Mercury in dead-space.
Bulb A. 30.6650	26.4225	4.2425
30.6691	26.4225	4.2425
30.6614	26.4225	4.2389
Mean volume of dead-space, 0.313.		
Bulb B. The volume of the dead-space to the mark on the stem was 0.312 ; it is unnecessary to give further details.		
Bulb C. The volume of the dead-space was the same as in bulb C.		

In calculating the results, the thermometers were considered as divided into three sections.

	A.	B.	C.
	cub. centims.	cub. centims.	cub. centims.
The bulb and stem to the level of the bottom of the auxiliary thermometer. Volume	12.216	25.976	26.714
The stem for the length corresponding to that of the bulb of the auxiliary thermometer. Volume .	0.035	0.044	0.044
The remaining portion of the stem and dead-space. Volume.	0.280	0.287	0.287

The Volume of the Auxiliary Thermometer.

The volume of the bulb and dead-space were determined by the methods already referred to. As great accuracy was not required, details need not be given.

Volume of bulb to *c* (fig. 3) 2.124 cub. centims.
Volume of dead-space and stem 0.204 „

9. *Calculation of the Results.*

The results were calculated by means of the formula :

$$P_0 \left\{ \frac{V + V_s}{273.03} + \frac{V_{ds}}{273 + T_{ds}} \right\} = P \left\{ \frac{V (1 - \alpha (273 - T))}{T} + \frac{V_s}{273 + T_s} + \frac{V_{ds}}{273 + T'_{ds}} \right\},$$

Where P_0 is the pressure on the gas when the bulb and stem were immersed in ice,

P is the pressure on the gas when the bulb is at the temperature to be measured,

V is the volume of the bulb at 0°C .,

V_s is the volume of the stem from B to C,

V_{ds} is the volume of the dead-space,

α is the coefficient of expansion of the glass,

273.03 is the melting-point of ice on the gas scale, the reciprocal of the pressure coefficient of the gas with which the thermometer is filled,

T is the temperature to be measured,

T_{ds} and T'_{ds} are the temperatures of the dead-space in degrees Centigrade,

T_s is the temperature of the stem calculated from the readings of the auxiliary thermometer.

In calculating T_s , the following formula was employed :

$$P_0 \left\{ \frac{V}{273} + \frac{V_{ds}}{273 + T_{ds}} \right\} = P \left\{ \frac{V}{T_s} + \frac{V_{ds}}{273 + T'_{ds}} \right\}.$$

Since for each series of observations the ice-point remained constant, the left-hand side of each equation could be expressed by a constant for each filling of the thermometer.

10. *Vapour Pressures of Liquid Oxygen on the Constant-Volume Hydrogen Scale.*

(I.) March, 1901. Thermometer A (12 cub. centims. bulb).

Ice-Point of Thermometer.

Pressure on gas (corr.).	Temperature of dead- space.	Constant.
millims. 844.65	$^\circ \text{C}$. 15.30	38.7114

The constant for the auxiliary thermometer was in this experiment determined by measuring the pressure when the bulb and dead-space were at the same temperature

Pressure on gas.	Temperature of bulb and dead-space.	Constant.
millims. 824.3	° C. 18.0	6.59

Vapour Pressures of Liquid Oxygen.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	Calculated.
millims.	° C.	° H. scale.	millims.	° H. scale.	° H. scale.
266.9	16.60	132	411.3	84.60	84.62
267.25	16.80	132	416.6	84.70	84.73
267.80	16.80	134	424.6	84.86	84.88
268.30	16.82	136	430.5	85.04	84.99
268.40	16.83	136	432.6	85.07	85.04
269.10	17.00	139	443.8	85.20	85.26

(II.) Thermometer A, refilled with hydrogen.

Ice-Point, March 19th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	° C.	
Main thermometer	974.1	15.9	44.651
Auxiliary thermometer. . . .	887.5	15.0	7.53

Vapour Pressures of Liquid Oxygen.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	Calculated.
millims.	° C.	° H. scale.	millims.	° H. scale.	° H. scale.
(a) March, 20th, 1902.					
323·0	16·6	140	664·2	88·79	88·84
323·5	16·6	142	668·9	88·93	88·92
325·2	16·8	149	703·6	89·39	89·38
325·5	16·8	152	708·3	89·48	89·44
(b) March 21st, 1902.					
293·95	16·6	136	252·1	80·78	80·77
294·45	16·7	141	258·4	80·91	80·95
294·85	16·8	146	260·3	81·02	81·03
(c) March 24th, 1902.					
303·7	12·9	119	356·7	83·44	83·46
304·0	13·1	121	359·4	83·52	83·52
304·4	13·3	125	364·7	83·63	83·64
309·3	14·1	158	427·2	85·01	84·98
309·45	14·2	159	429·4		
309·7	14·3	159	432·5		
(d) March 25th, 1902 (Morning).					
301·7	14·6	143	330·1	82·75	82·75
301·1	14·6		324·9		
301·2	14·7		325·4		
(e) March 25th, 1902 (Afternoon).					
309·7	13·4	121	434·5	85·18	85·19
309·9			438·4		
310·15			445·4		

In this set of experiments a little mercury was accidentally introduced into the bulb of the auxiliary thermometer, which was taken as being somewhat shorter and of smaller capacity.

(III.) Thermometer B (26 cub. centims. bulb), filled with hydrogen, June 2nd.

Ice-Point of Thermometer.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	° C	
Main thermometer	952·55	17·75	91·717
Auxiliary thermometer . . .	954	17·75	8·093

Vapour Pressure of Liquid Oxygen.

June 3rd, 1902.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressure of liquid oxygen.	Temperature	
				Found.	Calculated.
millims. 293·75	° C. 19·1	° H. scale. 98	millims. 346·7	° H. scale. 83·24	° H. scale. 83·23

This result is the mean of two very concordant observations.

11. *The Vapour Pressures of Liquid Oxygen on the Constant-Volume Helium Scale.*

(I.) Thermometer A (12 cub. centims. bulb).

The helium, which had been obtained by heating clèveite, was purified by sparking with oxygen, and was afterwards passed through liquid oxygen condensed in a bulb and cooled to below -200° C. The gas was passed through a bulb immersed in liquid air on its way into the thermometer.

Ice-Point of the Thermometer.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	° C.	
Main thermometer	966·9	14·6	44·327

The auxiliary thermometer was still filled with hydrogen.

Vapour Pressure of Liquid Oxygen.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	Calculated.
millims.	° C.	° He. scale.	millims.	° He. scale.	° He. scale.
(a) March 13th, 1902.					
324·5	18·2	199	764·0	90·36	90·25
326·45	18·3	208	685·5	89·22	89·22
(b) March 17th, 1902.					
322·9	17·8	165	677·5	89·10	89·12
323·2	17·8	169	682·1	89·18	89·18
323·5	18·0	176	686·7	89·27	89·25
323·6	18·0	180	688·2	89·29	89·28

(II.) Thermometer A.

The thermometers were filled with helium purified by passing it through a coil immersed in liquid hydrogen at its boiling-point (Appendix III.).

Ice-Point of Thermometers, April 17th and March 25th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	° C.	
Main thermometer { March 25th	976·94	18·9	44·769
{ April 17th	976·42	15·5	44·756
Auxiliary thermometer	996	16	7·73

Vapour Pressure of Liquid Oxygen.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	Calculated.
millims.	° C.	° He. scale.	millims.	° He. scale.	° He. scale.
324·75	20·0	175	675·5	89·10	89·09
324·75	20·4	167	677·0	88·96	89·13

(III.) Thermometer B (26 cub. centims. bulb).

The thermometers were filled with helium purified by passing it through a coil cooled to $15^{\circ}\cdot 5$ by means of liquid hydrogen boiling under reduced pressure (Appendix III.).

Ice-Point of Thermometer, June 4th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	$^{\circ}$ C.	
Main thermometer	939·95	17·35	90·503
Auxiliary thermometer . . .	954	17·3	8·093

Vapour Pressure of Liquid Oxygen

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	Calculated.
millims.	$^{\circ}$ C.	$^{\circ}$ He. scale.	millims.	$^{\circ}$ He. scale.	$^{\circ}$ He. scale.
288·1	17·7	98	322·0	82·73	82·74
288·7	18·0	100	328·0	82·90	82·89

(IV.) Thermometer C (26·7 cub. centims. bulb).

The helium was from the same sample as was used in the last experiment.

Ice-Point of Thermometer, June 11th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	$^{\circ}$ C.	
Main thermometer	922·25	16·90	91·204

The auxiliary thermometer was not refilled.

Vapour Pressure of Liquid Oxygen.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid oxygen.	Temperature	
				Found.	Calculated.
millims.	° C.	° He. scale.	millims.	° He. scale.	° He. scale.
*307·70	16·0	103	750·5	90·11	90·08
†307·94	16·5	105	755·5	90·18	90·15
305·00	17·1	98	692·0	89·32	89·32
300·45	17·1	98	600·5	87·99	88·02
291·95	17·0	89	449·0	85·47	85·46
301·05	17·0	97	608·5	88·17	88·15
292·30	16·8	78	456·0	85·58	85·58

In this experiment the vacuum vessel surrounding the thermometer bulb, &c., was first of all filled with nearly pure liquid oxygen. The liquid was made to boil steadily by passing a current of gaseous oxygen into the liquid. Liquid air was added to the liquid oxygen to obtain the lower temperature. Between the first and second observations the manometer and bulb containing the pure oxygen were exhausted and refilled.

12. *Treatment of the Results.*

The figures given in the last column of the foregoing tables are the temperatures, corresponding to the observed pressures, taken from the smoothed vapour pressure curves shown in Plate 1. It will be observed that the points obtained by direct observation lie in every case very close to the curve, and that the difference between the observed and calculated temperatures rarely exceeds two or three hundredths of a degree.

The results obtained by direct observation of the vapour pressures were smoothed by the method of RAMSAY and YOUNG ('Phil. Mag.' 1886, vol. 21, p. 33; vol. 22, p. 37). This method consists in calculating the ratios of the absolute temperatures T_a , T_b , T'_a , T'_b , &c., for any pair of substances A and B corresponding to vapour pressures p , p' , &c., and plotting the ratios T_a/T_b , T'_a/T'_b , &c., against the temperature T_a , T'_a , as rectangular coordinates. The points so defined lie on a straight line, from which the temperature corresponding to any pressure p for the substance B can be calculated, by first finding the temperature T_a corresponding to that vapour pressure for the substance A, determining the value of the ratio T_a/T_b , and dividing the value of T_a by it. In smoothing our results we took water as the second substance A.

Two sets of ratios were obtained by this method, corresponding to the tempera-

* The mean of two observations.

† The mean of four observations.

tures of liquid oxygen on the hydrogen and helium scales. When the results were plotted against the absolute temperatures of water two parallel straight lines were obtained. From these lines the smoothed values of the ratios corresponding to those temperatures at which the vapour pressures of water have the values expressed in the following table were determined, and from them the corresponding temperatures of liquid oxygen on the scale of the two thermometers were calculated (see Plate 1).

13. *The Vapour Pressures of Liquid Oxygen.*

Pressures in millimetres of mercury.	Temperatures on the hydrogen scale.	Temperatures on the helium scale.
800	90°60	90°70
790	90°47	90°57
780	90°35	90°45
770	90°23	90°33
760	90°10	90°20
750	89°97	90°07
740	89°85	89°95
730	89°71	89°81
720	89°58	89°68
710	89°46	89°56
700	89°33	89°43
650	88°65	88°75
600	87°91	88°01
550	87°13	87°23
500	86°29	86°39
450	85°37	85°47
400	84°39	84°49
350	83°31	83°41
300	82°09	82°19
250	80°70	80°80
200	79°07	79°17
150	77°07	77°17

14. *Discussion of the Results.*

When the results which are tabulated in the preceeding table are plotted on a diagram, it will be observed that the vapour pressures of liquid oxygen on the scale of the two thermometers are expressed by two curves, and that the temperature corresponding to any particular pressure is always 0·1° higher on the helium scale than on the hydrogen scale. As we shall show later (Part III., p. 169), the divergence becomes still greater at lower temperatures.

Though the pressure coefficients for hydrogen and helium between the melting-point of ice and the boiling-point of water do not appear to differ appreciably, and though at the normal temperature these gases may be considered as nearly perfect, it

is not surprising that they exhibit a difference at lower temperatures. It must be remembered that while the critical point of hydrogen is about 35° abs., it is probable (Part III., p. 177) that the critical point of helium does not lie far from 10° abs. The temperature of liquid air, about 85° abs., expressed as a multiple of critical temperature of the gas, is 8 with regard to helium, and 2.5 with regard to hydrogen, which is at that temperature in a state corresponding to that of oxygen, of which the critical point is 150° abs., at the boiling point of water. At this temperature we have no reason to believe that oxygen behaves in any way as a perfect gas.

Further, as was first shown by WROBLEWSKI (*loc. cit.*, p. 136), and later by two of us (TRAVERS and SENTER, 'Brit. Assoc.' 1901), the coefficient of expansion of hydrogen at constant pressure between the normal temperature and the temperature of liquid air increases rapidly with rise of pressure. This tends to confirm the result which we now bring forward, *that at an initial pressure of 1 metre of mercury at the melting-point of ice, the constant volume hydrogen and helium scales differ by 0.1 at the boiling-point of liquid oxygen.* At lower initial pressures the difference between the two scales of temperature might possibly be smaller.

15. *The Probable Accuracy of the Results.*

It may at once be pointed out that in experiments of this kind the probable accuracy cannot be arrived at by any method of calculation. So far as the actual measurements of temperatures and pressures are concerned, they may easily be made to 1 in 10,000. In discussing the actual experiments we have, however, to consider such sources of error as may be due to the superheating of the liquid of which the vapour pressure is being determined; inequalities in the temperature of the liquid in which the thermometer was immersed; and errors due to impurities in the gases and liquids employed.

It will be observed that in the course of our work we have employed four different thermometers, the large thermometer used for determining the pressure coefficients of the gases between 0° and 100° C., and three smaller instruments of which the bulbs had capacities of 12 cub. centims. (A), 26 cub. centims. (B), and 26.7 cub. centims. (C), respectively. Though the large thermometer should give the most accurate results, and, indeed, differentiation of the equations employed in calculating the results indicates that the errors of measurement of a steady temperature of about 85° abs. should be accurate to 0.005, it appears to be impossible to maintain so large a thermometer bulb at a uniform and steady temperature. We have therefore considered that the results obtained by means of the smaller thermometers, which are very concordant among themselves, and differ only by 0.1° from the results obtained by means of the large thermometer, are the most accurate. The values of the boiling-point and vapour pressures of liquid hydrogen determined by means of

the same thermometers show an even closer agreement; this, as we shall presently show, is due to the fact that liquid hydrogen, unlike liquid oxygen, does not tend to become superheated.

It is a somewhat remarkable fact that pure oxygen when liquified can only with difficulty be made to boil steadily. By passing a rapid current of oxygen or air through the liquid a fairly steady temperature may be maintained, but if the current is stopped the temperature may rise more than one degree. This probably accounts for the fact that the temperatures found by us are in every case somewhat lower than those of other observers, who, without exception, measured the pressure on the mass of liquid in which the thermometer was immersed. This source of error we have, however, taken great pains to eliminate; for the accuracy of the actual measurements, the concordance of the observation made by means of the different thermometers furnishes a sufficient guarantee.

PART III.

On the Vapour Pressures of Liquid Hydrogen at Temperatures below its Boiling-Point on the Constant-Volume Hydrogen and Helium Scales.

By MORRIS W. TRAVERS, *D.Sc.*, *Fellow of University College, London,*
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1. *Previous Investigations.*

THE boiling-point of liquid hydrogen has been measured directly by DEWAR and indirectly by OLSZEWSKI. The former employed constant-volume hydrogen and helium thermometers, the latter, however, containing a mixture of helium and neon, obtained by a method which will be discussed later (p. 171). The results are as follows (*Proc. Roy. Soc.* Feb. 1901, vol. 68, p. 40):—

	Pressure of the gas in thermometer at 0° C.	Temperature of liquid hydrogen.	Atmospheric pressure.
	millims.	° C.	millims.
Hydrogen thermometer (a)	286·6	− 253·03	760·6
„ „ (b)	269·8	− 253·37	764·4
„ „ (c)	127·0	− 250·35	759·5
„ „ (d)	739·0	− 252·81	770·5
Helium thermometer (a)	728·0	− 252·68	765·0
„ „ (b)	728·0	− 252·84	770·0

The boiling-point of liquid oxygen was measured in every case as a check on the results.

The results obtained by means of the hydrogen thermometer filled under a pressure of 739 millims. of mercury differ from the mean of our observations by $0^{\circ}\cdot 1$. The difference between this result and those obtained in experiments (*a*), (*b*) and (*c*) is certainly due to experimental error, since the temperatures determined by means of a gas thermometer should rise when the pressure on the gas in the thermometer is reduced. Since the pressure on the gas in the thermometer was in each of these experiments very small, the experimental error was proportionally large, and much greater weight must be attached to the last measurement.

The results obtained by means of the helium thermometer are lower than those obtained by us, though the difference is not constant.

Pressure.	Temperature.		
	DEWAR.	T. and J.	Δ .
	$^{\circ}\text{C.}$		
770	$-252^{\circ}\cdot 84$	$-252^{\circ}\cdot 57$	$0\cdot 27$
765	$-252^{\circ}\cdot 68$	$-252^{\circ}\cdot 60$	$0\cdot 08$

OLSZEWSKI ('Phil. Mag.' 1895 (5), vol. 39, p. 199; vol. 40, p. 202) employed the following method: Hydrogen was compressed to 170 atmospheres into a steel cylinder, with a loose glass lining enclosing the platinum coil of a resistance thermometer, of which the resistance had been determined at 0° , -78° , $-182^{\circ}\cdot 5$ and -208°C. The cylinder was cooled in liquid air, boiling *in vacuo*, and the gas was then allowed to escape from the cylinder. When the pressure fell to one atmosphere the resistance of the coil was measured, and the temperature determined by linear extrapolation. As the result of numerous trials, he found the temperature to be $-243^{\circ}\cdot 5$.

Though the method gave accurate results for the boiling-point of oxygen, the results are of no value so far as the temperature of liquid hydrogen is concerned, as the variation of the resistance of platinum with the temperature is not linear.

Beyond determining the boiling-point of liquid hydrogen on the scales of the two thermometers, DEWAR ('Roy. Soc. Proc.' vol. 68, p. 360) finds the melting-point on the helium scale to be 16° : an earlier determination by him gives the melting pressure as 55 millims. (T. and J. find $14^{\circ}\cdot 1$ and 49 millims. respectively, p. 170.) He also gives the following values for the temperatures of solid hydrogen corresponding to very low pressures (*loc. cit.*):—

	Thermometer 2.	Thermometer 3.
Vapour pressures	764·4 millims., 30–40 millims.	759·5 millims., 30–40 millims.
Temperatures	– 253·37° C., – 258·66° C.	– 250·35° C., – 255·67° C.

As has already been pointed out (p. 156), the pressure on the gas in these thermometers was very small, and little weight can be attached to the results.

2. *The Liquefaction of Hydrogen.*

The liquid hydrogen employed in these researches was obtained by means of an apparatus devised and constructed by one of us two years ago. The apparatus as it was originally designed, has been described in the 'Philosophical Magazine' (1901. vol. 17, 411), and such slight modifications as have since been introduced are referred to in the treatise, TRAVERS on 'The Experimental Study of Gases,' p. 206. The hydrogen was obtained by the action of dilute sulphuric acid upon commercial granulated zinc, contained in a cylindrical lead vessel, of which the joints had been soldered in the oxyhydrogen blow-pipe. The gas was purified by passing it first through a solution of potassium permanganate; through two glass towers, each $1\frac{1}{2}$ metres in length, containing broken pumice over which a sulphuric acid solution of potassium bichromate ran continuously; through a third glass tower containing pumice kept moist with silver nitrate solution; and finally, through a wash bottle containing caustic potash solution, into the gasometer. In this way it was possible to obtain hydrogen from which but little solid separated during the process of liquefaction.*

In one experiment in which we failed to obtain liquid hydrogen, owing to the formation of solid air in the vacuum vessel enclosing the regenerator coil, we found that the packing of the low-pressure piston of the compressor had become so worn that air was being taken into the cylinder with the hydrogen. With compressors of the "Whitehead" type, in which the piston-rod does not pass through a gland, and the compression is only affected by the inward motion of the piston, special care must be taken that the piston packing or cup-fibre fits the cylinder.

In connection with the work on liquid hydrogen, we wish to express our thanks to Mr. HOLDING, Mechanic in the Chemical Department of University College, for his invaluable services.

* In connection with the preparation of the hydrogen, a curious observation was made which has not been explained. The silver nitrate solution which escaped at the bottom of the tower appeared almost immediately to deposit pure silver in the form of small bright metallic crystals. Very little silver appeared to form inside the tower, and it seems as if some compound were produced which decomposed in contact with the air. The hydrogen had already passed through two towers containing acid chromate solution, and it is possible that hydrogen, like oxygen, becomes more active when it takes part in a slow chemical reaction.

3. *The Manipulation of Liquid Hydrogen.*

In each experiment we employed about 400 centims. of liquid hydrogen, which, when sufficient liquid air (about 8 litres) had been accumulated to cool the apparatus, could be obtained in less than 1 hour from the time of commencing the operations.

Though perhaps it is little to be expected, it is a fact that it is much easier to make measurements of the boiling-point and vapour pressures of liquid hydrogen than of liquid oxygen. As we have already stated (p. 140), liquid air tends to become superheated, and only with difficulty can it be made to boil steadily. Liquid hydrogen, on the other hand, can be made to boil quite steadily at any temperature between its boiling-point and melting-point. Possibly this may be due to the presence of finely divided particles of solid air suspended in it, for it does not appear that the phenomenon of superheating is dependent upon any intrinsic property of a liquid.

That the value of the latent heat of vaporisation is very high (DEWAR, 'Roy. Soc. Proc.,' June 13th, 1901, vol. 68, 361) is also in favour of the experimenter; for as the boiling-point of hydrogen is more than sixty degrees lower than the temperature of liquid air, the operation of cooling a thermometer bulb, previously cooled in liquid air, to the boiling-point of liquid hydrogen, necessitates the absorption of a considerable quantity of heat.

When the vacuum vessel which received the liquid hydrogen from the liquefier was filled with the liquid it was at once removed from the apparatus, plugged with animal wool, and enclosed in an outer vessel containing liquid air. The mouth of a vessel containing liquid hydrogen cannot be left open, as in that case air enters the vessel, solidifies, and rapidly evaporates the liquid. A plug of natural wool is preferable to a cork or rubber stopper, for the interstices of wool become filled with cold vapour, and the plug acts as an excellent insulator, whereas the heat radiated from the solid stopper helps to evaporate the liquid.

4. *Method of Determining the Vapour Pressures of Liquid Hydrogen.*

The thermometers employed in these researches were identical with those used in determining the vapour pressures of liquid oxygen, and in nearly every case in which a measurement was made with liquid hydrogen, a point on the vapour-pressure curve of oxygen was also determined as a check upon the results. The apparatus as arranged for these experiments is shown in section in fig. 1. The thermometer has already been fully described in Part II. of this investigation, so that to simplify the description only the bulb *a* of the main thermometer, and the bulb *a'* of the auxiliary thermometer, by means of which the temperature of the stem of the main thermometer was determined, are shown in the present diagram.

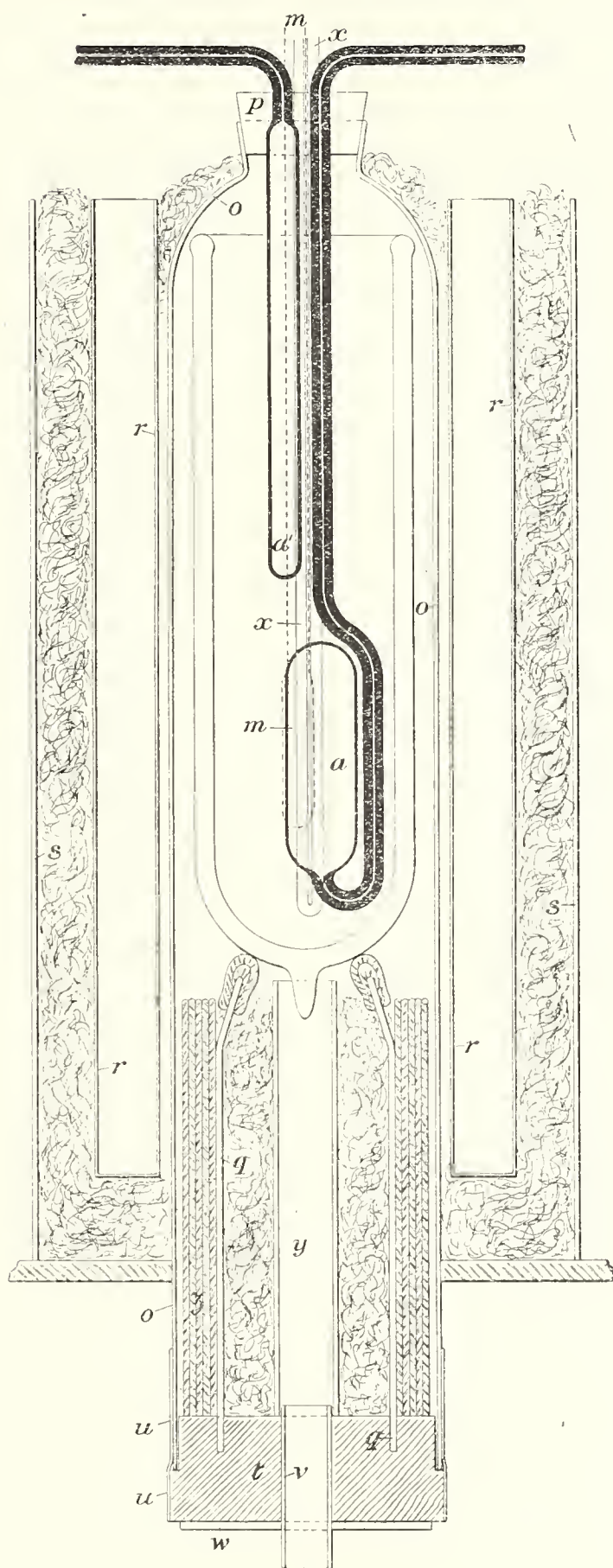


Fig. 1.

The stem of the main and auxiliary thermometers passed through two holes in a rubber stopper, *p*. Through one of two holes, pierced at right angles to the first, passed the stem of the bulb which contained pure hydrogen (pp. 138, 139) and communicated with a manometer. The capillary tube, *x*, which was sealed at its lower end, communicated with a compression apparatus containing helium, passed through the other hole. This tube was employed in an attempt to liquefy helium, which will be dealt with later. In order to make the diagram clearer, the glass walls and stem of the thermometer are blacked in, the compression tube is drawn in outline, and the tube and bulb for the pure liquid hydrogen is represented by the dotted line.

The thermometer bulb, &c., was enclosed within a wide glass tube, *o*, 75 millims. in diameter and 400 millims. long. This tube was open at the bottom for the introduction of the vacuum vessel containing the liquid hydrogen, and at the top was formed into a neck to fit the rubber stopper, *p*. Surrounding this tube for a considerable part of its length was an annular zinc vessel, open at the top, which could be filled with liquid air. This vessel was in turn surrounded with natural wool enclosed within a jacket of linoleum. The diameter of the linoleum jacket was such that it exactly filled the space between the vertical wooden supports to which the main and auxiliary thermometers were fixed (p. 141). The liquid-air jacket was not essential to the working of the apparatus, even when the liquid hydrogen in the vacuum vessel was boiling under reduced pressure at the extremely low temperature of 14° abs. It was thought that by using it an economy in the liquid hydrogen would be effected, but it is impossible to determine whether this was actually the case. Wool was packed into the top of the tube, *o*, and between the tubes which projected downwards through the rubber stopper, to shield off radiation from above.

The vacuum vessel containing the liquid was inserted from the lower and open end of the tube, *o*, so as to surround the thermometer bulb, and was kept in place by the arrangement shown in the figure. A stopper, *t*, made of boxwood, was turned on the lathe to fit the open end of the tube, *o*, and an air-tight junction between the two was made by means of a rubber sleeve, *u*. The sleeve was fixed permanently to the stopper, and the upper part of it could be rolled back, and then slipped over the tube, *o*, when the stopper was in position. A brass tube, *v*, passing through the stopper, could be connected with the exhaust pump by means of a rubber connection and lead tube; as the brass tube was soldered to a heavy brass plate, *w*, which was screwed to the outer surface of the stopper, it never became cold, and a good junction with the rubber tube was maintained during the experiment. It may be remarked here that in carrying out experiments at low temperatures, it is impossible to make air-tight junctions between frozen rubber and metal on account of the greater contraction of the latter.

The vacuum vessel was supported on four wires, *qq*, let into the wooden stopper and covered at the top with pads of cloth. Wire rings were soldered to them at

intervals, and round the cage so formed baize, z , was wound to form a plug a little smaller than the inner diameter of the tube, o . A glass tube, y , served as a continuation of the tube, o , and the space between y and z was packed with animal wool. By means of this plug the vacuum vessel was effectually shielded from radiation from below.

The method of filling the thermometer bulb has already been dealt with in a previous section of the work (p. 117). The ice point of the thermometer was of course determined before the tube, o , was placed in position.

Before commencing an experiment with liquid hydrogen a vacuum vessel containing liquid air was first introduced into the tube, o , and temporarily held in position by means of the plug; in this manner the bulb of the thermometer was cooled down to -185°C . When the vessel containing the liquid air was again lowered, the wool was removed from the mouth of the vacuum vessel containing the hydrogen, which was brought below the apparatus, and slowly raised into position by means of the plug. The rubber sleeve, u , was then turned upwards over the tube, o , and secured in position by means of a piece of wire.

The actual measurement of the vapour pressure was made in the same manner as has been described in the case of oxygen (p. 138). One observer read the pressures indicated by the manometer connected with the tube, m , in which pure hydrogen was liquefied, while the other took simultaneous readings of the main thermometer, of the auxiliary thermometer, and of the temperature of the dead space. So steadily did the hydrogen boil under normal pressure that this operation could be carried out without the least difficulty. It is noticeable that the vapour pressure of the pure hydrogen in the bulb, m , was always slightly higher than the barometric pressure. This is to be attributed to the presence of small quantities of impurities dissolved in the main quantity of hydrogen in the vacuum vessel, and confirms our reasons for adopting this method of measuring the vapour pressures.

In determining the vapour pressure of liquid hydrogen at lower pressure we employed the following method: The tube, v , was connected by a short rubber tube to a wide lead tube which communicated with the exhaust pump, with a large glass globe 300 millims. in diameter, with a mercury manometer, and with a fine adjustment stopcock, through which air could be admitted to the apparatus and the pressure so regulated.

The exhaust was maintained by means of a large two-cylinder 'Fleuss' pump, worked by a one horse-power electric motor. This pump, which was also used in connection with the apparatus for producing liquid hydrogen, was arranged so that the two cylinders could be worked in parallel to give a vacuum of 50 millims. of mercury, or in series to give a vacuum of less than 1 millim. of mercury. For reasons which will be entered into presently we have confined our measurements to pressure above 50 millims. of mercury. By opening and closing the fine adjustment cock, so as to admit more or less air into the apparatus, the pressure could be maintained

steady for a length of time amply sufficient for the equalisation of temperature throughout that part of the apparatus which was enclosed within the vacuum vessel.

5. *Calculation of the Results.*

The method of calculating the temperatures of liquid hydrogen was the same as has been described in Part II. of this work. It may be pointed out here that, at very low temperatures, the mass of gas in the stem of the thermometer becomes considerable, while the mass of gas in the dead space becomes nearly negligible. Taking a specific case :—

Constant for thermometer	44·769
Pressure of gas in thermometer	55·0 millims.
Volume of bulb	12·187
„ „ stem	0·035
„ „ dead-space	0·280
Temperature of stem	30° abs.
„ „ dead-space	288° „

The equation for determining the temperature T now becomes :—

$$44\cdot769 = 55\cdot0 \left\{ \frac{12\cdot187}{T} + \frac{0\cdot035}{30} + \frac{0\cdot280}{288} \right\},$$

where the second term within the bracket is equal to 0·0012, and the third term is less than 0·001. In both cases the correction is exceedingly small.

In calculating the volume of the bulb, the coefficient of expansion of glass between 0° and 253° C. was taken as 0·000019 for the following reasons :—The coefficient of expansion between 0° and 100° C. is 0·0000285, between 0° and – 185° it is 0·0000218; on extrapolation one obtains the value which we have taken for the coefficient. The following values for the volume of the bulb at – 253 are calculated for different coefficients :—

Coefficient.	Volume.
0·000019	25·850
0·000021	25·837

This difference of volume corresponds to a difference of 0·01 in the boiling-point of hydrogen.

6. *The Vapour Pressures of Liquid Hydrogen on the Constant-Volume Hydrogen Scale.*

(I.) Thermometer A (12 cub. centims. bulb).

Ice-Point of Thermometer, May 7th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
Main thermometer	millims. 953·05	° C. 15·85	43·679

The auxiliary thermometer was already filled with helium (Part II., § 11, II.), it was not refilled with hydrogen, for when the bulb of the main thermometer was cooled in liquid hydrogen boiling under reduced pressure, hydrogen might condense in the bottom of the bulb of the auxiliary thermometer.

Vapour Pressures of Liquid Hydrogen.

May 28th, 1902.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of hydrogen.	Temperature	
				Found.	Calculated.
millims.	° C.	° He. scale.	millims.	° H. scale.	° H. scale.
72·04	18·3	40	757·2	20·17	20·21
66·35	18·6	51	458·6	18·57	18·56
62·40	18·6	50	309·0	17·45	17·44
62·80	18·6	45	321·8	17·56	17·55
58·40	18·8	56	198·0	16·33	16·35

(II.) Thermometer B (26 cub. centims. bulb).

Ice-Point of Thermometer, June 4th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
Main thermometer	millims. 952·55	° C. 17·55	91·930
Auxiliary thermometer . . .	954	17·55	8·093

Vapour Pressures of Liquid Hydrogen.

June 4th, 1902.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid hydrogen.	Temperature	
				Found.	Calculated.
millims.	° C.	° H. scale.	millims.	° H. scale.	° H. scale.
71·83	18·1	27	766·6	20·28	20·25
71·83	18·2	29	766·6		

For the vapour pressure of liquid oxygen on this thermometer, see Part II., 10 (III.).

7. The Vapour Pressures of Liquid Hydrogen on the Constant-Volume Helium Scale.

(I.) Thermometer A (12 cub. centims. bulb). For preparation of helium (fraction 1), see pp. 172, 174.

Ice-Point of Thermometer, April 17th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	° C.	
Main thermometer	976·42	16·0	44·760
Auxiliary thermometer	996·5	18·9	7·73

Vapour Pressures of Liquid Hydrogen.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressure of liquid hydrogen.	Temperature	
				Found.	Calculated.
millims.	° C.	° He. scale.	millims.	° He. scale.	° He. scale.
April 18th, 1902.					
74·83	15·5	31	765·0	20·42	20·44
72·09	15·2	29	604·1	19·70	19·63
72·04	14·9	28	606·6	19·69	19·61
67·26	14·6	26	390·5	18·38	18·28
55·98	14·3	21	100·0	15·29	15·14
55·48	14·0	25	98·0	15·15	15·10
55·43	13·7	25	97·5	15·14	15·09
May 1st, 1902.					
74·75	15·6	41	759·2	20·41	20·41
68·70	15·3	33	449·0	18·76	18·70
55·20	15·2	27	96·8	15·07	15·08
55·30	15·2	27	75·8	14·55	14·67
52·90	15·1	27	69·4	14·44	—
52·20	15·1	27	62·0	14·25	14·38
55·05	14·8	35	95·3	15·01	15·06
51·30	14·7	31	55·2	14·00	14·22
55·00	14·7	91	95·6	15·01	15·05
51·90	15·0	29	59·2	14·16	14·32

For measurement of vapour pressures of liquid oxygen with this thermometer, see Part II., 11 (II.).

(II.) Thermometer B (26 cub. centims. bulb). Helium purified by passing through a coil cooled to 15°·5 abs. (see p. 174).

Ice-Point of Thermometer, June 4th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
	millims.	° C.	
Main thermometer	939·95	17·35	90·516
Auxiliary thermometer. . .	954·0	17·35	8·093

Vapour Pressures of Liquid Hydrogen.

June 13th, 1902.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid hydrogen.	Temperature	
				Found.	Calculated.
millims. 71·45	° C. 20·0	° He. 58	millims. 770·0	° He. 20·43	° He. 20·46

This result is the mean of two consecutive observations, which were nearly identical. The quantity of pure liquid hydrogen in the vapour pressure bulb (*m*, fig. 1) was increased between the two readings.

For the measurements of the vapour pressure of liquid oxygen with this thermometer, see Part II., 11 (III.).

III. Thermometer C (26·7 cub. centims. bulb). The thermometer was filled with helium from the same sample as was used in the last series of experiments.

Ice-Point of Thermometer, June 11th, 1902.

	Pressure on gas in thermometer.	Temperature of dead-space.	Constant.
Main thermometer	millims. 924·25	° C. 16·9	91·505
Auxiliary thermometer. . . .	954·0	17·3	8·093

Vapour Pressures of Liquid Hydrogen.

June 12th, 1902.

Pressure on gas in thermometer.	Temperature of dead-space.	Temperature of stem.	Vapour pressures of liquid hydrogen.	Temperature	
				Found.	Calculated.
millims.	° C.	° He.	millims.	° He.	° He.
*69·79	15·4	28	749·0	20·36	20·36
†55·45	16·1	26	169·2	16·17	16·19
†51·70	16·3	25	96·2	15·08	15·07
†50·95	16·4	24	83·0	14·86	14·82
†50·80	16·6	26	80·2	14·81	14·76
†50·55	16·6	30	76·3	14·74	14·70
53·70	16·4	33	129·8	15·65	15·63

For the measurement of the vapour pressures of liquid oxygen with this thermometer, see Part II., 11 (IV.).

8. *Treatment of the Results.—Their Probable Accuracy.*

As we shall presently find, the melting point of hydrogen is 14·1 on the helium scale, and consequently the whole of the observations recorded in the foregoing tables refer to liquid hydrogen only. As in the case of liquid oxygen (Part II., p. 151), the experimental results were smoothed by the method of RAMSAY and YOUNG, and the vapour-pressure curves for liquid hydrogen on the scales of the hydrogen and helium thermometers were plotted on a diagram as in Plate 1. The last columns of the preceding tables contain the temperatures read off the curves at points corresponding to the observed pressures.

It will be observed that at pressures near 760 millims. the points representing actual observation lie either on, or very close to, the curve. Of these points there are four on the curve representing the vapour pressures on the helium scale, obtained by means of three separate thermometers; there are two similar points on the curve representing the vapour pressures on the hydrogen scale, the results of observations with two different thermometers. We can therefore consider that the boiling-point of liquid hydrogen, which is 20°·22 on the hydrogen scale and 20°·41 on the helium scale, has been determined with a high degree of accuracy.

* The mean of two nearly coincident observations.

† In these observations the vapour pressures of hydrogen were measured on two manometers, connected with two bulbs containing pure liquid hydrogen, and immersed in the vacuum vessel surrounding the thermometer bulb to the depth of the top and the bottom of the latter respectively. The two sets of readings, of which the mean is given above, did not differ by more than 1 millim., indicating that the temperature of the thermometer bulb was uniform.

The following table shows the order of the agreement between the measurements of temperatures near the boiling-point.

Hydrogen Scale.

Thermometer.	Observed vapour pressures.	Observed temperature.	Temperature from smoothed curves.
	millims.	°	°
A	757·2	20·17	20·21
B	766·6	20·28	20·25

Helium Scale.

Thermometer.	Observed vapour pressures.	Observed temperature.	Temperature from smoothed curves.
	millims.	°	°
A	765·0	20·42	20·44
B	759·2	20·41	20·41
C	770·0	20·43	20·45
D	749·0	20·36	20·36

With regard to the measurements at lower pressures, the greatest difference between the observed and calculated temperatures in the first series of measurements with the helium thermometer exceeds 0·1. These errors can be accounted for by supposing that when the liquid hydrogen surrounding the thermometer bulb was made to boil under reduced pressure, the observations were made before the temperature of the thermometer bulb, and of the bulb containing the pure liquefied hydrogen, had become steady.

At pressures below 200 millims. it appeared to be much more easy to maintain a steady temperature than at intermediate pressures. In the last series of measurements, as has already been stated, two bulbs containing pure liquid hydrogen and connected with two manometers, were immersed at different levels in the vacuum vessel containing the liquid hydrogen surrounding the thermometer bulb, to the depth of the top and of the bottom of the thermometer bulb respectively. The pressures indicated by the manometers were read simultaneously and were never found to differ by more than 1 millim., proving that the temperature of the thermometer bulb was practically uniform.

In concluding these remarks on the accuracy of the results, it may be well to point out that the expression of differences of temperatures in degrees, particularly at low temperatures, in degrees, is somewhat apt to lead to a false conclusion. In dealing with the thermodynamic cycle, on which our idea of an absolute scale of temperature is

based, differences of temperature can only be expressed as fractions of the absolute temperature at which the processes in the cycle are performed. The accuracy of a thermometric measurement should then be as expressed not as ΔT but as $\Delta T/T$; this leads us to the conclusion that an error of $0^{\circ}01$ at the boiling-point of hydrogen is equivalent to an error of $0^{\circ}15$ at the normal temperature.

9. *The Vapour Pressures of Liquid Hydrogen.*

Pressures in millimetres of mercury.	Temperatures on the hydrogen scale.	Temperatures on the helium scale.
800	20 $\cdot$ 41	20 $\cdot$ 60
790	20 $\cdot$ 36	20 $\cdot$ 55
780	20 $\cdot$ 32	20 $\cdot$ 51
770	20 $\cdot$ 27	20 $\cdot$ 46
760	20 $\cdot$ 22	20 $\cdot$ 41
750	20 $\cdot$ 17	20 $\cdot$ 36
740	20 $\cdot$ 13	20 $\cdot$ 31
730	20 $\cdot$ 08	20 $\cdot$ 26
720	20 $\cdot$ 03	20 $\cdot$ 21
710	19 $\cdot$ 98	20 $\cdot$ 16
700	19 $\cdot$ 93	20 $\cdot$ 12
650	19 $\cdot$ 68	19 $\cdot$ 87
600	19 $\cdot$ 41	19 $\cdot$ 61
550	19 $\cdot$ 13	19 $\cdot$ 33
500	18 $\cdot$ 82	19 $\cdot$ 03
450	18 $\cdot$ 50	18 $\cdot$ 70
400	18 $\cdot$ 15	18 $\cdot$ 35
350	17 $\cdot$ 78	17 $\cdot$ 98
300	17 $\cdot$ 36	17 $\cdot$ 57
250	16 $\cdot$ 90	17 $\cdot$ 11
200	16 $\cdot$ 37	16 $\cdot$ 58
150	15 $\cdot$ 74	15 $\cdot$ 95
100	14 $\cdot$ 93	15 $\cdot$ 14
50	—	14 $\cdot$ 11

10. *Discussions of the Results.*

As has already been pointed out in Part II. of this research, the temperatures measured on the scale of the two thermometers differ by $0\cdot1$ at the temperature of liquid oxygen. It is not, therefore, surprising to find that at the temperatures of liquid hydrogen the two scales differ to an even greater extent. A glance at the vapour-pressure curves of liquid hydrogen on the helium and hydrogen scales will show that the temperatures on the helium scale lie almost exactly $0\cdot20$ above those on the hydrogen scale. Such a difference was, as we have already pointed out, to be expected (Part II., p. 152), but in the present state of our knowledge of the properties of these gases at low temperatures further discussion of these results is impossible.

APPENDIX I.—*On the Melting-Point of Hydrogen.*

DEWAR ('Roy. Soc. Proc.,' 1891, vol. 68, p. 360) states that the melting-point of hydrogen on the helium scale is 16° . The method by which the measurement was made has not been recorded; he had previously given the melting pressure as 55 millims. (see p. 106). Our experiments of the vapour pressures of hydrogen appeared to indicate that the melting-point lay considerably below the temperature given by DEWAR, for not only did the numerical results bear evidence that at temperatures close to 14° we were still dealing with liquid, but when after maintaining a pressure of 55 millims. in the apparatus for some time the vacuum vessel was found to contain liquid only. We therefore decided to carry out a separate experiment with a view to determining the exact melting-point.

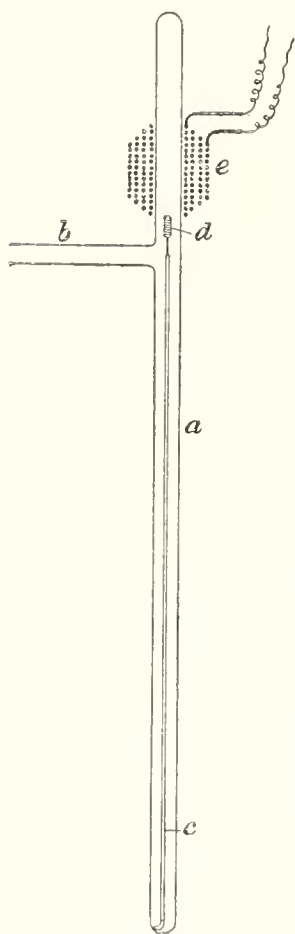


Fig. 2.

The apparatus employed in this experiment is shown in fig. 2. The tube *a*, which was 6 millims. in diameter, and 150 millims. long below the side tube *b*, passed through the rubber stopper *p* of the apparatus shown in fig. 1, so that the lower end of it lay at the level of the middle of the thermometer bulb. The tube *a* contained a very thin glass rod, *c*, bent at its lower end into a hook, and joined at the top to a piece of soft iron wire wound into a solid spiral, *d*. Round the tube *a* above the side tube a coil of insulated wire was wound forming a solenoid; by passing a current through the coil, the rod *c* with its iron head *d* could be given a vertical motion of about 3 millims.

The side tube *b* communicated with a mercury reservoir containing pure hydrogen from palladium, with a manometer of the type described in Part II. (fig. 2), and through a stop-cock with the mercury pump.

In carrying out the experiment the lower part of the tube *a* and the thermometer bulb were cooled in liquid hydrogen boiling under reduced pressure in the manner described on p. 160, and about 100 cub. centims. of gaseous hydrogen was allowed to enter and liquefy in it. At all temperatures down to $14^{\circ}\cdot 2$, which corresponded to the lowest pressure which we were able to obtain under the conditions of the experiment, the hydrogen appeared to remain liquid, for, on passing the current through the solenoid, the piece of iron *d* was seen to move upwards.

Maintaining the temperature outside *a* at from $14^{\circ}\cdot 2$ to $14^{\circ}\cdot 8$, the stop-cock connecting the side tube *b* with the mercury pump was opened for a moment. The pressure in the apparatus immediately fell to 49 millims., and remained steady there

while the stop-cock was open; on passing the current through the solenoid the piece of iron *d* did not move, indicating that its lower end was embedded in solid hydrogen. The stop-cock leading to the pump was then closed, and when the hydrogen in the tube *a* had melted it was again opened and the melting pressure was again determined. This operation was repeated several times; the melting pressure was invariably found to be 49 or 50 millims. This pressure corresponds to a temperature of $14^{\circ}1$ on the helium scale, which is the melting-point of hydrogen.

APPENDIX II.—*Note on the Preparation of Pure Hydrogen.*

It is obvious that the simplest method for the preparation of pure dry hydrogen would be to liquefy some of the gas in a bulb immersed in liquid hydrogen and allow the gas to evaporate directly into the apparatus which it is intended to fill. At the temperature of liquid hydrogen all possible impurities would be practically non-volatile, and the gas would not require further purification.

The hydrogen employed in these researches was obtained by the action of dilute sulphuric acid on pure platinised zinc. The gas was passed through a solution of potassium permanganate, and through a tube containing pentoxide of phosphorus into a tube fitted with a stop-cock containing pieces of palladium sponge. When the palladium was saturated, the stop-cock was closed and the tube connected with it was sealed to the apparatus for filling the thermometer. The gas from the palladium tube passed through a tube, about 40 centims. long, containing pentoxide of phosphorus, through a large bulb filled with glass beads and immersed in liquid air, to the inlet tube of the thermometer. The apparatus also communicated through a stop-cock with the mercury pump and with a mercury manometer. Before filling the thermometer the whole apparatus was exhausted and a considerable quantity of hydrogen allowed to escape from the palladium. The actual operations involved in filling the thermometer have already been described (p. 117); it is only necessary to state here that every precaution was taken to avoid contamination of the gas by moisture or by other impurities.

APPENDIX III.—*Note on the Preparation of Pure Helium.*

As has been pointed out by one of us (TRAVERS, 'The Experimental Study of Gases'), the constant-volume helium thermometer furnishes the only reliable means of measuring low temperatures. For the purpose of measuring the temperature of liquid hydrogen boiling under reduced pressure, it is of particular importance that the helium should be free not only from argon, but even from neon; for as that element has a vapour pressure of 12.8 millims. of mercury at the boiling-point of liquid hydrogen, and 2.4 millims. at $15^{\circ}65$ abs., its vapour pressure at 14° abs. would

probably be less than 1 millim. of mercury, and consequently a very small quantity of this impurity would be sufficient to vitiate the results.

As it is obviously impossible to separate neon completely from helium by cooling the gas to the temperature of liquid hydrogen, it is necessary to employ helium which is initially as free from that gas as is possible. DEWAR (*loc. cit.*, p. 155) has employed the gas from the Bath springs as a source of helium, but, as his own investigations show that the resulting helium contains 7.4 per cent. of neon, it is unsuitable for thermometric purposes, particularly for the measurement of the temperature, which can be obtained by boiling hydrogen under reduced pressure.

Some years ago it was shown (RAMSAY and TRAVERS, 'Roy. Soc. Proc.', vol. 60, p. 206) that by subjecting the gas from the mineral clèveite to the process of fractional diffusion, it was possible to separate from it a small quantity of argon and to reduce the density of the lighter fraction to 1.98 ($O = 16$). Later it was found by one of us, though the experiment has not been recorded, that when the gas from clèveite was passed through liquid oxygen, cooled to $-210^{\circ}C$. in a bulb immersed in liquid air boiling under reduced pressure, a small quantity of argon was removed from the gas, but apparently no neon. For this reason clèveite gas appeared to be the most suitable source for obtaining pure helium. The light helium obtained by the diffusion experiments was employed in these researches; it was purified by passing it at the normal pressure through a coil immersed in liquid hydrogen.

The apparatus employed will now be described.

The gas, of which we had about 180 cub. centims., was introduced through the syphon (fig. 3), into the gas-holder A, which contained mercury. The gas-holder was connected by a rubber tube with a mercury reservoir not shown in the figure, and in order to eliminate any risk of air entering the gas-holder through leakage of the rubber tube, the connection was made through a trap, D. The gas-holder communicated through the two-way stop-cock, B, with the syphon for introducing the gas, and with the refrigerating coil C, which led through a second stop-cock, B, to a second gas-holder similar to the first. The coil communicated with the mercury pump through the stop-cock E.

After thoroughly exhausting the coil, the stop-cock E was closed, and the gas admitted to it. A vacuum vessel containing about 300 cub. centims. of liquid hydrogen, surrounded by a larger vessel containing liquid air, was brought below the coil and raised till the latter was thoroughly immersed in the liquid. The stop-cocks B, and B, were then carefully opened, and by adjusting the level of the mercury reservoir connected with the gas-holders, the gas was passed three times backward and forward through the coil. When the whole of the gas had returned to the first gas-holder, the stop-cocks were closed. The gas in the gas-holder we called fraction 1.

About 100 cub. centims. of the original gas, measured under normal conditions, still remained in the coil, for at the boiling-point of liquid hydrogen a gas occupies only about $1/15$ of the volume which it fills at the normal temperature. On opening the

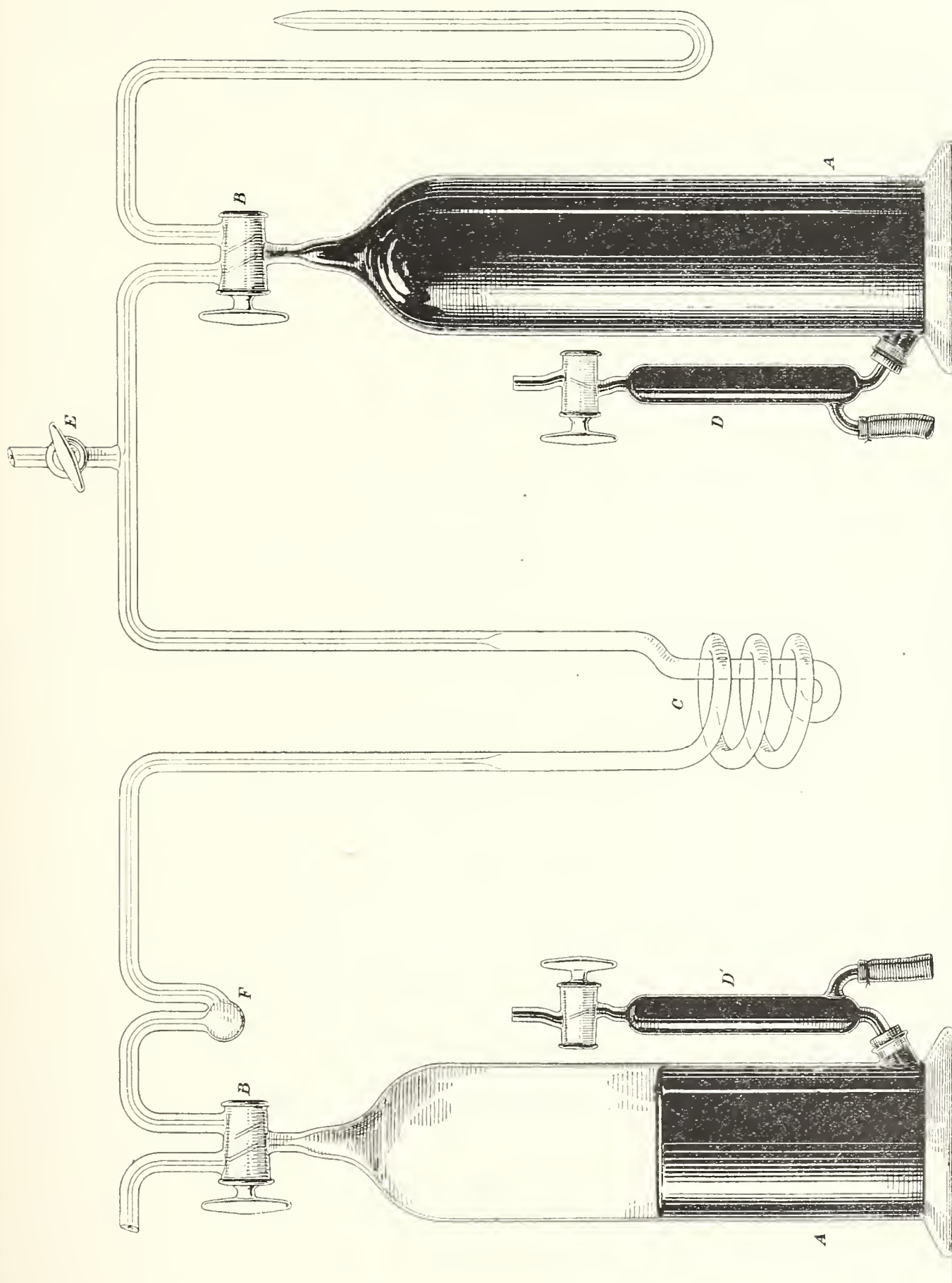


Fig. 3.

stop-cock E for a moment, about 70 cub. centims. of the gas passed into the pump. This fraction (2) was probably as pure as fraction 1, for the additional cooling produced by its sudden expansion would compensate for the tendency of any light impurity to evaporate; it was, however, kept separate (fraction 2).

When fraction 2 had been pumped off and collected, the stop-cock E was again opened, and the remaining volatile gas was pumped off and collected. It formed fraction 3, and consisted of about 25 cub. centims. of gas. Finally the liquid hydrogen was removed, and the residue in the coil, which volatilised and passed into the pump, was collected as fraction 4.

Fraction 4 was found to consist largely of nitrogen, which had probably entered the tube in which the gas had been stored since 1898 at the time it was last handled. This gas was sparked with oxygen over potash for some hours, and after removal of the excess of oxygen about 0.5 cub. centim. of gas was left; it was found on spectroscopic examination to consist of argon and krypton only, showing the spectrum of the latter with brilliancy.

The fact that clèveite helium yields krypton indicates that that gas must be present in the original mineral. As krypton is present in air to an extent not greater than 1 part in 1,000,000, the presence of that gas in the helium could not possibly be due to leakage of air into the apparatus. The quantity of krypton, like mercury vapour, which is necessary to give a brilliant spectrum in presence of argon, is exceedingly small, and possibly the total quantity of krypton in the gas might not exceed 0.01 cub. centim., or 0.005 per cent. on the original helium. It is interesting to note that when OLSZEWSKI attempted to liquefy helium by compressing it at -210°C ., and allowing it to expand (*Nature* 1896, vol. 62, p. 244), a small quantity of a white substance separated. It is intended on another occasion to investigate the heavy fraction of gas obtained in 1898 by the diffusion of helium.

The first fraction of gas, which we called fraction 1, was used in the first series of measurements of the boiling-point and vapour pressures of liquid hydrogen down to $15^{\circ}\cdot 5$ abs. Fractions 1 and 2 were subsequently mixed and passed a second time through a coil immersed in liquid hydrogen directly into the large constant-volume thermometer for the measurement of the pressure coefficient (p. 129), which was found to be 0.00366255.

For the second series of measurements of the vapour pressures of liquid hydrogen the helium was further purified by passing it through a coil immersed in liquid hydrogen boiling under reduced pressure. For this purpose the coil *c*, through which the helium was passed (fig. 3), was enclosed within an apparatus similar to that employed in determining the vapour pressure of hydrogen. The arrangement which is shown in fig. 4 requires little description. The vacuum vessel containing the liquid hydrogen was introduced from below into the wide (p. 160) tube so as to surround the coil and the bulb which contained the pure hydrogen, and was connected with a manometer by which the vapour pressure on the pure hydrogen, and from this

the temperature of the coil could be determined. By means of the exhaust pump the temperature of the liquid hydrogen was reduced to 110 millims. of mercury, corresponding to a temperature of $15^{\circ}3$ abs. Helium was then allowed to fill the coil at normal pressure, and by opening the stop-cock the gas was slowly admitted into the thermometers, which had previously been exhausted.*

APPENDIX IV.—*Note on the Vapour Pressures of Solid Neon.*

The following experiments are of interest both with regard to their application to the preparation of pure helium and to the light they throw on the homogeneity of neon.

In the first experiment the neon was condensed in a fractionating bulb (RAMSAY and TRAVERS, 'Phil. Trans.,' 1901, vol. 197, A, p. 51) which was connected with a manometer reading pressures below 100 millims. of mercury and, through a stop-cock, with a mercury pump. About 20 cub. centims. of neon were introduced into the bulb and condensed by cooling it by means of liquid hydrogen, boiling under the normal pressure ($20^{\circ}4$ on the helium scale). The vapour pressure observed by means of the manometer was 12.8 millims. of mercury. On opening the stop-cock, and allowing some of the neon to evaporate into the pump, the pressure fell momentarily, but on closing it the pressure again rose to 12.8 millims. The operation was repeated several times with the same result, proving that neon is a homogeneous substance. On removing the vacuum vessel containing the liquid hydrogen, a small quantity of solid neon was seen in the fractionating bulb.

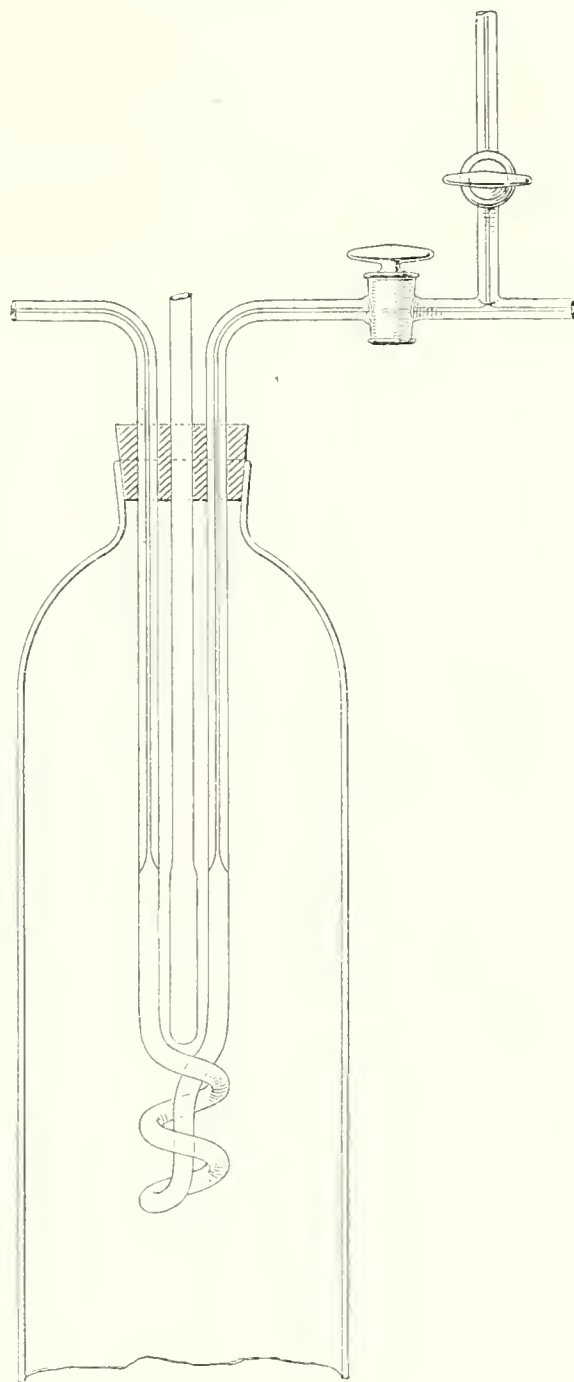


Fig. 4.

* NOTE, July 19th, 1902. —After filling the thermometer, the greater portion of the gas contained in the coil, which was maintained at $15^{\circ}3$ abs., was allowed to escape into the mercury pump, by opening the stop-cock communicating with it for a few seconds. When this gas had been pumped off, the stop-cock was again opened, and the remaining gas was collected separately. The last fraction, which consisted of about 10 cub. centims., was examined spectroscopically, but it appeared to contain no trace of neon.

In the second experiment, the neon was introduced into the bulb *m* of the apparatus employed in determining the vapour pressure of liquid hydrogen (fig. 1). At a temperature of $15^{\circ}\cdot65$ (helium scale), the vapour pressure of the neon was 2.4 millims. of mercury.

APPENDIX V.—*On the Probable Values of the Critical and Boiling-Points of Helium, and on an Attempt to Liquefy that Gas.*

As the foregoing results show, helium behaves at low temperature as a more perfect gas than hydrogen, and even if no further data were forthcoming, one would expect to find that its critical and boiling-points were lower than those of the latter gas. This, as we shall presently show, receives further confirmation from theoretical considerations and from experimental results.

In 1892, OLSZEWSKI (*loc. cit.*, p. 174) showed that when helium, under a pressure of 80 atmospheres, was cooled to -210°C. , and suddenly allowed to expand till the pressure fell to that of the atmosphere, no mist was seen in the compression tube. More recently ('Roy. Soc. Proc.', 1901, vol. 68, p. 360) DEWAR performed a similar experiment, cooling the compression tube in hydrogen reduced to its freezing-point. He did not, however, succeed in liquefying the gas, which he believed to have been cooled to 9° or 10° absolute, the temperature which he assigns to the upper limiting value of its critical point.

Though by compressing any gas, and subsequently allowing it to expand adiabatically, the relationship between the initial and final pressures p , p' and the initial and final temperatures T , T' can be found by the equation

$$\frac{T}{T'} = \left(\frac{p}{p'}\right)^{\frac{k-1}{k}},$$

when k is the ratio of the specific heats for the gas, this equation may not be applicable to the case in which a gas is compressed into a capillary tube and then allowed to expand, the thermal capacity of the gas being then very small compared with that of the walls of the containing tube.

It occurred to us that if we could cool a few centimetres of a compression-tube containing helium down to a temperature below the critical point of the gas, then, on compressing the gas, its volume would decrease and the pressure would rise till the vapour pressure of helium corresponding to that particular temperature was reached. It should then be possible, supposing the gas to be homogeneous, to decrease the volume of the helium without increasing the pressure, as a change in the volume of the gas would imply liquefaction in the cooled portion of the tube. Before describing our experiments we will first deal with the probable value of the critical and boiling-points of helium.

Of the five gases, helium, neon, argon, krypton, and xenon, the vapour pressures and critical constants of the last three only have been determined. They are as follows (RAMSAY and TRAVERS, 'Phil. Trans.,' A, 1901, vol. 197, p. 47).

	Atomic weight.	Boiling-point.	Critical point.	Ratio.
Helium	4	—	—	—
Neon	20	—	—	—
Argon	40	87 abs.	155 abs.	0·561
Krypton	82·5	121 „	210 „	0·576
Xenon	128	164 „	288 „	0·568

It will be observed that the ratios of the absolute boiling-points to the absolute critical points has a nearly constant value, the mean of the actual ratios being 0·57. This relationship is common among similar substances, for which the boiling and critical temperatures may be considered as corresponding states.

The boiling and critical points of neon are not known. We are certain, however, that the critical point lies below 60° abs., and, as the vapour pressure is 12·8 millims. at 20°·4 abs. the boiling-point must lie above 25° abs. If the ratio for the critical and boiling-points is the same as for the other gases, viz., 0·57, we may calculate the temperature of the boiling and critical points which will correspond.

Boiling-point	28°·5	. . .	31°	. . .	34°.
Critical point	50°	. . .	55°	. . .	60°.

In the paper in the 'Philosophical Transactions' already referred to, it was pointed out that argon, krypton, and xenon resemble one another, though krypton bears a stronger resemblance to neon and helium, forming, in fact, a link between the two sub-series of this group of elements. If now we plot (fig. 5) on a diagram the critical and boiling-points of the elements as ordinates and their atomic weights as abscissæ, the points representing argon, krypton, and xenon lie on a straight line. Taking 30 abs. as the boiling-point and 53 abs. as the critical point of neon, and plotting these points on the diagram against the atomic weight of neon, we find that the lines joining these points to the corresponding points for krypton, when prolonged, pass through the origin of the axes. If the points corresponding to helium lie also on these lines, the critical and boiling-points of this gas will be 10° and 6° abs. Even if we assign to the critical point of neon its upper limiting value 60° abs., and suppose the points on the diagram representing krypton, neon, and helium to lie on a curve, the critical point of helium cannot be far above 12° abs.

In our experiments the helium was compressed in the compression tube of an apparatus similar to that employed by CAILLETET or AMAGAT, only the pressure was transmitted to the surface of the mercury by means of compressed air, instead of

employing liquid and a pump. The compression tube is shown in fig. 6. The wider portion, A, which was made of boiler-gauge tube, had an internal diameter

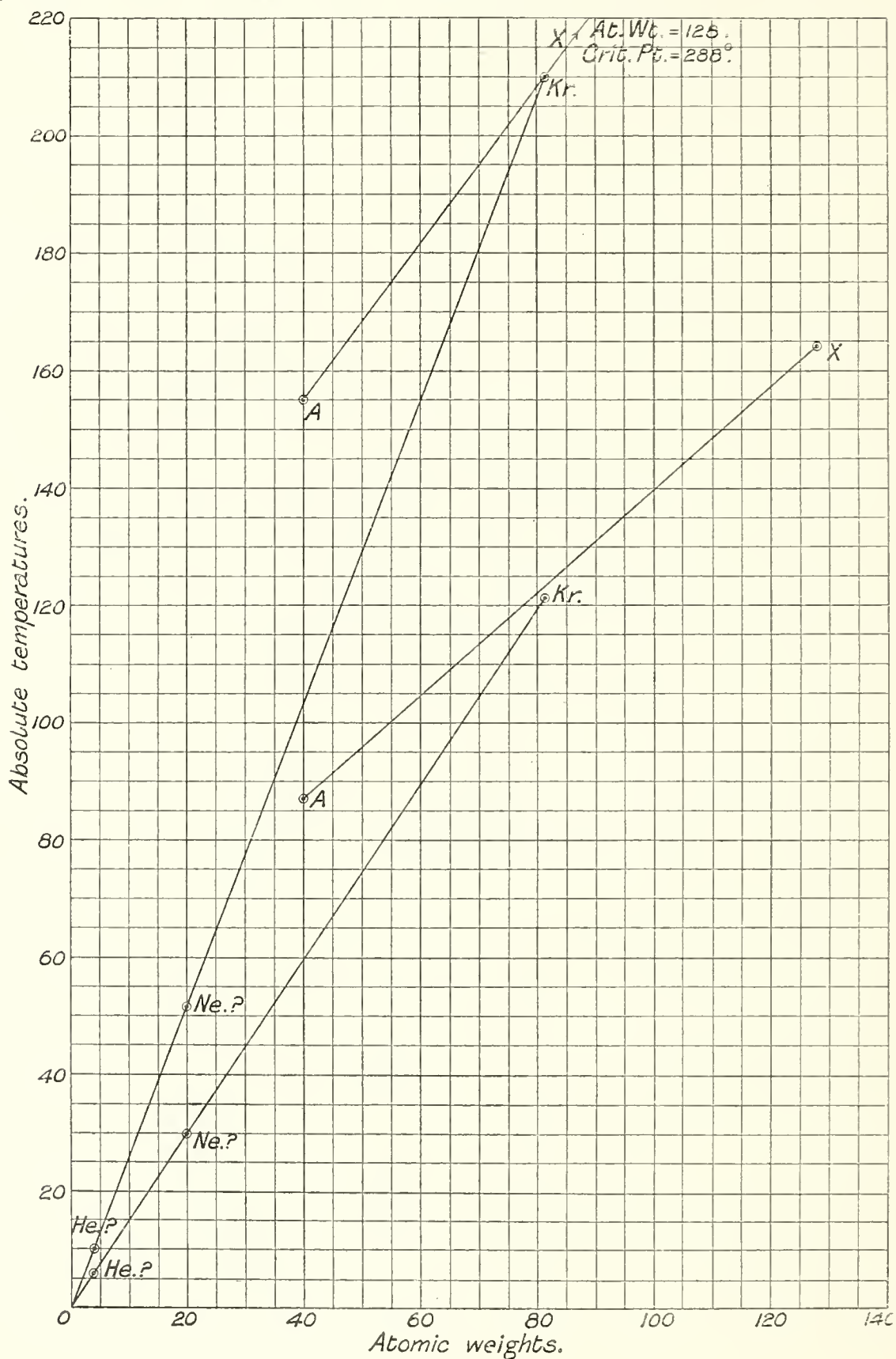


Fig. 5.

of 6 millims., and was capable of standing a pressure of 80 atmospheres, or more. At its lower end it entered the steel compression apparatus, and at the top it was

sealed to the capillary tube B, which had an internal diameter of 0.5 millim. When the lower portion of the tube B was cooled to the lowest temperature, it required a pressure of 60 atmospheres to compress the whole of the helium into the capillary tube.

The pressure on the helium was determined by means of two gauges of the Amagat type, filled with nitrogen, and indicating pressures from 2 to 12 atmospheres. and from 8 to 60 atmospheres respectively.

The compression tube was filled with pure helium prepared in the manner described on p. 174 (fraction 1).

In our first experiment the capillary compression tube was enclosed together with the thermometer bulb, &c., in the apparatus shown in fig. 1 (see also p. 160). At temperatures between $20^{\circ}5$ and 14° abs. (helium scale), the pressure on the gas was slowly increased to 60 atmospheres, and then slowly reduced. Under all conditions the smallest change in the position of the mercury meniscus in the compression tube was always accompanied by a corresponding change in the pressure. This experiment was repeated on three occasions.

In another set of experiments, the capillary portion B of the compression tube passed through a rubber stopper E (fig. 6) into a small silvered vacuum vessel C, with a contracted mouth, enclosed within a wider glass tube D, sealed at the bottom to a tube G, 7 millims. in diameter, and fitting closely to the rubber stopper E. The upper part of the tube D contained a plug of natural wool wrapped in gauze, to shield the mouth of the vacuum vessel from radiation from above. A tube F passed through the stopper E, and was connected with a mercury manometer which served to measure the pressure in D.

The vacuum vessel C was filled with liquid hydrogen and placed in the tube D, which was then rapidly fitted to the rubber stopper E; the compression tube B had previously been cooled with liquid air. A vacuum vessel H containing liquid air was then brought outside the tube D as in fig. 6. The tube G was connected with the double 'Fleuss' pump, arranged with the cylinders in series (p. 161), and the pressure on the liquid hydrogen was reduced to 5 millims. of mercury, and maintained at that pressure for 20 minutes. Even at this temperature, which is probably not far below 13° abs., no evidence could be obtained that helium had liquefied.

The extremely permanent character of helium confirms our view that that gas is by

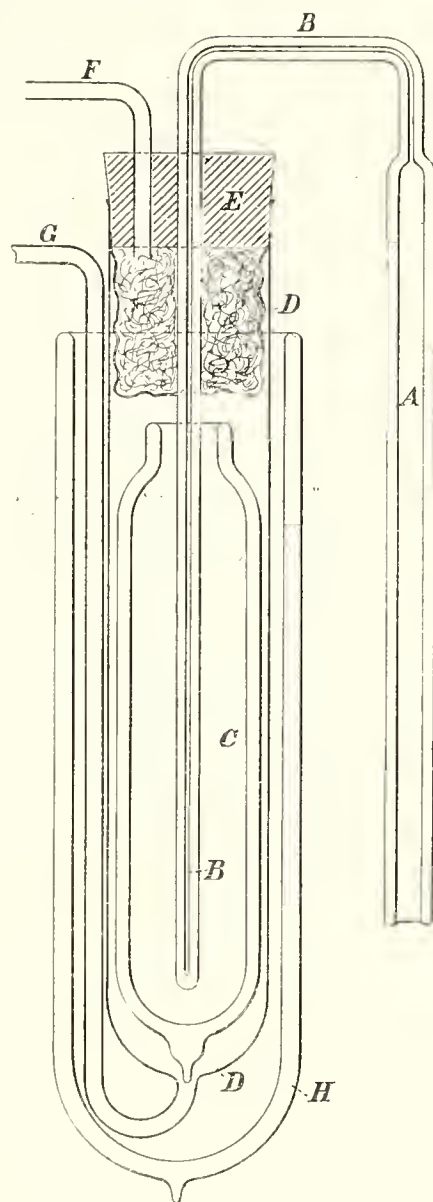


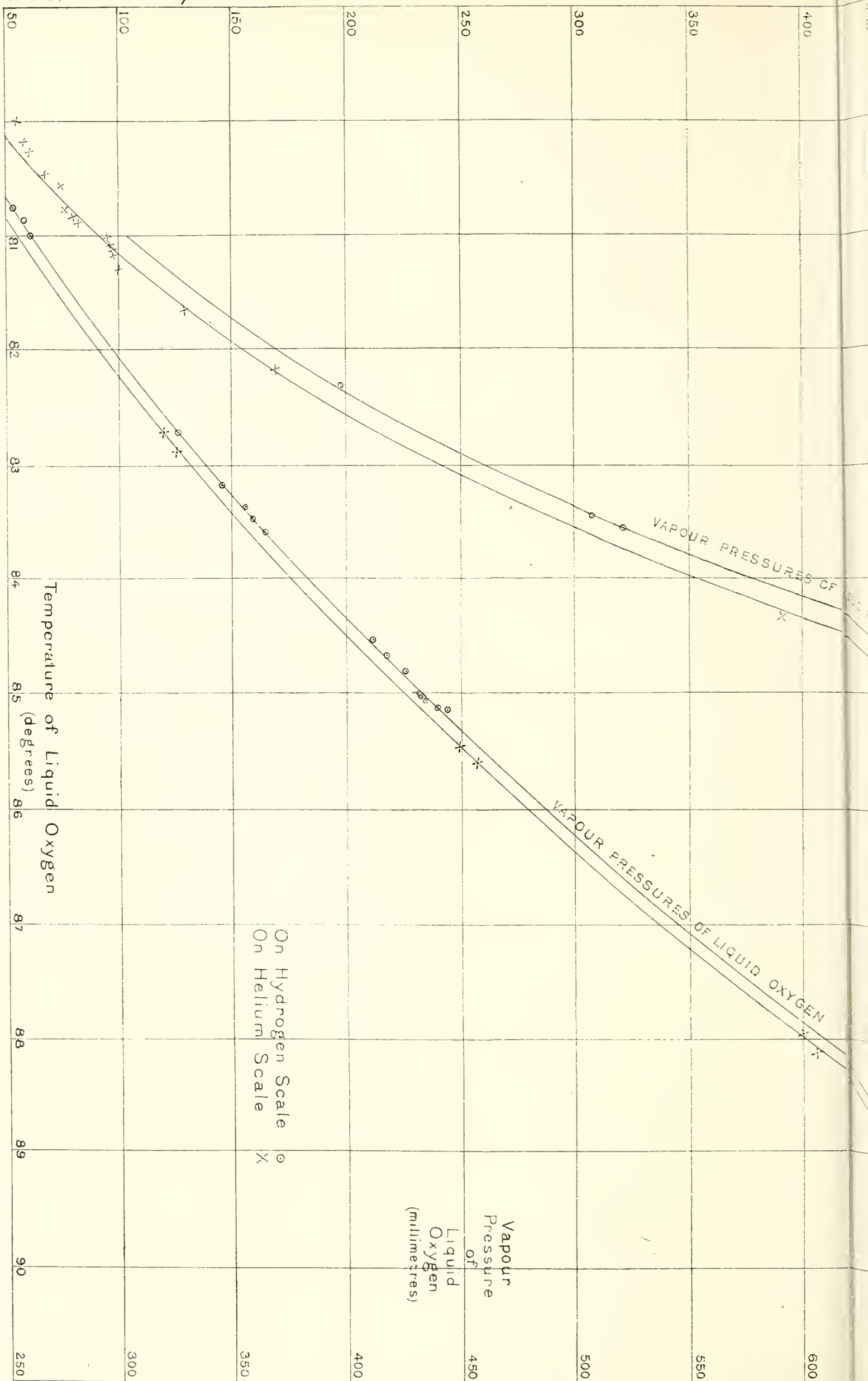
Fig. 6.

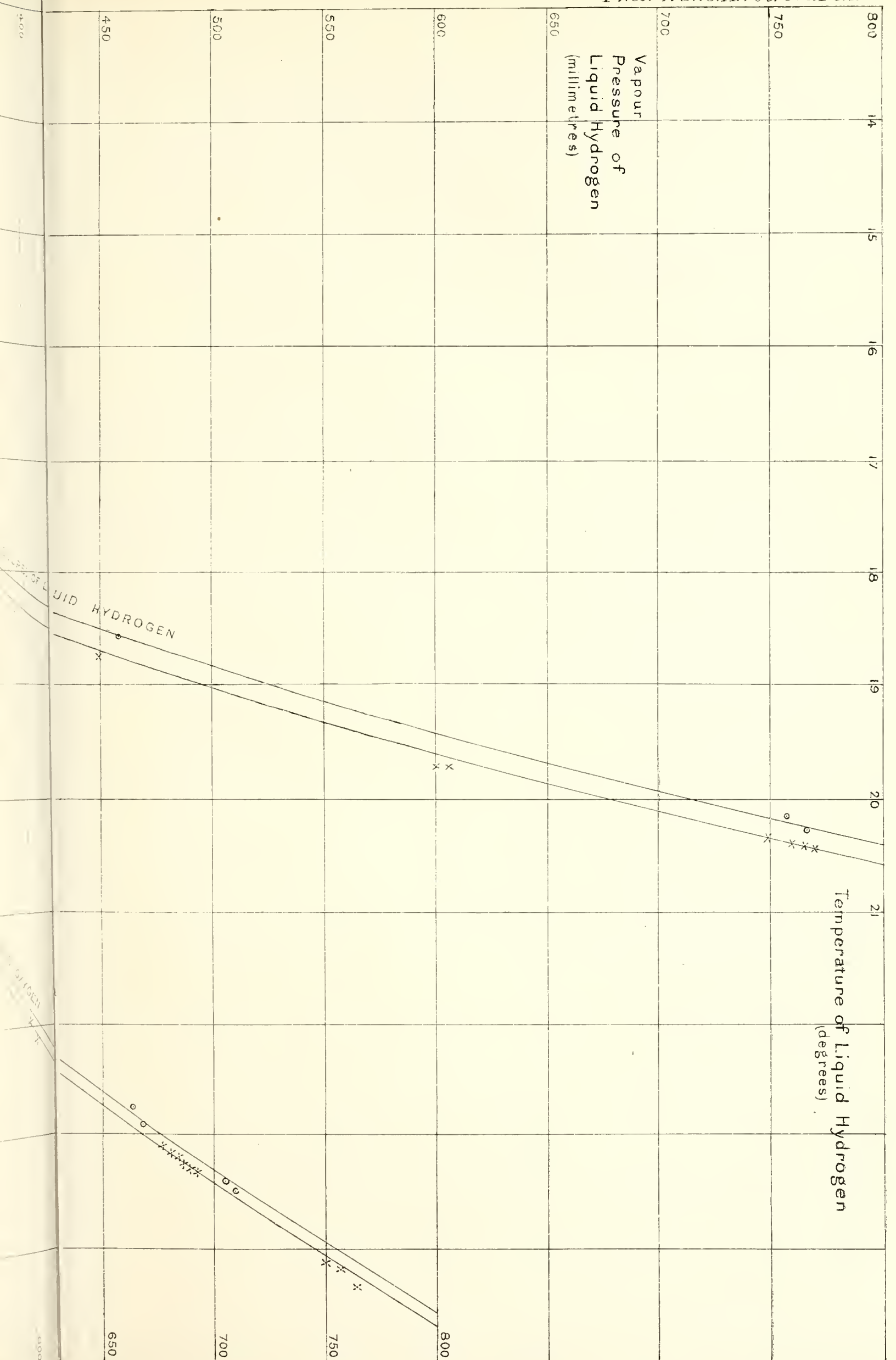
far the most perfect thermometric substance known to us. Possibly, even at the lowest temperature that we have arrived at, the temperatures measured on the constant-volume helium scale are not far from absolute measurements.

Turning to the question of liquefying helium, it would of course be possible to compress large quantities of the gas, and allowing it to expand adiabatically to at least produce a mist of helium. As the compression could only be effected in vessels with walls of considerable thickness, it would probably be impossible to produce any quantity of liquid in this way. With regard to the application of the regenerative process, which can so easily be employed in the case of air or of hydrogen, to the liquefaction of helium, we can say nothing; for apart from the experimental difficulties involved in the problem, we do not yet know whether helium becomes heated or cooled when allowed to expand freely at high or at low temperatures.

From this point of view, and for the correction of gas thermometers, the Joule-Thomson effect for helium should be determined over a wide range of temperature.

In conclusion, we wish to express our thanks to the Government-Grant Committee of the Royal Society for the assistance they have given us in carrying out this investigation.





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IV. *On some Definite Integrals, and a New Method of Reducing a Function of Spherical Co-ordinates to a Series of Spherical Harmonics.*

By ARTHUR SCHUSTER, F.R.S.

Received May 30,—Read June 5, 1902.

§ 1. *Introductory.*

THE following investigation deals with some definite integrals which are useful when it is desired to express a function of two angular variables by means of a series of spherical surface harmonics. An important theorem concerning these integrals leads to a method which considerably reduces the arithmetical labour involved in the reductions, and secures in practice the advantage of obtaining the numerical values of the coefficients of lower degrees independently of those of higher degrees.

The zonal harmonic of degree n is denoted by P_n and defined as usual by

$$P_n = \frac{1}{2^n 1 \cdot 2 \dots n} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n.$$

The tesseral harmonic of degree n and type σ is denoted by

$$T_n^\sigma = Q_n^\sigma (y_n^\sigma \cos \sigma\phi + h_n^\sigma \sin \sigma\phi),$$

where

$$Q_n^\sigma = \sin^\sigma \theta \frac{d^\sigma P_n}{d\mu^\sigma}.$$

In these equations θ represents an angle measured on a sphere from a point as pole and $\mu = \cos \theta$. The longitude measured from some standard meridian is denoted by ϕ .

The name given by HEINE, and translated by TODHUNTER as “Associated Functions of the First Kind,” is too cumbersome for use, and I propose to call these functions “Tesseral Functions.” The tesseral function is converted into a tesseral harmonic by the factor $\cos \sigma\phi$ or $\sin \sigma\phi$. The name may not perhaps appear to be appropriate, because it is only the factor which gives the function its “tesseral” character, but it is short and suggests at once the function it denotes.

The present investigation deals in great part with the definite integrals, taken over the surface of a sphere of unit radius, of the product of two tesseral functions and of

the product of a tesseral function and $\cos p\theta$ or $\sin p\theta$, the tesseral functions being all referred to the same axis. I shall denote, as usual, the factorial product of the numbers up to n by $n!$, but have found it necessary to introduce a separate notation for the products of successive even or successive odd numbers. I consequently define

$$n!! = n \cdot (n-2)!!.$$

Starting with $1!! = 1, \quad 2!! = 2,$

it follows that, for positive values of n ,

$$n!! = n \cdot n-2 \cdot n-4 \dots$$

where the last factor is either 1 or 2, according as n is odd or even.

We may extend the definition to negative values of the argument, for the successive substitution of $n = 1, n = -1, n = -3$, into the first equation, gives

$$(-1)!! = 1, \quad (-3)!! = -1, \quad (-5)!! = \frac{1}{3},$$

and generally if n is negative and odd,

$$n!! = (-1)^{\frac{-n-1}{2}} \frac{1}{(-n-2)!!}.$$

For $n = 2$, the original equation gives $0!! = 1$,

for $n = 0, \quad (-2)!! = \infty,$

and similarly for all negative and even values of n , $n!!$ is infinite. The ratio of two of these factorials of negative numbers is, however, finite, for it is easily shewn that if m and n be two negative numbers, whether even or odd,

$$\frac{m!!}{n!!} = (-1)^{\frac{n-m}{2}} \frac{(-n-2)!!}{(-m-2)!!}.$$

One of the advantages of a separate notation for what may be called the "alternate" or "double" factorial, is due to the fact that it often saves the inconvenience of different expressions for odd and even numbers.

§ 2. *Formulae of Transformation.*

It is convenient to collect together some equations which will often be required. Most of these equations will be found already in previous writings, such as HEINE'S 'Treatise' or ADAMS' 'Researches in Terrestrial Magnetism.'

As regards zonal harmonics, it is only necessary to quote the well-known relations:

$$(2n+1)\mu P_n = (n+1)P_{n+1} + nP_{n-1} \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$(2n+1)P_n = \frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

From these equations we derive the following :

$$(2n+1)\mu Q_n^\sigma = (n-\sigma+1)Q_{n+1}^\sigma + (n+\sigma)Q_{n-1}^\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$

$$(2n+1)\sin\theta Q_n^\sigma = Q_{n+1}^{\sigma+1} - Q_{n-1}^{\sigma+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

$$= (n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} - (n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1} \quad (C),$$

$$\frac{2\sigma Q_n^\sigma}{\sin\theta} = (n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1} + Q_{n+1}^{\sigma+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (D),$$

$$= Q_{n+1}^{\sigma+1} + (n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (E),$$

$$Q_n^\sigma - Q_{n-2}^\sigma = (n+\sigma-2)(n+\sigma-3)Q_{n-2}^{\sigma-2} - (n-\sigma+2)(n-\sigma+1)Q_n^{\sigma-2} \quad (F),$$

$$(2n+1)\frac{d}{d\mu}\sin^\rho\theta\frac{d^\sigma P_n}{d\mu^\sigma} \\ = (n+\sigma)(n+\sigma-\rho+1)\sin^{\rho-2}\theta\frac{d^\sigma P_{n-1}}{d\mu^\sigma} - (n-\sigma+1)(n-\sigma+\rho)\sin^{\rho-2}\theta\frac{d^\sigma P_{n+1}}{d\mu^\sigma} \quad (G),$$

$$2\sigma\frac{d}{d\mu}\sin^\rho\theta\frac{d^\sigma P_n}{d\mu^\sigma} = (2\sigma-\rho)\sin^\rho\theta\frac{d^{\sigma+1}P_n}{d\mu^{\sigma+1}} - \rho(n+\sigma)(n-\sigma+1)\sin^{\rho-2}\theta\frac{d^{\sigma-1}P_n}{d\mu^{\sigma-1}} \quad (H).$$

As special cases of (G) and (H) we may put ρ equal successively to σ , $\sigma+1$, and $\sigma+2$. The following equations are thus obtained :

$$(2n+1)\sin^2\theta\frac{d}{d\mu}Q_n^\sigma = (n+1)(n+\sigma)Q_{n-1}^\sigma - n(n+1-\sigma)Q_{n+1}^\sigma \quad (G_1),$$

$$2\sin\theta\frac{d}{d\mu}Q_n^\sigma = Q_n^{\sigma+1} - (n+\sigma)(n-\sigma+1)Q_n^{\sigma-1} \quad . \quad . \quad . \quad (H_1),$$

$$(2n+1)\sin\theta\frac{d}{d\mu}\sin\theta Q_n^\sigma = n(n+\sigma)Q_{n-1}^\sigma - (n+1)(n+1-\sigma)Q_{n+1}^\sigma \quad (G_2),$$

$$2\sigma\frac{d}{d\mu}\sin\theta Q_n^\sigma = (\sigma-1)Q_n^{\sigma+1} - (\sigma+1)(n+\sigma)(n-\sigma+1)Q_n^{\sigma-1} \quad (H_2),$$

$$(2n+1)\frac{d}{d\mu}\sin^2\theta Q_n^\sigma = (n-1)(n+\sigma)Q_{n-1}^\sigma - (n+2)(n-\sigma+1)Q_{n+1}^\sigma \quad (G_3),$$

$$2\sigma\frac{d}{d\mu}\sin^2\theta Q_n^\sigma = (\sigma-2)\sin\theta Q_n^{\sigma+1} - (\sigma+2)(n+\sigma)(n-\sigma+1)\sin\theta Q_n^{\sigma-1} \quad (H_3).$$

The formula (A) is well known and may be obtained by σ differentiations of (1), substituting in the result an equation derived from $\sigma-1$ differentiations of (2).

Equation (B) is the result of σ differentiations of (2), and multiplication by $\sin^{\sigma+1} \theta$. (C) may be proved by combining

$$\frac{d^\sigma}{d\mu^\sigma} (1 - \mu^2) \frac{dP_n}{d\mu} = (1 - \mu^2) \frac{d^{\sigma+1}P_n}{d\mu^{\sigma+1}} - 2\mu\sigma \frac{d^\sigma P_n}{d\mu^\sigma} - \sigma \cdot \sigma - 1 \frac{d^{\sigma-1}P_n}{d\mu^{\sigma-1}}$$

with the fundamental equation for zonal harmonics. The latter leads directly to

$$\frac{d^\sigma}{d\mu^\sigma} (1 - \mu^2) \frac{dP_n}{d\mu} = - \frac{d^{\sigma-1}}{d\mu^{\sigma-1}} n(n+1) \cdot P_n$$

from which we derive (by equating the two expressions on the right-hand sides):

$$(1 - \mu^2) \frac{d^{\sigma+1}P_n}{d\mu^{\sigma+1}} = 2\mu\sigma \frac{d^\sigma P_n}{d\mu^\sigma} - (n + \sigma)(n - \sigma + 1) \frac{d^{\sigma-1}P_n}{d\mu^{\sigma-1}},$$

or, after multiplication by $(1 - \mu^2)^{\frac{\sigma}{2}}$,

$$\sin \theta Q_n^{\sigma+1} = 2\mu\sigma Q_n^\sigma - (n + \sigma)(n - \sigma + 1) \sin \theta Q_n^{\sigma-1}.$$

If μQ_n^σ be now substituted from (A) and $\sin \theta Q_n^{\sigma-1}$ from (B), the equation (C) is obtained.

If $\sigma - 1$ be written for σ in (B) and $\sigma + 1$ for σ in (C) we may combine the two equations, so as to give (D) and (E). (F) is an important relation obtained from (B) and (C). The formulæ (G) and (H) are easily derived by direct differentiation and a few simple transformations.

$$\S 3. \int_{-1}^{+1} Q_n^\sigma d\mu.$$

If the equation (H₂) is integrated with respect to μ between the limits -1 and $+1$, the left-hand side vanishes at both limits: hence, after changing from $\sigma + 1$ to σ ,

$$\int_{-1}^{+1} Q_n^\sigma d\mu = \frac{\sigma}{\sigma-2} (n + \sigma - 1)(n - \sigma + 2) \int_{-1}^{+1} Q_n^{\sigma-2} d\mu,$$

and, by applying the same process to the right-hand side,

$$= \frac{\sigma}{\sigma-4} (n + \sigma - 1)(n + \sigma - 3)(n - \sigma + 2)(n - \sigma + 4) \int_{-1}^{+1} Q_n^{\sigma-4} d\mu.$$

Repetition of the same proceeding will ultimately lead for even values of σ to

$$\int_{-1}^{+1} Q_n^\sigma d\mu = \frac{\sigma}{2} \frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n + 1)!!} \int_{-1}^{+1} Q_n^2 d\mu.$$

But

$$\int_{-1}^{+1} Q_n^2 d\mu = \int_{-1}^{+1} (1 - \mu^2) \frac{d^2 P_n}{d\mu^2} d\mu = 2 \int_{-1}^{+1} \mu \frac{dP_n}{d\mu} d\mu = 2 \int_{-1}^{+1} \left(\frac{d\mu P_n}{d\mu} - P_n \right) d\mu = 2 \left[\mu P_n \right]_{-1}^{+1},$$

where the special case $n = 0$, for which the integral vanishes, is excluded.

Hence

$$\int_{-1}^{+1} Q_n^2 d\mu = 4, \text{ if } n \text{ be even; } = 0, \text{ if } n \text{ be odd;}$$

and finally

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma d\mu &= 2\sigma \frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n + 1)!!}, \text{ if } \sigma \text{ and } n \text{ be even;} \\ &= 0, \text{ if } \sigma \text{ be even and } n \text{ odd.} \end{aligned}$$

For odd values of σ , it is more convenient to use formula (G_3), which leads to :

$$\int_{-1}^{+1} Q_n^\sigma d\mu = \frac{(n + \sigma - 1)(n - 2)}{(n - \sigma)(n + 1)} \int_{-1}^{+1} Q_{n-2}^\sigma d\mu.$$

By repetition of the same process, we get for even values of n ultimately

$$\int_{-1}^{+1} Q_n^\sigma d\mu = \frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n + 1)!!} \frac{\sigma!!}{(\sigma - 3)!!} \frac{1}{(2\sigma - 2)!!} \int_{-1}^{+1} Q_{\sigma-1}^\sigma d\mu.$$

As the expression under the integral sign of the right-hand side vanishes, the value of the integral on the left-hand side is zero.

When n is odd we are ultimately led to

$$\int_{-1}^{+1} Q_n^\sigma d\mu = \frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n + 1)!!} \frac{(\sigma + 1)!!}{(\sigma - 2)!!} \frac{1}{(2\sigma - 1)!!} \int_{-1}^{+1} Q_\sigma^\sigma d\mu.$$

But

$$\int_{-1}^{+1} Q_\sigma^\sigma d\mu = (2\sigma - 1)!! \int_{-1}^{+1} \sin^\sigma \theta d\mu = (2\sigma - 1)!! \int_0^\pi \sin^{\sigma+1} \theta d\theta = \frac{\sigma!!}{(\sigma + 1)!!} (2\sigma - 1)!! \pi.$$

Collecting the results, we may put generally

$$\int_{-1}^{+1} Q_n^\sigma d\mu = c\sigma \frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n + 1)!!}, \text{ if } n + \sigma \text{ be even,}$$

where c is equal to 2 or π according as σ be even or odd. When $(n + \sigma)$ is odd, the integral vanishes.

The results of this paragraph allow us to represent a quantity which is constant over a sphere, in terms of a series of tesseral functions which are all of the same type, the type being arbitrary. It follows that we may express $\cos \sigma\lambda$ and $\sin \sigma\lambda$ in terms of a series of tesseral harmonics of type σ .

To find the values of the coefficients of such a series we put

$$\cos \sigma \phi = A_{\sigma}^{\sigma} T_{\sigma}^{\sigma} + A_{\sigma+2}^{\sigma} T_{\sigma+2}^{\sigma} + \dots A_n^{\sigma} T_n^{\sigma} + \dots$$

and integrating over unit sphere, after multiplying as usual by T_n^{σ} , we find

$$\int_0^{2\pi} \int_{-1}^{+1} Q_n^{\sigma} \cos^2 \sigma \phi d\phi d\mu = A_n^{\sigma} \int_0^{2\pi} \int_{-1}^{+1} (T_n^{\sigma})^2 d\phi d\mu = \frac{2\pi}{2n+1} \cdot \frac{(n+\sigma)!}{(n-\sigma)!} A_n^{\sigma}.$$

Substituting on the left-hand side the integral found above we obtain

$$A_n^{\sigma} = c\sigma \frac{2n+1}{2} \cdot \frac{(n-\sigma)!}{(n+\sigma)!} \cdot \sigma \frac{(n+\sigma-1)!!(n-2)!!}{(n-\sigma)!!(n+1)!!} = c\sigma \frac{2n+1}{2} \cdot \frac{(n-\sigma-1)!!(n-2)!!}{(n+\sigma)!!(n+1)!!},$$

where c is 2 or π according as σ and n are both even or both odd.

We obtain, for instance, in this way for $\sigma = 2$,

$$\frac{1}{2} = \frac{5}{1 \cdot 2 \cdot 3 \cdot 4} Q_2^2 + \frac{9}{3 \cdot 4 \cdot 5 \cdot 6} Q_4^2 + \frac{13}{5 \cdot 6 \cdot 7 \cdot 8} Q_6^2 + \dots$$

$$\S 4. \quad \int_{-1}^{+1} Q_n^{\sigma} \sin^{\lambda} \theta d\mu, \quad \int_{-1}^{+1} \mu Q_n^{\sigma} \sin^{\lambda} \theta d\mu.$$

If we put $\rho = \sigma + \lambda + 2$ in equation (G), and after changing from $n+1$ to n , integrate both sides, we obtain:

$$\begin{aligned} \int_{-1}^{+1} \sin^{\lambda} \theta Q_n^{\sigma} d\mu &= \frac{(n+\sigma-1)(n-\lambda-2)}{(n-\sigma)(n+\lambda+1)} \int_{-1}^{+1} \sin^{\lambda} \theta Q_{n-2}^{\sigma} d\mu \\ &= \frac{(n+\sigma-1)(n+\sigma-3)(n-\lambda-2)(n-\lambda-4)}{(n-\sigma)(n-\sigma-2)(n+\lambda+1)(n+\lambda-1)} \int_{-1}^{+1} \sin^{\lambda} \theta Q_{n-4}^{\sigma} d\mu. \end{aligned}$$

If $n - \sigma$ be odd, a continuation of this process reduces the degree of the tesseral function on the right-hand side until it becomes smaller than σ , and this will happen before any of the factors in the denominator become zero, so that in that case the integral on the left-hand side is zero.

If $n - \sigma$ be even, the integral on the right-hand side ultimately becomes

$$\int_{-1}^{+1} \sin^{\lambda} \theta Q_{\sigma}^{\sigma} d\mu = (2\sigma-1)!! \int_0^{\pi} \sin^{(\lambda+\sigma+1)} \theta d\theta = (2\sigma-1)!! \frac{(\lambda+\sigma)!!}{(\lambda+\sigma+1)!!} c,$$

where c is 2 or π according as $\lambda + \sigma$ is even or odd.

The integral to be determined now becomes

$$\begin{aligned} \int_{-1}^{+1} Q_n^{\sigma} \sin^{\lambda} \theta d\mu &= c \frac{(n+\sigma-1)!!(\sigma+\lambda)!!}{(n-\sigma)!!(n+\lambda+1)!!} [(n-\lambda-2)(n-\lambda-4)\dots(\sigma-\lambda)] \text{ (if} \\ &\quad n - \sigma \text{ be even)} \\ &= 0, \text{ if } n - \sigma \text{ be odd.} \end{aligned}$$

A little care is necessary in the interpretation of the square bracket on the right-hand side. All the factors of the product contained in it may be positive or negative, but when λ is intermediate between n and σ , some may be positive and some negative. In the latter case, one of the factors will be zero when $\sigma - \lambda$ is even, and in that case the integral on the right-hand side is zero. The above expression does not include the special case $n = \sigma$. By extending the notation of double factorials to negative numbers as defined in § 1, we may also write, including the cases when $n - \lambda$, or both $n - \lambda$ and $\sigma - \lambda$, are negative, or when $n = \sigma$,

$$\int_{-1}^{+1} Q_n^\sigma \sin^\lambda \theta d\mu = c \frac{(n + \sigma - 1)!!(\sigma + \lambda)!!(n - \lambda - 2)!!}{(n - \sigma)!!(\sigma - \lambda - 2)!!(n + \lambda + 1)!!} \text{ if } n - \sigma \text{ be even,}$$

$$= 0 \text{ if } n - \sigma \text{ be odd.}$$

The integral $\int_{-1}^{+1} \mu Q_n^\sigma \sin^\lambda \theta d\mu$ reduces to the one just determined with the help of equation (A). We thus find:

$$\int_{-1}^{+1} \mu Q_n^\sigma \sin^\lambda \theta d\mu = c \frac{(n + \sigma)!!(\sigma + \lambda)!!(n - \lambda - 3)!!}{(n - \sigma - 1)!!(\sigma - \lambda - 2)!!(n + \lambda + 2)!!} \text{ if } n - \sigma \text{ be odd,}$$

$$= 0 \text{ if } n - \sigma \text{ be even.}$$

The factor c takes, as before, the value 2 or π according as $\sigma + \lambda$ is even or odd.

For the special case $\sigma = 0$, the tesseral harmonics reduce to zonal harmonics, and the last equation becomes

$$\int_{-1}^{+1} \mu P_n \sin^\lambda \theta d\mu = c \frac{n!!\lambda!!(n - \lambda - 3)!!}{(n - 1)!!(-\lambda - 2)!!(n + \lambda + 2)!!} \text{ if } n \text{ be odd.}$$

If λ be even and $n > \lambda + 2$, the fraction on the right-hand side is zero, because in that case $(-\lambda - 2)!! = \infty$, and the numerator remains finite, but if $n < \lambda + 2$, the value of $\frac{(n - \lambda - 3)!!}{(-\lambda - 2)!!}$ remains finite whether λ be even or odd, and, in that case, the right-hand side may also be written (avoiding negative arguments of the factorials):

$$= (-1)^{\frac{n-1}{2}} c \frac{n!!\lambda!!\lambda!!}{(n - 1)!!(n + \lambda + 2)!!(\lambda - n + 1)!!}.$$

If n be odd and $n \equiv \lambda + 2$, we may transform the negative factorial and write the value of the integral

$$= (-1)^{\frac{\lambda+1}{2}} \pi \frac{n!!\lambda!!\lambda!!(n - \lambda - 3)!!}{(n - 1)!!(n + \lambda + 2)!}.$$

Similarly we obtain

$$\int_{-1}^{+1} P_n \sin^\lambda \theta d\mu = c \frac{(n-1)!! \lambda!! (n-\lambda-2)!!}{n!! (-\lambda-2)!! (n+\lambda+1)!!}, \quad \text{if } n \text{ be even, other-}$$

wise the integral is zero.

This includes, as particular cases,

$$\begin{aligned} \int_{-1}^{+1} P_n \sin^\lambda \theta d\mu &= 0 \text{ if } \lambda \text{ be even and } n > \lambda + 1, \\ &= (-1)^{\frac{\lambda+1}{2}} \pi \frac{(n-1)!! \lambda!! \lambda!! (n-\lambda-2)!!}{n!! (n+\lambda+1)!!} \text{ if } \lambda \text{ be odd and } n \equiv \lambda + 1. \\ &= (-1)^{\frac{n}{2}} c \frac{(n-1)!! \lambda!! \lambda!!}{n!! (\lambda+n+1)!! (\lambda-n)!!} \text{ if } n < \lambda + 1. \end{aligned}$$

Dr. W. D. NIVEN ('Phil. Trans.,' vol. 170 (1879 I.), p. 379) has already obtained an expression for the integral $\int_{-1}^{+1} P_n \sin^\lambda \theta d\mu$. His results are identical with the above, when allowing for the difference in the notation and after correcting an obvious misprint in the equation marked (16) on p. 388.

$$\S 5. \int_{-1}^{+1} P_\epsilon \frac{d^\rho P_n}{d\mu^\rho} d\mu.$$

If we write down the differential coefficients of P_n by means of the equation

$$dP_n/d\mu = (2n-1) P_{n-1} + (2n-5) P_{n-3} + \dots,$$

and repeat the same process ρ times, we obtain a series beginning with

$$d^\rho P_n/d\mu^\rho = (2n-1)(2n-3)\dots(2n-2\rho+1) P_{n-\rho} + \dots$$

There being no term containing a zonal harmonic of higher degree than $n-\rho$, we conclude that the above integral vanishes when $\epsilon > n-\rho$, and that when $\epsilon = n-\rho$

$$\int_{-1}^{+1} P_\epsilon \frac{d^\rho P_n}{d\mu^\rho} d\mu = \frac{2(2n-1)!!}{(2n-2\rho+1)!!}.$$

If $\epsilon < n-\rho$ we may transform the integral as follows :

$$\int_{-1}^{+1} P_\epsilon \frac{d^\rho P_n}{d\mu^\rho} d\mu = \frac{1}{2^\epsilon \epsilon!} \int_{-1}^{+1} \frac{d^\epsilon (\mu^2-1)^\epsilon}{d\mu^\epsilon} \cdot \frac{d^\rho P_n}{d\mu^\rho} d\mu.$$

After ϵ partial integrations, in which the first term always vanishes at both limits, the integral becomes

$$\frac{(-1)^\epsilon}{2^\epsilon \epsilon!} \int_{-1}^{+1} (\mu^2-1)^\epsilon \frac{d^{\rho+\epsilon} P_n}{d\mu^{\rho+\epsilon}} = \frac{1}{(2\epsilon)!!} \int_{-1}^{+1} \sin^{\epsilon-\rho} \theta Q_n^{\rho+\epsilon} d\mu.$$

The integral on the right-hand side has been found in the last paragraph. Writing $\rho + \epsilon = \sigma$ and $\epsilon - \rho = \lambda$, we note that $(n - \lambda - 2) = (n - \epsilon + \rho - 2)$ is necessarily positive as $\epsilon < n - \rho$, and that $\sigma - \lambda$ is positive and even. If, further, $n - \sigma$ be even $n - \lambda$ must be even also, because the difference between these quantities is $\sigma - \lambda = 2\rho$. We may now write the value of the integral

$$\int_{-1}^{+1} P_{\epsilon} \frac{d^{\rho} P_n}{d\omega^{\rho}} d\mu = 2 \frac{(n + \epsilon + \rho - 1)!! (n - \epsilon + \rho - 2)!!}{(n - \epsilon - \rho)!! (n + \epsilon - \rho + 1)!!} \frac{1}{(2\rho - 2)!!}$$

if $\epsilon \equiv n - \rho$ and $n + \rho + \epsilon$ is even,
 $= 0$, in all other cases.

If $\rho = 1$, the above reduces to

$$\int_{-1}^{+1} P_{\epsilon} \frac{dP_n}{d\mu} d\mu = 2.$$

$$\S 6. \int_{-1}^{+1} Q_n^{\sigma+2} Q_i^{\sigma} d\mu, \int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} d\mu, \int_{-1}^{+1} Q_n^{\sigma} Q_{n-2}^{\rho} d\mu.$$

Before discussing the general integral of the product of two tesseral functions, it is convenient to obtain the solution in a few special cases. When the type of the two tesserals differs by 2, we may transform the integrals as follows:

$$\begin{aligned} & \int_{-1}^{+1} (1 - \mu^2)^{\frac{\sigma+2}{2}} \frac{d^{\sigma+2} P_n}{d\mu^{\sigma+2}} (1 - \mu^2)^{\frac{\sigma}{2}} \frac{d^{\sigma} P_i}{d\mu^{\sigma}} d\mu \\ &= \int_{-1}^{+1} (1 - \mu^2)^{\sigma+1} \frac{d^{\sigma+2} P_n}{d\mu^{\sigma+2}} \cdot \frac{d^{\sigma} P_i}{d\mu^{\sigma}} d\mu \\ &= \frac{1}{2i+1} \int_{-1}^{+1} (1 - \mu^2)^{\sigma+1} \frac{d^{\sigma+2} P_n}{d\mu^{\sigma+2}} \left(\frac{d^{\sigma+1} P_{i+1}}{d\mu^{\sigma+1}} - \frac{d^{\sigma+1} P_{i-1}}{d\mu^{\sigma+1}} \right) d\mu. \end{aligned}$$

By the application of RODRIGUEZ'S theorem the right-hand side reduces to

$$\frac{(-1)^{\sigma+1} (i+\sigma)!}{2i+1 (i-\sigma)!} \int_{-1}^{+1} \frac{d^{\sigma+2} P_n}{d\mu^{\sigma+2}} \left[(i+\sigma+2) (i+\sigma+1) \frac{d^{\sigma-1} P_{i+1}}{d\mu^{\sigma+1}} - (i-\sigma) (i-\sigma-1) \frac{d^{\sigma-1} P_{i-1}}{d\mu^{\sigma+1}} \right] d\mu.$$

Integrating each term partially $\sigma + 1$ times we arrive at the equation

$$\begin{aligned} & \int_{-1}^{+1} Q_n^{\sigma+2} Q_i^{\sigma} d\mu \\ &= \frac{1}{2i+1} \frac{(i+\sigma)!}{(i-\sigma)!} \int_{-1}^{+1} \frac{dP_n}{d\mu} [(i+\sigma+2) (i+\sigma+1) P_{i+1} - (i-\sigma) (i-\sigma-1) P_{i-1}] d\mu. \end{aligned}$$

If $i + n$ be odd, or if $i > n$, the integrals on the right-hand side vanish. If $i < n$ and $i + n$ even, we must substitute,

$$\int_{-1}^{+1} \frac{dP_n}{d\mu} P_{i+1} = \int_{-1}^{+1} \frac{dP_n}{d\mu} P_{i-1} = 2, \text{ as found above.}$$

which gives

$$\int_{-1}^{+1} Q_n^{\sigma+2} Q_i^{\sigma} d\mu = 4 \frac{(i+\sigma)!}{(i-\sigma)!} (\sigma+1).$$

When $n = i$, that part of the integral which depends on P_{i+1} vanishes, so that in that case

$$\int_{-1}^{+1} Q_n^{\sigma+2} Q_n^{\sigma} d\mu = -\frac{2}{2n+1} \frac{(n+\sigma)!}{(n-\sigma-2)!}.$$

More particularly when $\sigma = 0$ we have the integrals,

$$\begin{aligned} \int_{-1}^{+1} Q_n^2 P_i d\mu &= 4, \text{ if } n+i \text{ is even and } i > n \\ &= 0 \text{ in all other cases,} \end{aligned}$$

$$\int_{-1}^{+1} Q_n^2 P_n d\mu = -2 \frac{n \cdot n-1}{2n+1}.$$

The result that $\int_{-1}^{+1} Q_n^{\sigma+2} Q_i^{\sigma} d\mu$ has zero value whenever $i > n$ can be extended to the more general integral $\int_{-1}^{+1} Q_n^{\sigma} Q_i^{\rho} d\mu$ provided then $\sigma > \rho$. To show this we need only consider the series,

$$Q_n^{\sigma} = A_0 Q_n^{\sigma-2} + A_2 Q_n^{\sigma-4} + \dots,$$

where the coefficients may be determined from the integrals found above. Multiplying both sides with $Q_i^{\sigma-4}$ and integrating, all the terms vanish when $i > n$ and hence $\int Q_n^{\sigma} Q_i^{\sigma-4} = 0$ in that case. From $\sigma - 4$ we may proceed to $\sigma - 6$ and so on.

To obtain the integral $\int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} d\mu$, we use formula (F), multiplying both sides by Q_n^{ρ} and integrating. We may without loss of generality take ρ to be smaller than σ . Utilising the result which has just been obtained, one integral on each side drops out and the equation becomes for even values of $\sigma + \rho$,

$$\begin{aligned} \int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} &= -(n-\sigma+1)(n-\sigma+2) \int_{-1}^{+1} Q_n^{\sigma-2} Q_n^{\rho} d\mu \\ &= (n-\sigma+1)(n-\sigma+2)(n-\sigma+3)(n-\sigma+4) \int_{-1}^{+1} Q_n^{\sigma-4} Q_n^{\rho} d\mu \\ &= (-1)^{\frac{\sigma-\rho}{2}} \frac{(n-\rho)!}{(n-\sigma)!} \int_{-1}^{+1} Q_n^{\sigma} Q_n^{\rho} d\mu \\ &= (-1)^{\frac{\sigma-\rho}{2}} \frac{(n+\rho)!}{(n-\sigma)!} \cdot \frac{2}{2n+1} \text{ if } \rho < \sigma \text{ and } \rho + \sigma \text{ even} \\ &= 0 \text{ if } \rho + \sigma \text{ is odd.} \end{aligned}$$

Reverting now to equation F and multiplying both sides with Q_{n-2}^ρ , we obtain

$$\begin{aligned}
 & \int_{-1}^{+1} Q_n^\sigma Q_{n-2}^\rho d\mu \\
 &= -(n - \sigma + 1) (n - \sigma + 2) \int_{-1}^{+1} Q_n^{\sigma-2} Q_{n-2}^\rho d\mu \\
 &\quad + (n + \sigma - 2) (n + \sigma - 3) \int_{-1}^{+1} Q_{n-2}^{\sigma-2} Q_{n-2}^\rho d\mu + \int_{-1}^{+1} Q_{n-2}^\sigma Q_{n-2}^\rho d\mu \\
 &= -(n - \sigma + 1) (n - \sigma + 2) \int_{-1}^{+1} Q_n^{\sigma-2} Q_{n-2}^\rho d\mu \\
 &\quad + (-1)^{\frac{\sigma-\rho+2}{2}} \frac{2}{2n-3} \left[(n + \sigma - 2) (n + \sigma - 3) \frac{(n-2+\rho)!}{(n-\sigma)!} - \frac{(n-2+\rho)!}{(n-\sigma-2)!} \right] \\
 &= -(n - \sigma + 1) (n - \sigma + 2) \int_{-1}^{+1} Q_n^{\sigma-2} Q_{n-2}^\rho d\mu - (-1)^{\frac{\sigma-\rho+2}{2}} 4 (\sigma - 1) \frac{(n + \rho - 2)}{(n - \sigma)!}.
 \end{aligned}$$

The integral on the right-hand side may again be transformed in the same manner, changing from σ to $\sigma - 2$. This leads to

$$\begin{aligned}
 \int_{-1}^{+1} Q_n^\sigma Q_{n-2}^\rho d\mu &= (n - \sigma + 1) (n - \sigma + 2) (n - \sigma + 3) (n - \sigma + 4) \int_{-1}^{+1} Q_n^{\sigma-4} Q_{n-2}^\rho d\mu \\
 &\quad + (-1)^{\frac{\sigma-\rho+2}{2}} 4 \frac{(n + \rho - 2)!}{(n - \sigma)!} \{(\sigma - 1) + (\sigma - 3)\}.
 \end{aligned}$$

If the same process be continued until the integral on the right-hand side becomes $\int_{-1}^{+1} Q_n^{\rho+2} Q_{n-2}^\rho d\mu$, the remaining terms on that side will consist of a series in arithmetical progression.

Adding this, we find

$$\begin{aligned}
 \int_{-1}^{+1} Q_n^\sigma Q_{n-2}^\rho d\mu &= (-1)^{\frac{\sigma-\rho+2}{2}} \left[\frac{(n-2-\rho)!}{(n-\sigma)!} \int_{-1}^{+1} Q_n^{\rho+2} Q_{n-2}^\rho d\mu \right. \\
 &\quad \left. + \frac{(n-2+\rho)!}{(n-\sigma)!} (\sigma + \rho + 2) (\sigma - \rho - 2) \right] \\
 &= (-1)^{\frac{\sigma-\rho+2}{2}} \left[\frac{(n-2-\rho)!}{(n-\sigma)!} \cdot \frac{(n-2+\rho)!}{(n-2-\rho)!} (4\rho + 4) \right. \\
 &\quad \left. + \frac{(n-2+\rho)!}{(n-\sigma)!} (\sigma + \rho + 2) (\sigma - \rho - 2) \right] \\
 &= (-1)^{\frac{\sigma-\rho+2}{2}} \frac{(n + \rho - 2)!}{(n - \sigma)!} (\sigma^2 - \rho^2), \quad \text{if } \sigma + \rho \text{ be even and } \rho < \sigma.
 \end{aligned}$$

If $\sigma + \rho$ is odd or if $\rho > \sigma$ the integral is zero.

$$\S 7. \int_{-1}^{+1} Q_n^\sigma P_i d\mu, \quad \int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu.$$

The integral of the product of a tesseral function and of a power of $\sin \theta$ having been obtained in § 4, the above integrals are found by expanding the tesseral function or the zonal harmonic in a series proceeding by powers of $\sin \theta$.

From the expression of a tesseral function of degree n and type σ , as it is generally given, viz.:

$$\sin^\sigma \theta [A_0 \mu^{n-\sigma} + A_2 \mu^{n-\sigma-2} + \dots],$$

it is seen by writing $x = \sin \theta$, $\mu = \sqrt{1-x^2}$, and expanding the binomials, that the term of lowest power will be $\sin^\sigma \theta$, and that if $n - \sigma$ be even, the term of highest power is $\sin^n \theta$.

The coefficients of the series, which are given by HEINE, are most easily obtained by going back to the differential equation:

$$n(n+1) Q_n^\sigma + \frac{d}{d\mu} (1-\mu^2) \frac{d}{d\mu} Q_n^\sigma = \frac{\sigma^2}{1-\mu^2} Q_n^\sigma.$$

Substituting $x = \sqrt{1-\mu^2}$, and changing variables, this becomes

$$n(n+1) Q_n^\sigma + \frac{1-2x^2}{x} \frac{dQ_n^\sigma}{dx} + (1-x^2) \frac{d^2 Q_n^\sigma}{dx^2} = \frac{\sigma^2}{x^2} Q_n^\sigma.$$

If the series

$$a_\sigma x^\sigma + a_{\sigma+2} x^{\sigma+2} + \dots + a_q x^q + a_{q+2} x^{q+2} + \dots$$

satisfies this differential equation, the coefficients a_{q+2} and a_q must be connected by the relation

$$a_q (n-q)(n+q+1) + a_{q+2} (q-\sigma+2)(q+\sigma+2) = 0,$$

as is seen by substitution.

The first coefficient is determined by the fact that it is equal to the value of $d^\sigma P_n / d\mu^\sigma$ when $\mu = 1$. This quantity is known to be equal to $\frac{(n+\sigma)!}{(n-\sigma)! (2\sigma)!} \frac{1}{(2\sigma)!}$. The other coefficients may now be determined in terms of this, and we find in this way for Q_n^σ the series

$$\frac{(n+\sigma)!}{(n-\sigma)! (2\sigma)!} \frac{x^\sigma}{(2\sigma)!} \left[1 - \frac{(n-\sigma)(n+\sigma+1)}{1 \cdot \sigma+1} \left(\frac{x}{2}\right)^2 + \frac{(n-\sigma)(n-\sigma-2)(n+\sigma+1)(n+\sigma+3)}{1 \cdot 2 \cdot \sigma+1 \cdot \sigma+2} \left(\frac{x}{2}\right)^4 + \dots \right].$$

The series breaks off with the term $x^{n-\sigma}$, when $n - \sigma$ is an even number, but also holds if $n - \sigma$ is odd.

The factor of x^λ in the series reduces to

$$(-1)^{\frac{1}{2}(\lambda-\sigma)} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!} \cdot \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!}.$$

When $(n-\sigma)$ is an odd number it will be more convenient to use a different series. Writing $Q = \mu N$, and changing the variable to $x = \sqrt{1-\mu^2}$ as before, the differential equation becomes

$$n-1 \cdot n+2 N \frac{(1-4x^2)}{x} \frac{dN}{dx} + (1-x^2) \frac{d^2N}{dx^2} = \frac{\sigma^2 N}{x^2},$$

and from this we find for Q_n^σ the series

$$\frac{(n+\sigma)!}{(n-\sigma)!(2\sigma)!!} \mu x^\sigma \left[1 - \frac{(n-\sigma-1)(n+\sigma+2)}{1 \cdot \sigma+1} \left(\frac{x}{2}\right)^2 + \frac{(n-\sigma-1)(n-\sigma-3)(n+\sigma+2)(n+\sigma+3)}{1 \cdot 2 \cdot \sigma+1 \cdot \sigma+2} \left(\frac{x}{2}\right)^4 - \dots \right].$$

The factor of μx^λ in the series is found to be

$$(-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!} \cdot \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!}.$$

In considering the integral $\int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu$, we may, without loss of generality, take ρ to be smaller than σ . If of the two quantities $i-\rho$ and $n-\sigma$, one is odd and the other even, the integral vanishes.

If $(n-\sigma)$ and $(i-\rho)$ are both even numbers, we may express Q_n^σ in terms of a series of powers of $\sin \theta$ and obtain in this way :

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu &= \sum_{\lambda=\sigma}^{\lambda=n} {}^{(2)}_{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} Q_i^\rho \sin^\lambda \theta d\mu \\ &= c \sum_{\lambda=\sigma}^{\lambda=n} {}^{(2)}_{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!(i+\rho-1)!!(i-\lambda-2)!!(\rho+\lambda)!!}{(n-\sigma-1)!!(n-\lambda)!!(i-\rho)!!(i+\lambda+1)!!(\rho-\lambda-2)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \end{aligned}$$

The symbol $\sum_{\lambda=\sigma}^{\lambda=n} {}^{(2)}_{\lambda}$ is intended to express that λ takes successively the values $\sigma, \sigma+2, \dots, n$, leaving out the alternate numbers. The constant c is equal to 2 or π , according as $\rho-\sigma$ is even or odd.

If $n-\sigma$ and $i-\rho$ are both odd, we find in the same way :

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu &= \sum_{\lambda=\sigma}^{\lambda=n-1} {}^{(2)}_{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \mu Q_i^\rho \sin^\lambda \theta d\mu \\ &= c \sum_{\lambda=\sigma}^{\lambda=n-1} {}^{(2)}_{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!(i+\rho)!!(i-\lambda-3)!!(\rho+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!(i-\rho-1)!!(i+\lambda+2)!!(\rho-\lambda-2)!!} \frac{1}{(\lambda-\sigma)!!(\lambda+\sigma)!!} \end{aligned}$$

The value of c is the same as before.

Our result shows that when $\rho-\sigma$ is even, the integral vanishes when $i > n$. For when $\rho-\sigma$ is even, $\rho-\lambda-2$ will also be even, for all values of λ , and

it will be negative because σ is the smallest value of λ and σ is larger than ρ . $1/(\rho - \lambda - 2)!!$ must therefore be zero, and all the terms of the sum vanish unless there is a negative even factorial in the numerator. The only factorial which can become negative is $(i - \lambda - 2)!!$, but n being the highest value of λ , it will not be negative when $i > n$. For the special case, $i = n$, all the terms of the sum vanish except the last, for which $\lambda = n$. We have in that case

$$\frac{(i - \lambda - 2)!!}{(\rho - \lambda - 2)!!} = \frac{(i - n - 2)!!}{(\rho - n - 2)!!} = \frac{(-2)!!}{(\rho - n - 2)!!} = (-1)^{\frac{n-\rho}{2}} (n - \rho)!!,$$

and the integral reduces to $(-1)^{\frac{\sigma-\rho}{2}} \frac{2}{2n+1} \frac{(n+\rho)!}{(n-\sigma)!}$, as already found in § 6.

The same expression is found when $n - \sigma$ is odd.

Another and generally more convenient expression for $\int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu$ is found by expanding Q_i^ρ instead of Q_n^σ in terms of powers of $\sin \theta$. We thus obtain, when $n - \sigma$ and $i - \rho$ are both even,

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu = c \sum_{\lambda=\rho}^{\lambda=i} (-1)^{\frac{\lambda-\rho}{2}} \frac{(n+\sigma-1)!!(i+\rho)!!(i+\lambda-1)!!(n-\lambda-2)!!}{(n-\sigma)!!(i-\rho-1)!!(i-\lambda)!!(n+\lambda+1)!!} \frac{1}{(\lambda-\rho)!!(\lambda+\rho)!!} \frac{(\sigma+\lambda)!!}{(\sigma-\lambda-2)!!},$$

and when $n - \sigma$ and $i - \rho$ are both odd,

$$= c \sum_{\lambda=\rho}^{\lambda=i-1} (-1)^{\frac{\lambda-\rho}{2}} \frac{(n+\sigma)!!(i+\rho-1)!!(n-\lambda-3)!!(i+\lambda)!!}{(n-\sigma-1)!!(i-\rho)!!(n+\lambda+2)!!(i-\lambda-1)!!} \frac{1}{(\lambda-\rho)!!(\lambda+\rho)!!} \frac{(\sigma+\lambda)!!}{(\sigma-\lambda-2)!!}.$$

Writing

$$A_0 = \frac{(\sigma+\rho)(\sigma+\rho-2)\dots(\sigma-\rho+2)(\sigma-\rho)}{(2\rho)!!},$$

$$A_2 = \frac{(\sigma+\rho+2)(\sigma+\rho)\dots(\sigma-\rho)(\sigma-\rho-2)}{2!!(2\rho+2)!!},$$

$$A_4 = \frac{(\sigma+\rho+4)(\sigma+\rho+2)\dots(\sigma-\rho-2)(\sigma-\rho-4)}{4!!(2\rho+4)!!},$$

we may put the integral into the following form :

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho d\mu, \text{ if } (n - \sigma) \text{ even, } (i - \rho) \text{ even, and excluding } n = i \\ = c \frac{(i+\rho)!}{(i-\rho)!} \frac{(n+\sigma-1)!!}{(n-\sigma)!!} \frac{(n-\rho-2)!!}{(n+\rho+1)!!} \Sigma,$$

where

$$\Sigma = A_0 - A_2 \frac{(i+\rho+1)(i-\rho)}{(n-\rho-2)(n+\rho+3)} + A_4 \frac{(i+\rho+1)(i+\rho+3)(i-\rho)(i-\rho-2)}{(n-\rho-2)(n-\rho-4)(n+\rho+3)(n+\rho+5)} \dots$$

where $c = 2$, if $\sigma - \rho$ is even, and $c = \pi$, if $\sigma - \rho$ is odd,

the last term of the series is

$$A_{i-\rho} \frac{[(i+\rho+1)(i+\rho+3)\dots(2i-1)][(i-\rho)(i-\rho-2)\dots 2]}{[(n-\rho-2)(n-\rho-4)\dots(n-i)][(n+\rho+3)(n+\rho+5)\dots(n+i+1)]},$$

but the series breaks off before the end, if $\sigma - \rho$ is even and $i > \sigma$ owing to the factor A vanishing. The number of terms in that case is $\sigma - \rho - 1$.

Similarly when $(n - \sigma)$ and $(i - \rho)$ are both odd

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\sigma d\mu = c \frac{(i+\rho)!}{(i-\rho)!} \frac{(n+\sigma)!!(n-\rho-3)!!}{(n-\sigma-1)!!(n+\rho+2)!!} \Sigma,$$

where

$$\Sigma = A_0 = A_2 \frac{(i+\rho+2)(i-\rho-1)}{(n-\rho-3)(n+\rho+4)} + A_4 \frac{(i+\rho+2)(i+\rho+4)(i-\rho-1)(i-\rho-3)}{(n-\rho-3)(n-\rho-5)(n+\rho+4)(n+\rho+6)} \dots$$

where c and the coefficients A have the same value as before. The last term of the series is

$$A_{i-\rho-1} \frac{[(i+\rho+2)(i+\rho+4)\dots(2i-1)][(i+\rho+1)(i-\rho-3)\dots 2]}{[(n-\rho-3)(n-\rho-5)\dots(n-i)][(n+\rho+4)(n+\rho+6)\dots(n+i+1)]}.$$

To obtain $\int Q_n^\sigma P_i d\mu$, we need only put $\rho = 0$ in the previous investigation. This gives :

$$A_0 = \sigma, \quad A_2 = \frac{\sigma+2 \cdot \sigma \cdot \sigma-2}{2 \cdot 2}, \quad A_4 = \frac{\sigma+4 \cdot \sigma+2 \cdot \sigma \cdot \sigma-2 \cdot \sigma-4}{2 \cdot 4 \cdot 2 \cdot 4},$$

$$\int_{-1}^{+1} Q_n^\sigma P_i d\mu = 2 \frac{(n+\sigma-1)!!(n-2)!!}{(n-\sigma)!!(n+1)!!} \left[A_0 - A_2 \frac{i \cdot i+1}{n-2 \cdot n+3} + A_4 \frac{i \cdot i-2 \cdot i+1 \cdot i+3}{n-2 \cdot n-4 \cdot n+3 \cdot n+5} \dots \right],$$

if i and σ is even, and

$$\int_{-1}^{+1} Q_n^\sigma P_i d\mu = 2 \frac{(n+\sigma)!!(n-3)!!}{(n-\sigma-1)!!(n+2)!!} \left[A_0 - A_2 \frac{i-1 \cdot i+2}{n-3 \cdot n+4} + A_4 \frac{i-1 \cdot i-3 \cdot i+2 \cdot i+4}{n-3 \cdot n-5 \cdot n+4 \cdot n+6} \dots \right],$$

if σ is even and i odd. Thus, if i be even

$$\int_{-1}^{+1} Q_n^2 P_i d\mu = 4,$$

$$\int_{-1}^{+1} Q_n^4 P_i d\mu = 8 [(n+3 \cdot n-2) - 3(i \cdot i+1)],$$

$$\int_{-1}^{+1} Q_n^6 P_i d\mu = 12 [(n+5 \cdot n+3 \cdot n-2 \cdot n-4) - 8(i \cdot i+1 \cdot n+5 \cdot n-4) + 10(i \cdot i-2 \cdot i+1 \cdot i+3)],$$

and if i be odd :

$$\int_{-1}^{+1} Q_n^2 P_i d\mu = 4,$$

$$\int_{-1}^{+1} Q_n^4 P_i d\mu = 8 [(n+4 \cdot n-3) - 3(i+2 \cdot i-1)],$$

$$\int_{-1}^{+1} Q_n^6 P_i d\mu = 12 [(n+6 \cdot n+4 \cdot n-3 \cdot n-5) - 8(i+2 \cdot i-1 \cdot n+6 \cdot n-5) + 10(i+2 \cdot i+4 \cdot i-1 \cdot i-3)].$$

The equations do not hold for $i = n$ as has already been explained.

$$\S 8. \int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu, \quad \int_{-1}^{+1} Q_n^\sigma \cos p\theta d\mu.$$

These important integrals are obtained in two different forms according as Q_n^σ or the trigonometrical functions are expressed in terms of the powers of $\sin \theta$. In the former case the problem reduces itself to the evaluation of integrals of the form

$$\int_0^\pi \sin^\lambda \theta \sin p\theta d\rho \quad \text{and} \quad \int_0^\pi \sin^\lambda \theta \cos \theta \sin p\theta d\theta.$$

It may easily be proved that

$$\begin{aligned} \int_{-1}^{+1} \sin^\lambda \theta \sin p\theta d\mu &= (-1)^{\frac{p-1}{2}} c \frac{(\lambda+1)!}{(\lambda+1+p)!!(\lambda+1-p)!!} \text{ when } p \text{ is odd,} \\ &= 0 \text{ when } p \text{ is even.} \end{aligned}$$

$$\begin{aligned} \int_{-1}^{+1} \sin^\lambda \theta \cos p\theta d\mu &= (-1)^{\frac{p}{2}} c \frac{(\lambda+1)!}{(\lambda+1+p)!!(\lambda+1-p)!!} \text{ when } p \text{ is even,} \\ &= 0 \text{ when } p \text{ is odd.} \end{aligned}$$

$$\begin{aligned} \int_{-1}^{+1} \sin^{\lambda-1} \theta \cos \theta \sin p\theta d\mu &= (-1)^{\frac{p+2}{2}} c p \frac{\lambda!}{(\lambda+1+p)!!(\lambda+1-p)!!} \text{ when } p \text{ is even,} \\ &= 0 \text{ when } p \text{ is odd.} \end{aligned}$$

$$\begin{aligned} \int_{-1}^{+1} \sin^{\lambda-1} \theta \cos \theta \cos p\theta d\mu &= (-1)^{\frac{p-1}{2}} c p \frac{\lambda!}{(\lambda+1+p)!!(\lambda+1-p)!!} \text{ when } p \text{ is odd,} \\ &= 0 \text{ when } p \text{ is even.} \end{aligned}$$

In these equations c is equal to 2 or π according as $(p+\lambda)$ is even or odd.

From the results of § 7 we may now write, if $n-\sigma$ be even and p odd,

$$\begin{aligned}
& \int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu \\
&= \sum_{\lambda=\sigma}^{\lambda=n} \binom{2}{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \sin^\lambda \theta \sin p\theta \, d\mu \\
&= c \sum_{\lambda=\sigma}^{\lambda=n} \binom{2}{\lambda} (-1)^{\frac{\lambda+p-\sigma-1}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{(\lambda+1)!}{(\lambda+1+p)!!(\lambda+1-p)!!}.
\end{aligned}$$

When p and $n - \sigma$ are both even the integral vanishes.

If $(n - \sigma)$ be odd and p even,

$$\begin{aligned}
& \int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu \\
&= \sum_{\lambda=\sigma}^{\lambda=n-1} \binom{2}{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \cos \theta \sin^\lambda \theta \sin p\theta \, d\mu \\
&= cp \sum_{\lambda=\sigma}^{\lambda=n-1} \binom{2}{\lambda} (-1)^{\frac{\lambda+p-\sigma+2}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{(\lambda+1)!}{(\lambda+2+p)!!(\lambda+2-p)!!}.
\end{aligned}$$

It will be noticed that the integral is zero whenever $(n + \sigma + p)$ is even. Hence if $\sin p\theta$ is expressed in terms of tesseral functions, p being even, only odd values of n occur when σ is even, and only even values when σ is an odd number. The reverse will be the case when p is odd.

We find similarly, if $(n - \sigma)$ is odd and p odd,

$$\begin{aligned}
& \int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu \\
&= \sum_{\lambda=\sigma}^{\lambda=n-1} \binom{2}{\lambda} (-1)^{\frac{\lambda-\sigma}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \int_{-1}^{+1} \cos \theta \sin^\lambda \theta \cos p\theta \, d\mu \\
&= cp \sum_{\lambda=\sigma}^{\lambda=n-1} \binom{2}{\lambda} (-1)^{\frac{\lambda+p-\sigma-1}{2}} \frac{(n+\sigma-1)!!(n+\lambda)!!}{(n-\sigma)!!(n-\lambda-1)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{(\lambda+1)!}{(\lambda+2+p)!!(\lambda+2-p)!!},
\end{aligned}$$

and when $n - \sigma$ and p are both even,

$$\begin{aligned}
& \int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu \\
&= c \sum_{\lambda=\sigma}^{\lambda=n} \binom{2}{\lambda} (-1)^{\frac{\lambda-\sigma+p}{2}} \frac{(n+\sigma)!!(n+\lambda-1)!!}{(n-\sigma-1)!!(n-\lambda)!!(\lambda-\sigma)!!(\lambda+\sigma)!!} \frac{(\lambda+1)!}{(\lambda+1+p)!!(\lambda+1-p)!!},
\end{aligned}$$

when $n - \sigma + p$ is odd the integrals vanish for all values of p .

We may now prove the theorem which has been mentioned in the first paragraph. If σ and p are both even, $\int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu$ is zero, unless n is odd. In that case the quantity $(\lambda + 2 - p)!!$ which occurs in the denominator will be even, as λ takes up

successively the values $\sigma, \sigma + 2$, &c. The highest value which $\lambda + 2 - p$ can take is $n + 1 - p$, and this is negative when $p > n + 1$. It follows that

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu = 0 \quad \text{when } p > n + 1 \text{ and } \sigma \text{ is even.}$$

The restriction that p is even is not necessary, because when p is odd n will be even and the factorial in the denominator, according to the above equations, is $(\lambda + 1 - p)!!$. This again will be infinitely large whenever it is negative, or taking the highest value of λ , which is now n , whenever $p > n + 1$.

We prove exactly in the same way that

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta d\mu = 0 \quad \text{when } p > n + 1 \text{ and } \sigma \text{ is odd.}$$

The value of the integrals in the general case may conveniently be expressed by two series Σ_1 and Σ_2 defined by

$$\begin{aligned} \frac{(n-p+1)!!(n+p+1)!!(n-\sigma)!}{(2n-1)!!(n+1)!} \Sigma_1 &\equiv 1 - \frac{(n+\sigma)(n-\sigma)(n+p+1)(n-p+1)}{2 \cdot n+1 \cdot n \cdot 2n-1} \\ &+ \frac{(n+\sigma)(n+\sigma-2)(n-\sigma)(n-\sigma-2)(n+p+1)(n+p-1)(n-p+1)(n-p-1)}{2 \cdot 4 \cdot n+1 \cdot n \cdot n-1 \cdot n-2 \cdot 2n-1 \cdot 2n-3} \dots \\ \frac{(n-p+1)!!(n+p+1)!!(n-\sigma)!}{(2n-1)!!n!p} \Sigma_2 &\equiv 1 - \frac{(n+\sigma-1)(n-\sigma-1)(n+p+1)(n-p+1)}{2 \cdot n \cdot n-1 \cdot 2n-1} \\ &+ \frac{(n+\sigma-1)(n+\sigma-3)(n-\sigma-1)(n-\sigma-3)(n+p+1)(n+p-1)(n-p+1)(n-p-1)}{2 \cdot 4 \cdot n \cdot n-1 \cdot n-2 \cdot n-3 \cdot 2n-1 \cdot 2n-3} \dots \end{aligned}$$

Then

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu &= (-1)^{\frac{n-\sigma+p-1}{2}} c \Sigma_1 \text{ if } p \text{ be odd and } n + \sigma \text{ is even.} \\ &= (-1)^{\frac{n-\sigma+p+1}{2}} c \Sigma_2 \text{ if } p \text{ be even and } n + \sigma \text{ is odd.} \\ &= 0 \text{ whenever } n + p + \sigma \text{ is even.} \end{aligned}$$

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma \cos p\theta d\mu &= (-1)^{\frac{n-\sigma+p}{2}} c \Sigma_1 \text{ if } p \text{ be even and } n + \sigma \text{ is even.} \\ &= (-1)^{\frac{n-\sigma+p+2}{2}} c \Sigma_2 \text{ if } p \text{ be odd and } n + \sigma \text{ is odd.} \\ &= 0 \text{ whenever } n + p + \sigma \text{ is odd.} \end{aligned}$$

The constant c takes the value 2 or π , according as $p + n$ is even or odd. The series Σ_1 and Σ_2 break off in all cases ultimately when one of the factors of the products $(n - \sigma) \cdot (n - \sigma - 2) \dots$ or $(n - \sigma - 1) (n - \sigma - 3) \dots$ become zero. When $n - p$ is odd, the series may break off before it has reached its full number of

terms, owing to one of the factors in the product $(n - p + 1) (n - p - 1) \dots$ becoming zero. In the special case $n = \sigma$ the value of Σ_1 reduces to

$$\frac{(2n - 1)!! (n + 1)!}{(n - p + 1)!! (n + p + 1)!!}.$$

The above equations are inconvenient for numerical calculations, unless $n - \sigma$ be small, or $n - p$ be even and small. We obtain quite different and generally more convenient expressions, if we begin by expressing $\sin p\theta$ or $\cos p\theta$ in terms of a series proceeding by powers of $\sin \theta$.

Let

$$B_0 = 1; B_1 = p; B_2 = \frac{p-1}{1} \cdot \frac{p+1}{2}; B_3 = \frac{p-2}{1} \cdot \frac{p}{2} \cdot \frac{p+2}{3};$$

and generally

$$B^\lambda = \frac{(p - \lambda + 1)(p - \lambda + 3) \dots (p + \lambda - 3)(p + \lambda - 1)}{\lambda!}$$

$$= \frac{(p + \lambda - 1)!!}{(p - \lambda - 1)!!} \frac{1}{\lambda!}.$$

Also put

$$C_0 = 1; C_1 = p; C_2 = p \cdot \frac{p}{2}; C_3 = p \cdot \frac{p-1}{1} \cdot \frac{p+1}{2}; C_4 = p \cdot \frac{p-2}{1} \cdot \frac{p}{2} \cdot \frac{p+2}{3}.$$

$$C_\lambda = p \cdot \frac{(p - \lambda + 2)(p - \lambda + 4) \dots (p + \lambda - 4)(p + \lambda - 2)}{\lambda!}$$

$$= \frac{p}{\lambda!} \cdot \frac{(p + \lambda - 2)!!}{(p - \lambda)!!}$$

Then the well-known expressions for the trigonometrical functions of the multiples of an angle may be written :

If p be even :

$$\frac{\sin p\theta}{\cos \theta} = B_1 \sin \theta - B_3 \sin^3 \theta + B_5 \sin^5 \theta - \dots \pm B_{p-1} \sin^{p-1} \theta,$$

$$\cos p\theta = C_0 - C_2 \sin^2 \theta + C_4 \sin^4 \theta - \dots \pm C_p \sin^p \theta;$$

and if p be odd :

$$\sin p\theta = C_1 \sin \theta - C_3 \sin^3 \theta + C_5 \sin^5 \theta - \dots \pm C_p \sin^p \theta,$$

$$\frac{\cos p\theta}{\cos \theta} = B_0 - B_2 \sin^2 \theta + B_4 \sin^4 \theta - \dots \pm B_{p-1} \sin^{p-1} \theta.$$

We derive from this, p being even :

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu = \sum_{\lambda=1}^{\lambda=p-1} (-1)^{\frac{\lambda-1}{2}} B_\lambda \int_{-1}^{+1} Q_n^\sigma \sin^\lambda \theta \cos \theta \, d\mu,$$

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu = \sum_{\lambda=0}^{\lambda=p} (-1)^{\frac{\lambda}{2}} C_\lambda \int_{-1}^{+1} Q_n^\sigma \sin^\lambda \theta \, d\mu,$$

and p being odd :

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu = \sum_{\lambda=1}^{\lambda=p} (-1)^{\frac{\lambda-1}{2}} C_\lambda \int_{-1}^{+1} Q_n^\sigma \sin^\lambda \theta \, d\mu,$$

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu = \sum_{\lambda=0}^{\lambda=p-1} (-1)^{\frac{\lambda}{2}} B_\lambda \int_{-1}^{+1} Q_n^\sigma \sin^\lambda \theta \cos \theta \, d\mu.$$

Hence from the results of § 4, if p be even and $(n - \sigma)$ odd :

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu = c \sum_{\lambda=1}^{\lambda=p-1} (-1)^{\frac{\lambda-1}{2}} B_\lambda \frac{(n + \sigma)!! (n - \lambda - 3)!! (\sigma + \lambda)!!}{(n - \sigma - 1)!! (n + \lambda + 2)!! (\sigma - \lambda - 2)!!}.$$

The constant c is 2 or π , according as σ is odd or even.

If p be odd and $n - \sigma$ even, and with the same meaning of c :

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu = c \sum_{\lambda=1}^{\lambda=p} (-1)^{\frac{\lambda-1}{2}} C_\lambda \frac{(n + \sigma - 1)!! (n - \lambda - 2)!! (\sigma + \lambda)!!}{(n - \sigma)!! (n + \lambda + 1)!! (\sigma - \lambda - 2)!!}.$$

Similarly if p and $n - \sigma$ are both even,

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu = c \sum_{\lambda=0}^{\lambda=p} (-1)^{\frac{\lambda}{2}} C_\lambda \frac{(n + \sigma - 1)!! (n - \lambda - 2)!! (\sigma + \lambda)!!}{(n - \sigma)!! (n + \lambda + 1)!! (\sigma - \lambda - 2)!!},$$

and if p and $n - \sigma$ be both odd,

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu = c \sum_{\lambda=0}^{\lambda=p-1} (-1)^{\frac{\lambda}{2}} B_\lambda \frac{(n + \sigma)!! (n - \lambda - 3)!! (\sigma + \lambda)!!}{(n - \sigma - 1)!! (n + \lambda + 2)!! (\sigma - \lambda - 2)!!}.$$

In the two last equations c is equal to 2 when σ is even and equal to π when σ is odd. In the interpretation of the summation a little care is required when $\sigma - \lambda - 2$ and $n - \lambda - 2$ or $n - \lambda - 3$ are negative, as the alternate factorials are infinite for even values of the argument.

To put the equations into a form useful for numerical calculations, it is convenient to designate by separate letters the following six series :—

$$M_n^\sigma \equiv C_1 \cdot \sigma - 1 \cdot \sigma + 1 - C_3 \frac{\sigma - 3 \cdot \sigma - 1 \cdot \sigma + 1 \cdot \sigma + 3}{n - 3 \cdot n + 4} \\ + C_5 \frac{\sigma - 5 \cdot \sigma - 3 \cdot \sigma - 1 \cdot \sigma + 1 \cdot \sigma + 3 \cdot \sigma + 5}{n - 3 \cdot n - 5 \cdot n + 4 \cdot n + 6} - \dots$$

$$S_n^\sigma \equiv B_1 \cdot \sigma - 1 \cdot \sigma + 1 - B_3 \frac{\sigma - 3 \cdot \sigma - 1 \cdot \sigma + 1 \cdot \sigma + 3}{n - 4 \cdot n + 5} \\ + B_5 \frac{\sigma - 5 \cdot \sigma - 3 \cdot \sigma - 1 \cdot \sigma + 1 \cdot \sigma + 3 \cdot \sigma + 5}{n - 4 \cdot n - 6 \cdot n + 5 \cdot n + 7} - \dots$$

$$U_n^\sigma \equiv C_0 \cdot \sigma - C_2 \frac{\sigma - 2 \cdot \sigma \cdot \sigma + 2}{n - 2 \cdot n + 3} + C_4 \frac{\sigma - 4 \cdot \sigma - 2 \cdot \sigma \cdot \sigma + 2 \cdot \sigma + 4}{n - 2 \cdot n - 4 \cdot n + 3 \cdot n + 5} - \dots$$

$$V_n^\sigma \equiv B_0 \sigma - B_2 \frac{\sigma - 2 \cdot \sigma \cdot \sigma + 2}{n - 3 \cdot n + 4} + B_4 \frac{\sigma - 4 \cdot \sigma - 2 \cdot \sigma \cdot \sigma + 2 \cdot \sigma + 4}{n - 3 \cdot n - 5 \cdot n + 4 \cdot n + 6} - \dots$$

$$K_n^\sigma \equiv C_n - C_{n+2} \frac{n + 2 - \sigma \cdot n + 2 + \sigma}{2 \cdot 2n + 3} + C_{n+4} \frac{n + 2 - \sigma \cdot n + 4 - \sigma \cdot n + 2 + \sigma \cdot n + 4 + \sigma}{2 \cdot 4 \cdot 2n + 3 \cdot 2n + 5} - \dots$$

$$N_n^\sigma \equiv B_{n-1} - B_{n+1} \frac{n + 1 - \sigma \cdot n + 1 + \sigma}{2 \cdot 2n + 3} + B_{n+3} \frac{n + 1 - \sigma \cdot n + 3 - \sigma \cdot n + 1 + \sigma \cdot n + 3 + \sigma}{2 \cdot 4 \cdot 2n + 3 \cdot 2n + 5} - \dots$$

The series are all continued until they break off, which may happen either because the factors B, C, or one of the other factors takes zero value.

With the help of the functions just defined, the integrals now take the form

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu &= \pi \frac{(n + \sigma - 1)!! (n - 3)!!}{(n - \sigma)!! (n + 2)!!} M_n^\sigma \text{ if } \sigma \text{ be even, } p \text{ odd, } n \text{ even.} \\ &= \pi \frac{(n + \sigma)!! (n - 4)!!}{(n - \sigma - 1)!! (n + 3)!!} S_n^\sigma \text{ if } \sigma \text{ be even, } p \text{ even, } n \text{ odd.} \\ &= 2 \left[\frac{(n + \sigma - 1)!! (n - 3)!!}{(n - \sigma)!! (n + 2)!!} M_n^\sigma + (-1)^{\frac{\sigma-1}{2}} \frac{(n + \sigma)!}{(2n + 1)!!} K_n^\sigma \right] \\ &\quad \text{if } \sigma \text{ be odd, } p \text{ odd, } n \text{ odd.} \\ &= 2 \left[\frac{(n + \sigma)!! (n - 4)!!}{(n - \sigma - 1)!! (n + 3)!!} S_n^\sigma + (-1)^{\frac{\sigma-1}{2}} \frac{(n + \sigma)!}{(2n + 1)!!} N_n^\sigma \right] \\ &\quad \text{if } \sigma \text{ be odd, } p \text{ even, } n \text{ even.} \\ &= 0 \text{ if } n + \sigma + p \text{ be even.} \end{aligned}$$

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu &= 2 \left[\frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n - 1)!!} U_n^\sigma + (-1)^{\frac{\sigma}{2}} \frac{(n + \sigma)!}{(2n + 1)!!} K_n^\sigma \right] \\ &\quad \text{if } \sigma \text{ be even, } p \text{ even, } n \text{ even.} \\ &= 2 \left[\frac{(n + \sigma)!! (n - 3)!!}{(n - \sigma - 1)!! (n + 2)!!} V_n^\sigma + (-1)^{\frac{\sigma}{2}} \frac{(n + \sigma)!}{(2n + 1)!!} N_n^\sigma \right] \\ &\quad \text{if } \sigma \text{ be even, } p \text{ odd, } n \text{ odd.} \end{aligned}$$

$$= \pi \frac{(n + \sigma - 1)!! (n - 2)!!}{(n - \sigma)!! (n + 1)!!} U_n^\sigma \text{ if } \sigma \text{ be odd, } p \text{ even, } n \text{ odd.}$$

$$= \pi \frac{(n + \sigma)!! (n - 3)!!}{(n - \sigma - 1)!! (n + 2)!!} V_n^\sigma \text{ if } \sigma \text{ be odd, } p \text{ odd, } n \text{ even.}$$

$$= 0 \text{ if } n + \sigma + p \text{ be odd.}$$

By comparing these last equations with the results obtained when the same integrals were expressed in terms of the series Σ_1 and Σ_2 , we may derive the following special values :

$$\begin{aligned} M_n^\sigma &= 0 \text{ if } \sigma \text{ be even, } p \text{ odd, } n \text{ even and } p \equiv n + 3, \\ S_n^\sigma &= 0 \text{ ,, } \sigma \text{ ,, ,, } p \text{ even, } n \text{ odd ,, } p \equiv n + 3, \\ U_n^\sigma &= 0 \text{ ,, } \sigma \text{ ,, odd. } p \text{ ,, } n \text{ ,, ,, } p \equiv n + 3. \\ V_n^\sigma &= 0 \text{ ,, } \sigma \text{ ,, ,, } p \text{ odd, } n \text{ even ,, } p \equiv n + 3. \end{aligned}$$

The two series marked N_n^σ and K_n^σ had to be introduced because the series for M_n^σ , S_n^σ , U_n^σ , V_n^σ break off as soon as one of the factors becomes zero. But in the original summation which gave rise to, *e.g.*, M_n^σ , viz. :

$$\sum_{\lambda=1}^{\lambda=p} C_\lambda \frac{(n + \sigma - 1)!! (n - \lambda - 2)!! (\sigma + \lambda)!!}{(n - \sigma)!! (n + \lambda + 1)!! (\sigma - \lambda - 2)!!},$$

$(\sigma - \lambda - 2)!!$ begins to be infinite when $\lambda = \sigma$, and hence the terms of the series will drop out until $(n - \lambda - 2)$ is negative, *i.e.* until $\lambda \equiv n$. For higher values of λ , $\frac{(n - \lambda - 2)!!}{(\sigma - \lambda - 2)!!}$ will be finite again. There is, therefore, a second portion of the series not included under M_n^σ , and it is this second portion which appears as K_n^σ .

§ 9. Relations between M_n^σ , S_n^σ , U_n^σ , V_n^σ .

Certain relations exist between the series which serve to express the integrals $\int_{-1}^{+1} Q_n^\sigma \cos p\theta d\mu$ and $\int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu$, and these relations are important for the calculation of numerical tables.

Starting from the equation (F, § 2)

$$Q_n^\sigma - Q_{n-2}^\sigma = (n + \sigma - 2)(n + \sigma - 3) Q_{n-2}^{\sigma-2} - (n - \sigma + 2)(n - \sigma + 1) Q_n^{\sigma-2},$$

we may multiply by $\cos p\theta$ or $\sin p\theta$ and integrate on both sides. Expressing the result by means of the above series, we obtain the following set of equations :

$$\left. \begin{aligned} (n + \sigma - 1)(n - 3) M_n^\sigma - (n - \sigma)(n + 2) M_{n-2}^\sigma \\ &= (n + \sigma - 2)(n + 2) M_{n-2}^{\sigma-2} - (n - \sigma + 1)(n - 3) M_n^{\sigma-2} \\ (n + \sigma)(n - 4) S_n^\sigma - (n - \sigma - 1)(n + 3) S_{n-2}^\sigma \\ &= (n + \sigma - 3)(n + 3) S_{n-2}^{\sigma-2} - (n - \sigma + 2)(n - 4) S_n^{\sigma-2} \\ (n + \sigma - 1)(n - 2) U_n^\sigma - (n - \sigma)(n + 1) U_{n-2}^\sigma \\ &= (n + \sigma - 2)(n + 1) U_{n-2}^{\sigma-2} - (n - \sigma + 1)(n - 2) U_n^{\sigma-2} \\ (n + \sigma)(n - 3) V_n^\sigma - (n - \sigma - 1)(n + 2) V_{n-2}^\sigma \\ &= (n + \sigma - 3)(n + 2) V_{n-2}^{\sigma-2} - (n - \sigma + 2)(n - 3) V_n^{\sigma-2} \end{aligned} \right\} \text{ (K).}$$

Other relations are obtained as follows :

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma \sin p\theta \, d\mu &= \int_0^\pi Q_n^\sigma \sin \theta \sin p\theta \, d\theta = \frac{1}{p} \int_0^\pi \cos p\theta \frac{d}{d\theta} Q_n^\sigma \sin \theta \, d\theta \\ &= -\frac{1}{p} \int_{-1}^{+1} \cos p\theta \frac{d}{d\mu} Q_n^\sigma \sin \theta \, d\mu, \end{aligned}$$

and making use of (H_2) , § 2,

$$= \frac{1}{2p\sigma} \int_{-1}^{+1} \cos p\theta [(\sigma+1)(n+\sigma)(n-\sigma+1) Q_n^{\sigma-1} - (\sigma-1) Q_n^{\sigma+1}] \, d\mu.$$

If σ be even and n odd, the integral on the left-hand side depends on S_n^σ , while that on the right-hand side depends on $U_n^{\sigma-1}$ and $U_n^{\sigma+1}$. For σ even and n even, the left-hand side may be expressed by M_n^σ , and the right-hand side by $V_n^{\sigma-1}$ and $V_n^{\sigma+1}$. Treating the integral $\int_{-1}^{+1} Q_n^\sigma \cos p\theta \, d\mu$ in the same way, and collecting the results, we find :

$$\left. \begin{aligned} 2p\sigma M_n^\sigma &= (\sigma+1)(n+\sigma)(n-\sigma+1) V_n^{\sigma-1} - (\sigma-1)(n+\sigma+1)(n-\sigma) V_n^{\sigma+1} \\ 2p\sigma S_n^\sigma &= (n-2)(n+3) [(\sigma+1) U_n^{\sigma-1} - (\sigma-1) U_n^{\sigma+1}] \\ 2p\sigma U_n^\sigma &= \frac{1}{(n-2)(n+3)} [(\sigma-1)(n+\sigma+1)(n-\sigma) S_n^{\sigma+1} \\ &\quad - (\sigma+1)(n+\sigma)(n-\sigma+1) S_n^{\sigma-1}] \\ 2p\sigma V_n^\sigma &= (\sigma-1) M_n^{\sigma+1} - (\sigma+1) M_n^{\sigma-1} \end{aligned} \right\} (L).$$

We may also connect together the different values which the same function assumes for different values of p .

Starting from the identity

$$\begin{aligned} \int_{-1}^{+1} Q_n^\sigma [\sin(p+1)\theta - \sin(p-1)\theta] \, d\mu \\ = 2 \int_{-1}^{+1} Q_n^\sigma \sin p\theta \cos \theta \, d\mu \end{aligned}$$

we find, transforming the right-hand side with the help of equation (A), § 2, the integral to be

$$= \frac{1}{2n+1} \int_{-1}^{+1} [(n-\sigma+1) Q_{n+1}^\sigma + (n+\sigma) Q_{n-1}^\sigma] \sin p\theta \, d\mu.$$

Three other equations may be obtained corresponding to this ; by substituting the $+$ for the $-$ sign in the left-hand sign, or by changing the sine function into the cosine function. Each of these three equations gives two relations according as σ is even or odd. The following eight relations are thus obtained :

$$\frac{2n+1}{2} (S_n^\sigma(p+1) + S_n^\sigma(p-1)) = (n-2) M_{n+1}^\sigma(p) + (n+3) M_{n-1}^\sigma(p),$$

$$\begin{aligned} \frac{2n+1}{2} (M_n^\sigma(p+1) + M_n^\sigma(p-1)) &= \frac{(n+\sigma+1)(n-\sigma+1)}{n+4} S_{n+1}^\sigma(p) \\ &+ \frac{(n+\sigma)(n-\sigma)}{n-3} S_n^\sigma(p), \end{aligned}$$

$$\begin{aligned} \frac{2n+1}{2} (M_n^\sigma(p+1) - M_n^\sigma(p-1)) &= (n+\sigma+1)(n-1) U_{n+1}^{\sigma+1}(p) \\ &- (n-\sigma)(n+2) U_{n-1}^{\sigma+1}(p), \end{aligned}$$

$$\begin{aligned} \frac{2n+1}{2} (S_n^\sigma(p+1) - S_n^\sigma(p-1)) &= (n+\sigma+2)(n-2) V_{n+1}^{\sigma+1}(p) \\ &- (n-\sigma-1)(n+3) V_{n-1}^{\sigma+1}(p), \end{aligned}$$

$$\begin{aligned} \frac{2n+1}{2} (U_n^\sigma(p+1) + U_n^\sigma(p-1)) &= \frac{(n+\sigma+1)(n-\sigma+1)}{n+3} V_{n+1}^\sigma(p) \\ &+ \frac{(n+\sigma)(n-\sigma)}{n-2} V_{n-1}^\sigma(p), \end{aligned}$$

$$\frac{2n+1}{2} (V_n^\sigma(p+1) + V_n^\sigma(p-1)) = (n-1) U_{n+1}^\sigma(p) + (n+2) U_{n-1}^\sigma(p),$$

$$\frac{2n+1}{2} (U_n^\sigma(p-1) - U_n^\sigma(p+1)) = \frac{n+\sigma+1}{n+3} M_{n+1}^{\sigma+1}(p) - \frac{n-\sigma}{n-2} M_{n-1}^{\sigma+1}(p),$$

$$\frac{2n+1}{2} (V_n^\sigma(p-1) - V_n^\sigma(p+1)) = \frac{n+\sigma+2}{n+4} S_{n+1}^{\sigma+1}(p) - \frac{n-\sigma-1}{n-3} S_{n-1}^{\sigma+1}(p).$$

§ 10. Numerical Calculations of the Four Series.

For the calculation of the numerical values of the series $M_n^\sigma(p)$, the special case $\sigma = 0$ was taken as starting-point. In this particular case the well-known integral

$$\int_{-1}^{+1} P_n \sin p\theta d\mu = -\pi p \frac{(n+p-2)!!(n-p-2)!!}{(n+p+1)!!(n-p+1)!!},$$

which holds when $n+p$ is odd and $n \equiv p-1$, gives

$$M_n^0 = - \frac{n!!(n+2)!!}{(n-1)!!(n-3)!!} p \frac{(n+p-2)!!(n-p-2)!!}{(n+p+1)!!(n-p+1)!!} \text{ when } n \text{ is even and } p \text{ odd.}$$

The values of M_n^0 were calculated by this equation for all values of p up to and including $p = 11$, and for all values of n up to and including 12. The first of the equations K, § 9, was then applied, and substituting $\sigma = 2$, $n = 2$, and $M_2^0 = 0$, the value of M_2^2 was calculated. Similarly putting $n = 4, 6$, &c., the values of M_n^2 for even values of n were all tabulated. By successive steps I similarly found M_n^4 , M_n^6 , &c., up

to M_{12}^{12} . The last value so obtained, and some of the intermediate ones, were calculated independently from the series defining M , numerical errors being easily detected in this way. The values of S_n^σ , U_n^σ , V_n^σ were dealt with in a similar manner. As a final check, the values of S_n^σ and V_n^σ were obtained from U_n^σ and M_n^σ by means of the second and fourth of the equations (L).

§ 11. *Application of the Previous Results to the Expression of Functions by Means of a Series of Spherical Harmonics.*

The results obtained in this investigation lead to a new method of calculating the coefficients of the series of spherical harmonics which represent a function F of spherical co-ordinates. If the values of this function are given for all points of a sphere, the coefficients of the term involving T_n^σ is known to depend solely on the integral $\oint F T_n^\sigma dS$ taken over the surface of the sphere. This theoretically perfect proceeding has to be modified in practice when the values of F are known only at definite points, so that the intermediate positions have to be evaluated by a process of interpolation on the supposition that F is continuous everywhere.

All methods which have been applied so far, suffer from the serious inconvenience that the method of least squares is applied in such a way as to make the value of the coefficients of lower degrees depend on the number of terms which are taken into account. That is to say, the coefficients are not independent of each other as they ought to be.

A method suggested by F. E. NEUMANN, which is free from this defect, introduces other complications, and has never, as far as I know, been applied in practice.

The theorem of § 8 offers a simple solution of the practical difficulties, and reduces the whole problem to an expansion by means of FOURIER'S analysis, which can be carried out either by the well-known process of calculation, or by mechanical means.

Let F be expressed, in the first place, for different circles of latitude as a series proceeding by cosines and sines of the longitude and of its multiples. This first step is common to all methods.

The result may be expressed symbolically by

$$F = \kappa^0 + \kappa' \cos \phi + \kappa^2 \cos 2\phi + \dots + K' \sin \phi + K^2 \sin 2\phi + \dots,$$

where κ^0 , κ' , K' , K^2 . . . are functions of the colatitude. If their values are known at $q - 1$ equidistant circles of latitude, and at the poles, we may determine the coefficients α_p^σ and α_q^σ , which satisfy the equations

$$\kappa^\sigma = \alpha_0^\sigma + \alpha_1^\sigma \cos \theta + \alpha_2^\sigma \cos 2\theta + \dots + \alpha_p^\sigma \cos p\theta + \dots + \alpha_q^\sigma \cos q\theta,$$

$$K^\sigma = \alpha_0^\sigma + \alpha_1^\sigma \cos \theta + \alpha_2^\sigma \cos 2\theta + \dots + \alpha_p^\sigma \cos p\theta + \dots + \alpha_q^\sigma \cos q\theta.$$

If we give to θ the successive values $0, \pi/q, 2\pi/q \dots q\pi/q$, we shall have $q + 1$

equations to determine $q + 1$ coefficients. The solution is conducted according to well-known rules, making use of the proposition that, if p and s are integer numbers smaller than q ,

$$0 = \frac{1}{2} + \cos \frac{p\pi}{q} \cos \frac{s\pi}{q} + \cos \frac{2p\pi}{q} \cos \frac{2s\pi}{q} + \dots \\ + \cos \frac{(q-1)p\pi}{q} \cos \frac{(q-1)s\pi}{q} + \frac{1}{2} \cos p\pi \cos s\pi.$$

If p and s are equal to each other the sum on the right-hand side is equal to $\frac{1}{2}q$.

The factor $\frac{1}{2}$ of the first and last term should be noted. If we designate by $\kappa_0, \kappa_1, \kappa_2, \dots, \kappa_q$, the value of κ for $\theta = 0$, and at the successive circles of latitude, it follows that

$$\frac{q}{2} a_p^\sigma = \frac{1}{2} (\kappa_0 + \kappa_q \cos p\pi) + \sum_{s=1}^{s=q-1} \kappa_s^\sigma \cos sp\pi/q.$$

The condition under which the coefficients are determined is that the coefficients of the Fourier series higher than a_q^σ are zero. It will be noted that, even if we suppose some of the lower coefficients to vanish, the equations will still give *those* values for the remaining coefficients which, according to the method of least squares, fit in best with the assumed values of κ , but that in the calculations half weight only is given to the values at the two poles.

[The above method of obtaining the coefficients a_p^σ , which is identical with that in common use, when the range of θ is 2π , is convenient whenever the function to be analysed is known at every point of the sphere, so that the coefficients κ and K may be determined for a sufficient number of equidistant circles of latitude. But other methods are available, and hence the process of obtaining the coefficients of the series of spherical harmonics which I am endeavouring to explain, is not restricted to cases where the original function is known everywhere; provided it is continuous, as well as its derivatives over the surface of the sphere. Graphical interpolation may be employed to determine the function at unknown points with sufficient accuracy, and there are several good mechanical devices in existence, by means of which at any rate the lowest and most important coefficients of the Fourier series may be found. It will, in some cases, materially help this process of interpolation if it is remembered that continuity at the pole involves the vanishing of all the values of κ and K except κ^0 .—August 2, 1902.]

Having calculated the coefficients, the reduction to spherical harmonics is made, by substitution of

$$\cos p\theta = A_\sigma^\sigma Q_\sigma^\sigma + A_{\sigma+1}^\sigma Q_{\sigma+1}^\sigma + \dots \quad A_{p-1}^\sigma Q_{p-1}^\sigma + \dots \quad A_n^\sigma Q_n^\sigma + \dots,$$

where the values of A_n^σ may be calculated and tabulated once for all. By the theorem of § 8, all coefficients vanish up to and inclusive of A_{p-2}^σ if σ be odd, so that

the spherical harmonic of degree n will only depend on the Fourier coefficients a_p , for which p is equal to or smaller than $n + 1$. When σ is even we secure the same advantage by developing κ^σ and K^σ in terms of the sine functions, and we write in that case :

$$\kappa^\sigma = b_1^\sigma \sin \theta + b_2^\sigma \sin 2\theta + \dots \quad b_p^\sigma \sin p\theta + \dots$$

$$K^\sigma = \beta_1^\sigma \sin \theta + \beta_2^\sigma \sin 2\theta + \dots \quad \beta_p^\sigma \sin p\theta + \dots$$

and

$$\sin p\theta = B_{p-1}^\sigma Q_{n-1}^\sigma + B_{p+1}^\sigma Q_{n+1}^\sigma + \dots \quad B_p^\sigma Q_n^\sigma + \dots$$

To calculate finally a coefficient such as B_n^σ we proceed in the usual way, thus :

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu = \frac{2B_p^\sigma}{2n+1} \frac{(n+\sigma)!}{(n-\sigma)!}.$$

Therefore

$$\begin{aligned} B_p^\sigma &= \frac{2n+1}{2} \frac{(n-\sigma-1)!! (n-3)!!}{(n+\sigma)!! (n+2)!!} \pi M_n^\sigma \text{ when } n \text{ is even and } \sigma \text{ is even} \\ &= \frac{2n+1}{2} \frac{(n-\sigma)!! (n-4)!!}{(n+\sigma-1)!! (n+3)!!} \pi S_n^\sigma \text{ when } n \text{ is odd and } \sigma \text{ is even.} \end{aligned}$$

Similarly

$$\begin{aligned} A_p^\sigma &= \frac{2n+1}{2} \frac{(n-\sigma)!! (n-3)!!}{(n+\sigma-1)!! (n+2)!!} \pi V_n^\sigma \text{ when } n \text{ is even and } \sigma \text{ is odd} \\ &= \frac{2n+1}{2} \frac{(n-\sigma-1)!! (n-2)!!}{(n+\sigma)!! (n+1)!!} \pi U_n^\sigma \text{ when } n \text{ is odd and } \sigma \text{ is odd.} \end{aligned}$$

Combining equations, we find that if g_n^σ denote the coefficient of $t_n^\sigma Q_n^\sigma \cos \sigma\phi$ in the development of F , and h_n^σ the coefficient of $t_n^\sigma Q_n^\sigma \sin \sigma\phi$,

$$\begin{aligned} g_n^\sigma &= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma)!! (n-3)!!}{(n+\sigma-1)!! (n+2)!!} \pi \sum_{p=1}^{p=n+1} a_p^\sigma V_n^\sigma(p), \text{ when } n \text{ is even and } \sigma \text{ odd and } p \text{ odd} \\ &= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma-1)!! (n-2)!!}{(n+\sigma)!! (n+1)!!} \pi \sum_{p=0}^{p=n+1} a_p^\sigma U_n^\sigma(p), \text{ when } n \text{ is odd and } \sigma \text{ odd and } p \text{ even} \\ &= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma-1)!! (n-3)!!}{(n+\sigma)!! (n+2)!!} \pi \sum_{p=1}^{p=n+1} b_p^\sigma M_n^\sigma(p) \text{ when } n \text{ is even and } \sigma \text{ even and } p \text{ odd} \\ &= \frac{2n+1}{2t_n^\sigma} \frac{(n-\sigma)!! (n-4)!!}{(n+\sigma-1)!! (n+3)!!} \pi \sum_{p=2}^{p=n+1} b_p^\sigma S_n^\sigma(p) \text{ when } n \text{ is odd and } \sigma \text{ even and } p \text{ even.} \end{aligned}$$

To obtain h_n^σ , substitute α and β for a and b .

The coefficient t_n^σ is introduced because the quantity here designated by Q_n^σ is not uniformly accepted as the standard form for a spherical harmonic. For some purposes

it is more convenient to take the form of GAUSS, for which THOMSON and TAIT have adopted the symbol Θ_n^σ , and which is connected with Q_n^σ by the relation

$$\Theta_n^\sigma = \frac{(n - \sigma)!}{(2n - 1)!!} Q_n^\sigma$$

Another constant, which has been applied by ADOLF SCHMIDT, is based on the consideration that for numerical work it is inconvenient to deal with functions the average values of which differ considerably from each other. This author therefore takes a function $r_n^\sigma \Theta_n^\sigma$ as basis of calculation, and determines r_n^σ , so that the average value of the square of $r_n^\sigma \Theta_n^\sigma \cos \sigma \phi$ over a sphere of unit radius is equal to unity. The function R_n^σ defined in this way is connected with Θ_n^σ and Q_n^σ by

$$R_n^\sigma = (2n - 1)!! \sqrt{\frac{\epsilon_\sigma (2n + 1)}{(n + \sigma)! (n - \sigma)!}} \Theta_n^\sigma = \sqrt{\frac{\epsilon_\sigma (2n + 1) (n - \sigma)!}{(n + \sigma)!}} Q_n^\sigma$$

where ϵ_σ is equal to 1 for $\sigma = 0$ and equal to 2 for all other values of σ .

If a function is to be expanded in terms of R_n^σ , we must therefore write

$$t_n^\sigma = \sqrt{\frac{\epsilon_\sigma (2n + 1) (n - \sigma)!}{(n + \sigma)!}}$$

For numerical work the introduction of R_n^σ in place of Q_n^σ possesses undoubted advantages. Although at first I was reluctant to adopt the additional complication due to the introduction of a square root and the addition of yet another function to those given by previous writers, I found that the inconvenience of tabulating values differing considerably in magnitude from each other was very great, and I therefore felt myself almost compelled during the course of the investigation to adopt SCHMIDT's function as above defined.

We now substitute t_n^σ into the equations for g_n^σ and h_n^σ and write :

$$\begin{aligned} v_n^\sigma &= \frac{\pi}{2} \frac{(n - 3)!!}{(n + 2)!!} \sqrt{\frac{2n + 1}{\epsilon_\sigma}} \sqrt{\frac{(n + \sigma)!! (n - \sigma)!!}{(n + \sigma - 1)!! (n - \sigma - 1)!!}} V_n^\sigma, \\ u_n^\sigma &= \frac{\pi}{2} \frac{(n - 2)!!}{(n + 1)!!} \sqrt{\frac{2n + 1}{\epsilon_\sigma}} \sqrt{\frac{(n + \sigma - 1)!! (n - \sigma - 1)!!}{(n + \sigma)!! (n - \sigma)!!}} U_n^\sigma, \\ m_n^\sigma &= \frac{\pi}{2} \frac{(n - 3)!!}{(n + 2)!!} \sqrt{\frac{2n + 1}{\epsilon_\sigma}} \sqrt{\frac{(n + \sigma - 1)!! (n - \sigma - 1)!!}{(n + \sigma)!! (n - \sigma)!!}} M_n^\sigma, \\ s_n^\sigma &= \frac{\pi}{2} \frac{(n - 4)!!}{(n + 3)!!} \sqrt{\frac{2n + 1}{\epsilon_\sigma}} \sqrt{\frac{(n + \sigma)!! (n - \sigma)!!}{(n + \sigma - 1)!! (n - \sigma - 1)!!}} S_n^\sigma. \end{aligned}$$

where $\epsilon_0 = 1$ and $\epsilon_\sigma = 2$ if $\sigma > 0$.

By substitution of the values of V_n^σ , &c., we may also write more simply

$$v_n^\sigma = \frac{1}{2\epsilon_\sigma} \int_{-1+1}^{+1} R_n^\sigma \cos p\theta d\mu; \text{ if } \sigma \text{ be odd, } p \text{ odd, } n \text{ even.}$$

$$u_n^\sigma = \frac{1}{2\epsilon_\sigma} \int_{-1+1}^{+1} R_n^\sigma \cos p\theta d\mu; \text{ if } \sigma \text{ ,, ,, } p \text{ even, } n \text{ odd.}$$

$$m_n^\sigma = \frac{1}{2\epsilon_\sigma} \int_{-1}^{+1} R_n^\sigma \sin p\theta d\mu; \text{ if } \sigma \text{ be even, } p \text{ odd, } n \text{ even.}$$

$$s_n^\sigma = \frac{1}{2\epsilon_0} \int_{-1}^{+1} R_n^\sigma \sin p\theta d\mu; \text{ if } \sigma \text{ ,, ,, } p \text{ even, } n \text{ odd.}$$

It would thus appear to be unnecessary to have separate symbols for u_n^σ and v_n^σ or for m_n^σ and s_n^σ , but as it is convenient to tabulate separately the integrals for odd and even values of p , the retention of all four symbols facilitates reference.

We may now obtain the final coefficients by summation thus:

$$\begin{aligned} g_n^\sigma &= \sum_{p=1}^{p=n+1} a_p^\sigma v_n^\sigma, \text{ when } n \text{ is even, } \sigma \text{ odd, and } p \text{ odd,} \\ &= \sum_{p=0}^{p=n+1} a_p^\sigma u_n^\sigma, \text{ ,, } n \text{ ,, odd, } \sigma \text{ ,, ,, } p \text{ even,} \\ &= \sum_{p=1}^{p=n+1} b_p^\sigma m_n^\sigma, \text{ ,, } n \text{ ,, even, } \sigma \text{ even, ,, } p \text{ odd,} \\ &= \sum_{p=2}^{p=n+1} b_p^\sigma s_n^\sigma, \text{ ,, } n \text{ ,, odd, } \sigma \text{ ,, ,, } p \text{ even,} \end{aligned}$$

with similar equations for h_n^σ , a and b being replaced by α and β .

The values of v_n^σ , u_n^σ , m_n^σ and s_n^σ are given in Tables V.—VIII. and their logarithms in Tables IX.—XII., for values of n , σ and p up to 12 inclusive. By means of these tables we may, for instance, write down at once the coefficients as far as the third degree as follows:

$$\begin{aligned} g_0^\circ &= \cdot 785398 b_1^\circ \\ g_1^\circ &= \cdot 680175 b_2^\circ \\ g_2^\circ &= - \cdot 219525 b_1^\circ + \cdot 658575 b_3^\circ \\ g_3^\circ &= - \cdot 259746 b_2^\circ + \cdot 649365 b_4^\circ \\ g_1' &= \cdot 680175 a_0' - \cdot 340087 a_2' \\ g_2' &= \cdot 380229 a_1' - \cdot 380229 a_3' \\ g_3' &= \cdot 159061 a_0' + \cdot 318123 a_2' - \cdot 397653 a_4' \\ g_2^2 &= \cdot 570345 b_1^2 - \cdot 190115 b_3^2 \\ g_3^2 &= \cdot 502996 b_2^2 - \cdot 251498 b_4^2 \\ g_3^3 &= \cdot 616042 a_0^3 - \cdot 410695 a_2^3 + \cdot 102672 a_4^3. \end{aligned}$$

The calculation of the ten coefficients g up to the third degree involves therefore the determination of sixteen Fourier coefficients. There is no h coefficient of zero type, and there are therefore only six h coefficients up to the third degree which depend upon twelve Fourier coefficients. In general there will be $\frac{1}{2}(n+1)(n+2)$ g coefficients, and $\frac{1}{2}(n+1)n$ h coefficients up to and including the n^{th} degree, giving a total of $(n+1)^2$ coefficients in the series of spherical harmonics. To determine these completely, it would be necessary to calculate $(n+1)^2$ Fourier coefficients for the quantities g and n for the quantities h , giving a total of $(n+1)(n+2)$ Fourier coefficients. The tables accompanying this paper being calculated for values of n up to 12 are therefore sufficient to determine 169 coefficients in the series of spherical harmonics, by means of a simple computation after the determination of 182 coefficients of the Fourier series. These latter coefficients may be evaluated by mechanical devices.

§ 12. *Special Application to the Theory of Terrestrial Magnetism.*

The advantages of the proposed method of obtaining the coefficients of a series of spherical harmonics are considerably increased when the quantities to be represented by such a series are not given directly, but by means of their differential coefficients. This is the case when the magnetic potential has to be calculated by means of the observed magnetic forces. If X and Y represent the components of magnetic force resolved towards the geographical north and east respectively, the magnetic potential is determined by means of the equations

$$dV/d\theta = X; \quad dV/d\phi = -Y \sin \theta.$$

If no electric currents of sufficient intensity traverse the earth's surface, a function V can be found which satisfies both equations. If $Y \sin \theta$ be obtained in a series proceeding by spherical harmonics, then all the terms which depend on the longitude ϕ are at once expressed in a similar series, as the integration according to ϕ leaves each term in the standard form. The treatment of the other component acting along the meridian involves, however, serious difficulties, and it is not necessary here to enter into the question as to the more or less complicated methods by means of which V has been hitherto derived in the standard form from X .

The method based on the results of the preceding investigation avoids these difficulties. For odd values of σ , X is expressed in a series, each term of which has the form $\cos n\theta \sin m\phi$ or $\cos n\theta \cos m\phi$, while for even values of σ , X is expressed in terms of the form $\sin n\theta \sin m\phi$ and $\sin n\theta \cos m\phi$. After integration with respect to θ , the formula of § 11 will determine the required coefficients. We may treat the eastern force similarly, obtaining a series proceeding by $\cos p\theta$ or $\sin p\theta$ according as σ is odd or even. $Y \sin \theta$ is then derived in a series proceeding

by $\sin p\theta$ or $\cos p\theta$, and after integration with respect to ϕ the equations of § 11 may be applied. The spheroidal shape of the earth may also, if necessary, be easily taken into account, but I reserve the discussion of this matter until the completion of some calculations, on which I am at present engaged, will allow me to compare the results obtained by the method I advocate with those obtained in other ways.

Great pains have been taken to guard against numerical errors, and it is hoped that all numbers given in the tables are correct, with the proviso, however, that the last decimal place in Tables V. and XII. is uncertain by one unit, and wrong possibly even occasionally by two units. Professor CORE has assisted me in the numerical work.

TABLE I.—Values of $V_n^\sigma(p)$.

	$n; p:$	1.	3.	5.	7.	9.	11.
$\sigma = 1$	2	1	-1	—	—	—	—
	4	1	$\frac{5}{2}$	$-\frac{7}{2}$	—	—	—
	6	1	$\frac{7}{3}$	$\frac{21}{5}$	$-\frac{33}{5}$	—	—
	8	1	$\frac{6}{5}$	$\frac{66}{35}$	$\frac{429}{70}$	$-\frac{143}{14}$	—
	10	1	$\frac{55}{49}$	$\frac{143}{98}$	$\frac{715}{294}$	$\frac{2431}{294}$	$-\frac{4199}{294}$
	12	1	$\frac{13}{12}$	$\frac{325}{252}$	$\frac{221}{126}$	$\frac{4199}{1584}$	$\frac{4199}{396}$
$\sigma = 3$	4	3	$-\frac{9}{2}$	$\frac{3}{2}$	—	—	—
	6	3	1	$-\frac{23}{3}$	$\frac{11}{3}$	—	—
	8	3	2	$-\frac{3}{2}$	$-\frac{65}{6}$	$\frac{13}{2}$	—
	10	3	$\frac{117}{49}$	$\frac{93}{98}$	$-\frac{213}{98}$	$-\frac{1377}{98}$	$\frac{969}{98}$
	12	3	$\frac{31}{12}$	$\frac{179}{108}$	$-\frac{17}{270}$	$-\frac{323}{90}$	$-\frac{9367}{540}$
$\sigma = 5$	6	5	-9	5	-1	—	—
	8	5	-2	-10	$\frac{19}{2}$	$-\frac{5}{2}$	—
	10	5	$\frac{5}{7}$	$-\frac{83}{14}$	$-\frac{19}{2}$	$\frac{1003}{70}$	$-\frac{323}{70}$
	12	5	$\frac{25}{12}$	$-\frac{35}{12}$	$-\frac{239}{30}$	$-\frac{247}{30}$	$\frac{1159}{60}$
$\sigma = 7$	8	7	-14	10	$-\frac{7}{2}$	$\frac{1}{2}$	—
	10	7	$-\frac{41}{7}$	$-\frac{145}{14}$	$\frac{31}{2}$	$-\frac{107}{14}$	$\frac{19}{14}$
	12	7	$-\frac{7}{4}$	$-\frac{121}{12}$	$-\frac{133}{30}$	$\frac{193}{10}$	$-\frac{763}{60}$
$\sigma = 9$	10	9	$-\frac{135}{7}$	$\frac{225}{14}$	$-\frac{15}{2}$	$\frac{27}{14}$	$-\frac{3}{14}$
	12	9	$-\frac{41}{4}$	$-\frac{325}{36}$	$\frac{371}{18}$	$-\frac{29}{2}$	$\frac{173}{30}$
$\sigma = 11$	12	11	$-\frac{99}{4}$	$\frac{275}{12}$	$-\frac{77}{6}$	$\frac{9}{2}$	$-\frac{11}{12}$

TABLE II.—Values of $U_n^\sigma(p)$.

	$n; p:$	0.	2.	4.	6.	8.	10.	12.
$\sigma = 1$	1	1	$-\frac{1}{2}$	—	—	—	—	—
	3	1	2	$-\frac{5}{2}$	—	—	—	—
	5	1	$\frac{5}{4}$	$\frac{7}{2}$	$-\frac{21}{4}$	—	—	—
	7	1	$\frac{28}{25}$	$\frac{42}{25}$	$\frac{132}{25}$	$-\frac{429}{50}$	—	—
	9	1	$\frac{15}{14}$	$\frac{66}{49}$	$\frac{429}{196}$	$\frac{715}{98}$	$-\frac{2431}{196}$	—
	11	1	$\frac{22}{21}$	$\frac{715}{588}$	$\frac{715}{441}$	$\frac{2431}{882}$	$\frac{4199}{441}$	$-\frac{4199}{252}$
$\sigma = 3$	3	3	-2	$\frac{1}{2}$	—	—	—	—
	5	3	$\frac{7}{4}$	$-\frac{11}{2}$	$\frac{9}{4}$	—	—	—
	7	3	$\frac{12}{5}$	$\frac{2}{15}$	$-\frac{44}{5}$	$\frac{143}{30}$	—	—
	9	3	$\frac{37}{14}$	$\frac{10}{7}$	$-\frac{39}{28}$	$-\frac{169}{14}$	$\frac{221}{28}$	—
	11	3	$\frac{58}{21}$	$\frac{167}{84}$	$\frac{3}{7}$	$-\frac{17}{6}$	$-\frac{323}{21}$	$\frac{323}{28}$
$\sigma = 5$	5	5	$-\frac{15}{4}$	$\frac{3}{2}$	$-\frac{1}{4}$	—	—	—
	7	5	$\frac{4}{5}$	$-\frac{38}{5}$	$\frac{28}{5}$	$-\frac{13}{10}$	—	—
	9	5	$\frac{5}{2}$	$-\frac{26}{7}$	$-\frac{241}{28}$	$\frac{145}{14}$	$-\frac{85}{28}$	—
	11	5	$\frac{10}{3}$	$-\frac{95}{84}$	$-\frac{137}{21}$	$-\frac{1717}{210}$	$\frac{323}{21}$	$-\frac{323}{60}$
$\sigma = 7$	7	7	$-\frac{28}{5}$	$\frac{14}{5}$	$-\frac{4}{5}$	$\frac{1}{10}$	—	—
	9	7	$-\frac{1}{2}$	$-\frac{62}{7}$	$\frac{253}{28}$	$-\frac{53}{14}$	$\frac{17}{28}$	—
	11	7	2	$-\frac{199}{28}$	$-\frac{127}{21}$	$\frac{199}{14}$	$-\frac{57}{7}$	$\frac{19}{12}$
$\sigma = 9$	9	9	$-\frac{15}{2}$	$\frac{39}{7}$	$-\frac{45}{8}$	$\frac{5}{14}$	$-\frac{1}{8}$	—
	11	9	-2	$-\frac{265}{28}$	$\frac{85}{7}$	$-\frac{293}{42}$	$\frac{43}{21}$	$-\frac{1}{4}$
$\sigma = 11$	11	11	$-\frac{66}{7}$	$\frac{165}{28}$	$-\frac{55}{21}$	$\frac{11}{14}$	$-\frac{1}{7}$	$\frac{1}{84}$

TABLE III.—Values of $M_n^\sigma(p)$.

	$n; p:$	1.	3.	5.	7.	9.	11.
$\sigma = 0.$	0	-1	—	—	—	—	—
	2	-1	3	—	—	—	—
	4	-1	$-\frac{15}{2}$	$\frac{35}{2}$	—	—	—
	6	-1	$-\frac{21}{5}$	-21	$\frac{231}{5}$	—	—
	8	-1	$-\frac{18}{5}$	$-\frac{66}{5}$	$-\frac{129}{10}$	$\frac{1287}{14}$	—
	10	-1	$-\frac{165}{40}$	$-\frac{515}{98}$	$-\frac{715}{142}$	$-\frac{7293}{98}$	$\frac{46189}{294}$
	12	-1	$-\frac{13}{4}$	$-\frac{1625}{252}$	$-\frac{221}{18}$	$-\frac{4199}{154}$	$-\frac{4199}{36}$
$\sigma = 2.$	2	3	-1	—	—	—	—
	4	3	$\frac{33}{2}$	$-\frac{21}{2}$	—	—	—
	6	3	11	39	-33	—	—
	8	3	10	22	$\frac{143}{2}$	$-\frac{143}{2}$	—
	10	3	$\frac{471}{40}$	$\frac{1833}{98}$	$\frac{507}{14}$	$\frac{11271}{98}$	$-\frac{12597}{98}$
	12	3	$\frac{113}{12}$	$\frac{625}{36}$	$\frac{527}{18}$	$\frac{323}{6}$	$\frac{6137}{36}$
$\sigma = 4.$	4	15	$-\frac{15}{2}$	$\frac{5}{2}$	—	—	—
	6	15	31	-37	11	—	—
	8	15	38	34	$-\frac{169}{2}$	$\frac{65}{2}$	—
	10	15	$\frac{285}{7}$	$\frac{723}{14}$	$\frac{375}{14}$	$-\frac{2091}{14}$	$\frac{969}{14}$
	12	15	$\frac{505}{12}$	$\frac{715}{12}$	$\frac{1717}{30}$	$\frac{323}{30}$	$-\frac{13889}{60}$
$\sigma = 6.$	6	35	-21	7	-1	—	—
	8	35	42	-74	$\frac{59}{2}$	$-\frac{15}{2}$	—
	10	35	$\frac{465}{7}$	$\frac{47}{14}$	$-\frac{1765}{14}$	$\frac{1377}{14}$	$-\frac{323}{14}$
	12	35	$\frac{315}{4}$	$\frac{635}{12}$	$-\frac{1607}{30}$	$-\frac{1691}{10}$	$\frac{11039}{60}$
$\sigma = 8.$	8	63	-42	18	$-\frac{9}{2}$	$\frac{1}{2}$	—
	10	63	$\frac{353}{7}$	$-\frac{1629}{14}$	$\frac{1191}{14}$	$-\frac{411}{14}$	$\frac{57}{14}$
	12	63	$\frac{371}{4}$	$-\frac{565}{12}$	$-\frac{865}{6}$	$\frac{1079}{6}$	$-\frac{973}{12}$
$\sigma = 10.$	10	99	$-\frac{495}{7}$	$\frac{495}{14}$	$-\frac{165}{14}$	$\frac{33}{14}$	$-\frac{3}{14}$
	12	99	$\frac{187}{4}$	$-\frac{1925}{12}$	$\frac{869}{6}$	$-\frac{113}{6}$	$\frac{211}{12}$
$\sigma = 12.$	12	143	$-\frac{429}{4}$	$\frac{715}{12}$	$-\frac{143}{6}$	$\frac{13}{2}$	$-\frac{13}{12}$

TABLE IV.—Values of $S_n^\sigma(p)$.

	$n; p:$	2	4.	6.	8.	10.	12.
$\sigma=0$	1	-2	—	—	—	—	—
	3	-2	5	—	—	—	—
	5	-2	$-\frac{56}{5}$	$\frac{126}{5}$	—	—	—
	7	-2	-6	$-\frac{198}{7}$	$\frac{1287}{21}$	—	—
	9	-2	$-\frac{176}{35}$	$-\frac{429}{35}$	$-\frac{1144}{21}$	$\frac{2431}{21}$	—
	11	-2	$-\frac{65}{14}$	$-\frac{65}{7}$	$-\frac{442}{21}$	$-\frac{20995}{231}$	$\frac{4199}{22}$
$\sigma=2$	3	6	-3	—	—	—	—
	5	6	24	-18	—	—	—
	7	6	$\frac{46}{3}$	$\frac{154}{3}$	$-\frac{143}{3}$	—	—
	9	6	$\frac{96}{7}$	$\frac{195}{7}$	$\frac{624}{7}$	$-\frac{663}{7}$	—
	11	6	$\frac{183}{14}$	$\frac{163}{7}$	$\frac{306}{7}$	$\frac{969}{7}$	$-\frac{323}{2}$
$\sigma=4$	5	30	-24	6	—	—	—
	7	30	$\frac{110}{3}$	$-\frac{190}{3}$	$\frac{65}{3}$	—	—
	9	30	48	33	-120	51	—
	11	30	$\frac{105}{2}$	57	$\frac{192}{5}$	$-\frac{969}{5}$	$\frac{969}{10}$
$\sigma=6$	7	70	-70	30	-5	—	—
	9	70	32	-123	80	-17	—
	11	70	$\frac{145}{2}$	-27	$-\frac{842}{5}$	$\frac{779}{5}$	$-\frac{399}{10}$
$\sigma=8$	9	126	-144	81	-24	3	—
	11	126	$\frac{9}{2}$	-183	174	-69	$\frac{21}{2}$
$\sigma=10$	11	198	$-\frac{495}{2}$	165	-66	15	$-\frac{3}{2}$

TABLE V.— $v_n^\sigma = \frac{1}{4} \int_{-1}^{+1} R_n^\sigma \cos p\theta d\mu$.

σ	$n; p:$	1.	3.	5.	7.	9.	11.
$\sigma=1$	2	.380229	—	—	—	—	—
	4	.116421	.291052	— .407473	—	—	—
	6	.063364	.088709	.266128	— .418201	—	—
	8	.041506	.049808	.078269	.254374	—	—
	10	.029936	.033601	.043681	.072803	— .423957	—
	12	.022921	.024831	.029561	.040203	.247529	.427550
$\sigma=3$	4	.308021	— .462031	.154011	—	—	—
	6	.180337	.060112	— .460860	.220412	—	—
	8	.120909	.080606	— .026869	— .436615	.261969	—
	10	.088128	.070143	.027877	— .063848	— .412763	.290463
	12	.067865	.058439	.037493	— .001424	— .081187	— .392403
$\sigma=5$	6	.257385	— .463293	.257385	— .051477	—	—
	8	.187600	— .075040	— .375200	.356140	— .093800	—
	10	.140757	.020108	— .166898	— .267439	.403370	— .129899
	12	.109922	.045801	— .064121	— .175142	— .181005	.424665
$\sigma=7$	8	.221971	— .443913	.317102	— .110986	.015855	—
	10	.181681	— .152018	— .268813	.402293	— .198366	.035224
	12	.146379	— .036595	— .210856	— .092707	.403588	— .265922
$\sigma=9$	10	.195951	— .419896	.349913	— .163293	.041990	— .004665
	12	.172490	— .196447	— .173022	.395023	— .277900	.092101
$\sigma=11$	12	.176003	— .396007	.366673	— .205337	.072001	— .014667

TABLE VI.— $u_n^\sigma = \frac{1}{4} \int_{-1}^{+1} R_n^\sigma \cos p\theta d\mu$.

$n; p:$	0.	2.	4.	6.	8.	10.	12.
$\sigma=1$	1	—	—	—	—	—	—
	3	·340087	—	—	—	—	—
	5	·318123	·397653	—	—	—	—
	7	·098522	·275861	·413791	—	—	—
	9	·055015	·082523	·259357	·421455	—	—
	11	·036794	·046256	·075166	·250553	·425940	—
$\sigma=3$		·025739	·031298	·041731	·070913	·245076	·428884
	3	—	—	—	—	—	—
	5	·616042	·102672	·191548	—	—	—
	7	·255398	·468228	·449220	·243328	—	—
	9	·153143	·006806	·048959	·424306	·277431	—
	11	·105449	·050214	·011205	·074077	·402128	·301596
$\sigma=5$		·078434	·051978	—	—	—	—
	5	—	—	—	—	—	—
	7	·571086	·171326	·028554	—	—	—
	9	·282177	·428908	·316038	·073366	—	—
	11	·185519	·137814	·319358	·384290	·112637	—
		·135311	·030606	·176548	·221265	·416242	·145685
$\sigma=7$	7	—	—	—	—	—	—
	9	·538359	·215344	·061527	·007691	—	—
	11	·290384	·367424	·374832	·157044	·025186	—
		·201669	·204756	·174231	·409512	·234595	·045616
	9	—	—	—	—	—	—
	11	·513121	·214343	·091629	·020362	·002036	—
$\sigma=9$		·291820	·306874	·393725	·226199	·066393	·008106
	9	—	—	—	—	—	—
	11	—	—	—	—	—	—
		—	—	—	—	—	—
	9	—	—	—	—	—	—
	11	—	—	—	—	—	—
$\sigma=11$		—	—	—	—	—	—
	11	·492809	·264005	·117335	·035201	·006400	·000533
		—	—	—	—	—	—
	11	—	—	—	—	—	—
		—	—	—	—	—	—
	11	—	—	—	—	—	—

TABLE VII.— $m_\sigma = \frac{1}{2\epsilon_\sigma} \int_{-1}^{+1} R_n^\sigma \sin \rho \theta d\rho; \epsilon_0 = 1; \epsilon_\sigma = 2 \text{ if } \sigma > 0.$

	$n; p:$	1.	3.	5.	7.	9.	11.
$\sigma = 0$	0	.785398	—	—	—	—	—
	2	— .219525	.658575	—	—	—	—
	4	— .036816	— .276117	.644271	—	—	—
	6	— .015827	— .058074	— .290369	.638812	—	—
	8	— .006918	— .024904	— .065224	— .296770	.635936	—
	10	— .004037	— .013592	— .029450	— .068717	— .300391	.634159
$\sigma = 2$	12	— .002595	— .008435	— .016736	— .031865	— .070766	— .302716
	2	.570345	— .190115	—	—	—	—
	4	.082322	.452772	— .288127	—	—	—
	6	.030056	.110206	.390729	.330617	—	—
	8	.014883	.049610	.109141	.354708	.354708	—
	10	.008642	.027689	.053878	.010432	.331293	.370269
$\sigma = 4$	12	.005541	.017393	.032067	.054078	.099433	.314872
	4	.544509	— .272254	.054451	—	—	—
	6	.164624	.340222	— .406071	.120724	—	—
	8	.078046	.197717	.176905	.439661	.169100	—
	10	.044511	.120817	.153246	.079485	.443206	.205388
	12	.028277	.079333	.112323	.107893	.020297	— .436380
$\sigma = 6$	6	.520106	— .312062	.104021	— .014860	—	—
	8	.202631	.243157	— .428420	.228684	.043421	—
	10	.110160	.209079	.010566	— .396801	.309572	.072616
	12	.068548	.154233	.103638	— .104911	.331186	.360336
$\sigma = 8$	8	.499435	— .332957	.142696	— .035674	.003964	—
	10	.225133	.168020	— .410966	.300467	.103688	.014380
	12	.131741	.193952	— .098457	— .300774	.376055	.469556
$\sigma = 10$	10	.481977	— .344269	.172134	— .057379	.011476	— .001043
	12	.235552	.110288	— .378441	.341678	.162386	.041481
$\sigma = 12$	12	.467044	— .350283	.194601	— .077841	.021229	— .003538

TABLE VIII.— $s^{\sigma} \int_{-1}^{+1} R_{\sigma}^{\sigma} \sin p\theta d\mu; \epsilon_0 = 1; \epsilon_{\sigma} = 2 \text{ if } \sigma > 0.$

	$n; p:$	2.	4.	6.	8.	10.	12.
$\sigma = 0$	1	.680175	—	—	—	—	—
	3	— .259746	.649365	—	—	—	—
	5	— .508764	— .284908	.641042	—	—	—
	7	— .020794	— .062381	— .294084	.637181	—	—
	9	— .010970	— .027582	— .067230	— .298801	.634953	—
	11	— .006638	— .015410	— .030821	— .069860	— .301668	.633503
$\sigma = 2$	3	.502996	— .251498	—	—	—	—
	5	.104266	.417062	— .312797	—	—	—
	7	.043315	.110695	.370588	— .344117	—	—
	9	.023011	.052596	.106836	.341875	— .363242	—
	11	.013975	.030445	.054236	.101817	.322420	— .376156
$\sigma = 4$	5	.451483	— .361187	.090297	—	—	—
	7	.203168	.248316	— .428909	.146732	—	—
	9	.110869	.177390	.121956	— .443476	.188477	—
	11	.068190	.119333	.129562	.046369	— .440509	.220254
$\sigma = 6$	7	.402871	— .402871	.172659	— .028776	—	—
	9	.239504	.109488	— .420843	.273719	— .058165	—
	11	.151841	.157264	— .058567	— .365287	.337956	— .086550
$\sigma = 8$	9	.362831	— .414664	.233249	— .069111	.008639	—
	11	.251159	.008970	— .364778	.346838	— .137539	.020930
$\sigma = 10$	11	.330211	— .412764	.275176	— .110070	.025016	— .002502

TABLE IX.— $\text{Log } v_n^\sigma = \log \left\{ \frac{1}{4} \int_{-1}^{+1} R_n^\sigma \cos p\theta d\mu \right\}$.

σ	$n; p:$	1.	3.	5.	7.	9.	11.
$\sigma = 1$	2	9.5800455	—	—	—	—	—
	4	9.0660312	—	—	—	—	—
	6	8.8018414	—	—	—	—	—
	8	8.6181138	—	—	—	—	—
	10	8.4761883	—	—	—	—	—
	12	8.3602405	—	—	—	—	—
$\sigma = 3$	4	9.4885802	—	—	—	—	—
	6	9.2560839	—	—	—	—	—
	8	9.0834581	—	—	—	—	—
	10	8.9451143	—	—	—	—	—
	12	8.8316471	—	—	—	—	—
	14	8.7316471	—	—	—	—	—
$\sigma = 5$	6	9.4105833	—	—	—	—	—
	8	9.2732328	—	—	—	—	—
	10	9.1484712	—	—	—	—	—
	12	9.0410838	—	—	—	—	—
	14	8.9410838	—	—	—	—	—
	16	8.8410838	—	—	—	—	—
$\sigma = 7$	8	9.3462968	—	—	—	—	—
	10	9.2593087	—	—	—	—	—
	12	9.1654792	—	—	—	—	—
	14	9.0716492	—	—	—	—	—
	16	8.9778192	—	—	—	—	—
	18	8.8839892	—	—	—	—	—
$\sigma = 9$	10	9.2921481	—	—	—	—	—
	12	9.2367633	—	—	—	—	—
	14	9.1813785	—	—	—	—	—
	16	9.1259937	—	—	—	—	—
	18	9.0706089	—	—	—	—	—
	20	8.9652241	—	—	—	—	—
$\sigma = 11$	12	9.2455204	—	—	—	—	—
	14	9.1901356	—	—	—	—	—
	16	9.1347508	—	—	—	—	—
	18	9.0793660	—	—	—	—	—
	20	9.0239812	—	—	—	—	—
	22	8.9685964	—	—	—	—	—

The minus sign has been inserted where v_n^σ is negative.

TABLE X.— $\text{Log } u_n^\sigma = \log \frac{1}{4} \int_{-1}^{+1} R_n^\sigma \cos p\theta d\mu$.

	$n; p:$	0.	2.	4.	6.	8.	10.	12.
$\sigma=1$	1	9.8326206	—	—	—	—	—	—
	3	9.2015646	—	—	—	—	—	—
	5	8.8966219	—	—	—	—	—	—
	7	8.6912647	—	—	—	—	—	—
	9	8.5358191	—	—	—	—	—	—
$\sigma=3$	11	8.4105944	—	—	—	—	—	—
	3	9.7896102	—	—	—	—	—	—
	5	9.4072166	—	—	—	—	—	—
	7	9.1850978	—	—	—	—	—	—
	9	9.0230421	—	—	—	—	—	—
$\sigma=5$	11	8.8945021	—	—	—	—	—	—
	3	9.7567015	—	—	—	—	—	—
	5	9.4505217	—	—	—	—	—	—
	7	9.2683891	—	—	—	—	—	—
	9	9.1313325	—	—	—	—	—	—
$\sigma=7$	11	9.7310723	—	—	—	—	—	—
	3	9.4629721	—	—	—	—	—	—
	5	9.3046393	—	—	—	—	—	—
	7	9.102198	—	—	—	—	—	—
	9	9.4651149	—	—	—	—	—	—
$\sigma=9$	11	9.6926784	—	—	—	—	—	—
	3	9.7896102	—	—	—	—	—	—
	5	9.4072166	—	—	—	—	—	—
	7	9.1850978	—	—	—	—	—	—
	9	9.0230421	—	—	—	—	—	—
$\sigma=11$	11	8.8945021	—	—	—	—	—	—
	3	9.7567015	—	—	—	—	—	—
	5	9.4505217	—	—	—	—	—	—
	7	9.2683891	—	—	—	—	—	—
	9	9.1313325	—	—	—	—	—	—

The minus sign has been inserted where u_n^σ is negative.

TABLE XI.— $\log m_\sigma^\sigma = \log \frac{1}{2\epsilon_\sigma} \int_{-1}^{+1} R_\sigma^\sigma \cos \rho \theta d\mu$. $\epsilon_\sigma = 1$; $\epsilon_\sigma = 2$ if $\sigma > 0$.

	n ; ρ :	1.	3.	5.	7.	9.	11.
$\sigma = 0$	0	9.8950899	—	—	—	—	—
	2	-9.3414849	9.8186062	—	—	—	—
	4	-8.5660312	-9.4410925	9.8090692	—	—	—
	6	-8.1407316	-8.7639809	-9.4629509	9.8053736	—	—
	8	-7.8399625	-8.3962650	-8.8144084	-9.4724198	9.8034130	—
	10	-7.6060069	-8.1332947	-8.4690868	-8.8370636	-9.4776870	9.8021982
$\sigma = 2$	12	-7.4141931	-7.9260765	-8.2236460	-8.5033129	-8.8498183	-9.4810365
	2	9.7561369	-9.2790156	—	—	—	—
	4	8.9155162	9.6558778	-9.4595842	—	—	—
	6	8.4779326	9.0422040	9.5918759	-9.5193252	—	—
	8	8.1726861	8.6955648	9.0379875	9.5498708	-9.5498708	—
	10	7.9365977	8.4422012	8.7314128	8.0183564	9.5202127	-9.5685174
$\sigma = 4$	12	7.7436014	8.2403773	8.5060576	8.7330181	8.9975313	9.4981337
	4	9.7360052	-9.4349752	8.7360052	—	—	—
	6	9.2164933	9.5317637	-9.6086037	9.0817947	—	—
	8	8.8923524	9.2960447	9.2477400	-9.6431178	9.2281445	—
	10	8.6484713	9.0821269	9.1853903	8.9002833	-9.6466060	9.3125758
	12	8.4514358	8.8994547	9.0504693	9.0329935	8.3074257	-9.6398642
$\sigma = 6$	6	9.7160907	-9.4942420	9.0171207	-8.1720227	—	—
	8	9.3067061	9.3858874	-9.6318698	9.3592352	-8.6376994	—
	10	9.0420242	9.3203110	8.0239261	-9.5985729	9.4907621	-8.8610307
	12	8.8359967	9.0881793	9.0155212	-9.0208233	-9.5200723	9.5567071
	8	9.6984793	-9.5223881	9.1544113	-8.5523513	7.5981088	—
	10	9.3473544	9.2253601	-9.6138070	9.4777977	-9.0157277	8.1577608
$\sigma = 8$	12	9.1197216	9.2876950	-8.9932483	-9.4782406	9.4752512	9.2293127
	10	9.6830259	-9.5368979	9.2358679	-8.7587466	8.0597766	-7.0183840
	12	9.3683840	9.0425304	-9.5779983	9.5336173	-9.2105476	8.6178501
	10	9.6830259	-9.5368979	9.2358679	-8.7587466	8.0597766	-7.0183840
	12	9.3683840	9.0425304	-9.5779983	9.5336173	-9.2105476	8.6178501
	12	9.6693581	-9.5444194	9.2891469	-8.8912068	8.3269355	7.5187843

The minus sign has been inserted where m_n is negative.

TABLE XII.— $\log s^\sigma = \log \frac{1}{2\epsilon_\sigma} \int_{-1}^{+1} R_\sigma^\sigma \sin p\theta d\mu$; $\epsilon_0 = 1$: $\epsilon_\sigma = 2$ if $\sigma > 0$.

$n; p:$	2.	4.	6.	8.	10.	12.
$\sigma = 0$	1 9·8326206 3 -9·4145489 5 -8·7065163 7 -8·3179337 9 -8·0402060 11 -7·8220558	9·8124889 -9·4547043 -8·7950550 -8·4406207 -8·1978112	9·8068868 -9·4684709 -8·8275653 -8·4888412	9·8042629 -9·4753827 -8·8442288	9·8027417 -9·4795297	9·8017490
$\sigma = 2$	3 9·7015647 5 9·0181410 7 8·6366429 9 8·3619324 11 8·1453468	-9·4005347 9·6202009 9·0441281 8·7209543 8·4835186	-9·4952622 9·5688910 9·0287177 8·7342851	-9·5367063 9·5338677 9·0078189	-9·5601966 9·5084213	-9·5753680
$\sigma = 4$	5 9·6546416 7 9·3078542 9 9·0448101 11 8·8337222	-9·5577315 9·3950043 9·2489300 9·0767602	8·9556716 -9·6323652 9·0862027 9·1124758	9·1665250 -9·6468700 8·6662311	9·2752590 -9·6439547	9·3429247
$\sigma = 6$	7 9·6051663 9 9·3793134 11 9·1813900	-9·6051663 9·0393654 9·1966300	9·1371896 -9·6241205 -8·7676558	-8·4590383 9·4373054 -9·5626341	-8·7646641 9·5288595	-8·9372649
$\sigma = 8$	9 9·5596417 11 9·3998851	-9·6176967 7·9527901	9·3678192 -9·5620287	-8·8395454 9·5401268	7·9364557 -9·1384267	8·3207669
$\sigma = 10$	11 9·5187917	-9·6157017	9·4396104	-9·0416704	8·3982178	-7·3982178

The minus sign has been inserted where s_n^σ is negative.

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V. *Effects of Strain on the Crystalline Structure of Lead.*

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[PLATES 2-6.]

THE effects of strain upon the crystalline structure of metals and the subsequent effects of annealing at low temperatures have already been studied by EWING and ROSENHAIN,* and the work described in the following pages is in many ways a continuation of theirs. A variety of lead was obtained in which the crystalline structure was on a particularly large scale. This furnished a metal specially well adapted for experiments on the influence of strain on crystalline structure, and the author, on the suggestion of Professor EWING, was successful in obtaining single crystals of sufficient size to furnish a test piece possessing a uniform orientation throughout the part under test. As the effects of a strain could then be studied in individual crystals, the problem was greatly simplified.

The material used was a pure lead, commercially known as "chemical lead," its chief use being for purposes such as the lining of acid chambers. It is obtained by treating ordinary furnace lead in Pattinson crystallisation pots, and by this process practically all other metals which were originally alloyed with it are removed. The first sample obtained had been cut from a casting, and was about 6 inches by 4 inches in area by $1\frac{1}{2}$ inches deep. One of the large faces had been in contact with the air during solidification and the other with the bottom of the mould. The former showed in a very beautiful manner, and without any treatment, the crystalline structure of the material. The crystalline grains were clearly lined out by the slight differences of level at which they had formed, the boundaries showing up as fine lines. These grains varied from about $\frac{1}{2}$ sq. inch to 4 sq. inches in area and were of quite irregular shape. Many of them exhibited distinctive markings on their surfaces, these generally taking the form of slight ridges. In many cases one fairly

* 'Phil. Trans.,' A, vol. 193, 1899, pp. 353-377, and 'Phil. Trans.,' A, vol. 195, 1900, pp. 279-301.

large ridge ran down the centre of the crystal, and from this others branched away at right angles, giving the appearance of a sort of skeleton.

In order to see how the growth of the crystals had proceeded throughout the casting, the rough sides were smoothed with an ordinary wood plane, and the whole casting was subjected to a prolonged etching in dilute nitric acid. The effect of this treatment upon the upper surface was to bring out in a very striking manner the orientation of the different crystals. The surface of each crystal became covered with a number of geometrical pits, these pits being similar and similarly situated over its area, though varying in shape and position from one crystal to another. The crystals accordingly showed either bright or dark, as the light striking the sides of the pits was reflected or not into the eye. As the specimen was revolved the crystals flashed out one after another from almost black to a brilliant white. Fig. 1 (Plate 2) is a photograph of this surface in its natural size, and the different shades of the various crystals can be clearly seen. The appearances observed after etching are precisely similar in kind to those which are already well known as occurring when ordinary metallic surfaces are subjected to microscopic examination; but here they are exhibited on an exceptionally large scale.

The first effect of etching upon the sides of the specimen was to reveal a number of minute crystals all over the side surfaces, but as the action proceeded these disappeared, and the large crystalline structure of the lump could be seen. The crystals upon the upper surface of the casting were found to only descend to a depth of about $\frac{1}{4}$ inch, and a second system had grown vertically upwards from the bottom of the mould to meet them. The sections of the latter parallel to the upper surface of the casting were generally of rather smaller area than those upon the upper surface.

As will be seen in fig. 1, some of the crystals do not reflect the light quite equally from all parts of their surface, but seem to be shaded. The shaded appearance is only visible in certain illuminations, the orientation of the crystal appearing quite uniform in all others. When examined under the microscope, it was found that the edges of the etched pits upon them were not quite parallel in the various parts. Fig. 2 shows part of such a crystal magnified 12 diameters, and it can be seen that the edges of the pits between the two dark bands running across the centre are not quite parallel to those on the portions on either side; the difference is, however, slight. The effect is probably caused by a slight disturbance whilst the crystal was forming, one part being moved relatively to another when the metal between them was still molten. Such a disturbance might be brought about by external causes, such as a shaking of the mould, or by stresses set up as the metal cooled. The difference of orientation is still further brought out by the fact that the acid has acted more strongly where the two portions join than upon the surrounding area; this etching out has produced furrows, which appear as the dark bands in fig. 2. Such an action is always seen between two patches of different orientation, and

where the difference is great, as it is between two crystals, a furrow which can be easily seen without any magnification is produced. It may be accounted for by electrolytic differences between the two portions producing a more violent action where they meet.

The large size of the crystals in this casting suggested that it would be possible to obtain a fairly large sized specimen of lead having a uniform orientation throughout, so that the effects of strain upon such a specimen might be studied. To do this an attempt was made to obtain a casting in which individual crystals went right through from top to bottom, as they could then be separated one from another with a minimum amount of cutting. It was found to be very difficult to do this by direct casting, as crystallisation always seemed to take place from both top and bottom, thus producing a casting such as has been described above. The following method was therefore resorted to:—

The lead was melted in a flat sheet-iron dish over a small gas furnace, an iron plate being placed between the flame and the dish to equalise the temperature. The gas was then turned low and the surface of the metal was allowed to solidify, one corner, however, being kept molten by removing the crust with a hot iron spike. Owing to the contraction of the lead upon solidifying the growth of the various crystalline grains could be observed, a slight difference of level being formed between the solid and molten portions. The growth was, perhaps, best seen in those crystals which formed with two cubic axes parallel to the surface. Fig. 3 illustrates diagrammatically the manner of this growth. One or more parallel arms, such as are seen in the figure, shot out from the place where crystallisation started and gradually spread outwards into the molten metal. At certain intervals along these came others at right angles to them, then a third set from these parallel to the first, and so on, numerous successive sub-branches growing at right angles and gradually filling in the intervening spaces. The growth of an individual crystal proceeded in this manner until arrested by meeting either the sides of the dish or another crystal. The outlines of the branches remained even after the whole casting was cool, forming the ridges which have been already mentioned.

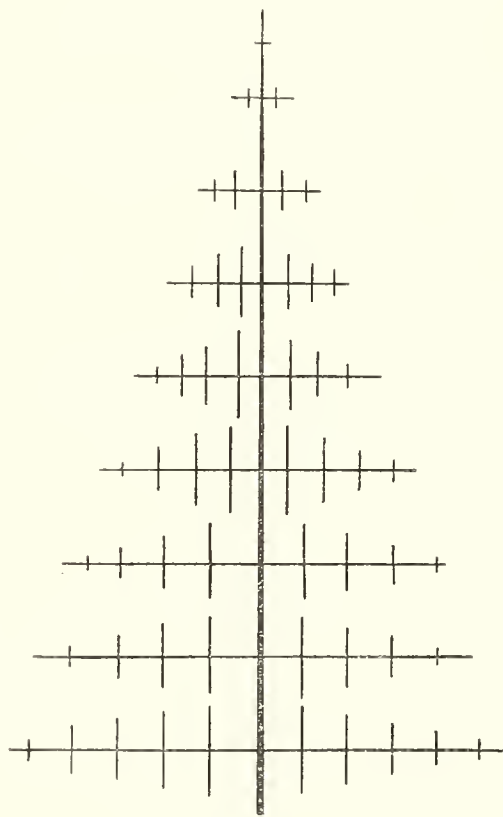


Fig. 3.

When a sufficiently strong crust had formed, the dish was lifted up and the still molten metal was poured out through the free corner. When cool, it was found that a certain portion round the edges of the dish had solidified to the full depth of the

metal, but that in the centre there was simply the upper crust, about $\frac{1}{8}$ inch to $\frac{1}{4}$ inch in thickness, and this could be easily cut away from the surrounding portions. The crystals forming the crust fell away from one another to a certain extent when unsupported by the metal beneath. This was due to the well-known fact that the metallic impurities form a eutectic with part of the lead and collect in the crystalline boundaries. This eutectic has a lower freezing-point than the pure lead, and hence remains molten after the main mass has solidified. The boundaries are, therefore, lines of weakness, and the crystals tend to fall apart when the supporting metal below is poured away.

The upper surface of the crust which had been in contact with the air was, apart from the skeletal markings, fairly plane, but the lower surface was covered with numerous small spiky projections, these being regular in shape and position over each individual crystal, but varying from one crystal to another. Fig. 4 (Plate 2) shows such a surface natural size, and the various crystalline grains can be easily distinguished by their different textures. It will be seen that in some cases certain spikes are raised rather above the others, forming ridges which run in different directions across the crystals. These correspond to those upon the upper surface which have been mentioned above. A portion of the same surface at the junction of three crystals is shown magnified to 10 diameters in fig. 5, and the different shape and orientation of the spikes upon each crystal can be clearly seen. These spikes have more or less curved surfaces, but generally roughly assume the form of octahedra. Some of them exhibit a somewhat wavy surface, as if the metal had solidified in successive layers. Such markings are probably due to a certain unsteadiness when the molten metal is poured away, the latter flowing two or three times over the solidified surface before finally leaving it.

The various crystals were now separated one from another with a fretsaw. The rough sides were smoothed by cutting away the spiky projections with a sharp knife, and the crystals were then etched with dilute nitric acid.

The etching consisted of two operations; the crystals were first placed in a 20 per cent. solution of nitric acid for about half an hour. This removed any roughness from the surface and produced a fairly smooth plane specimen. They were next immersed in a 5 per cent. solution, in order to produce the geometrical pits by means of which most of the following phenomena have been studied. In order to obtain a good development of such pits it was found necessary to prolong the etching for from one to two days; if particularly large pits were required the process was even longer. It was found in general that the stronger the acid solution the smaller were the pits, and *vice versa*. With a 5 per cent. solution a beautiful system of contiguous pits covering the entire surface of the specimen could be produced which was easily visible at a magnification of 20 diameters or less.

The shape of these etched pits has been found to be that of a portion of a negative cubo-octahedron. Such a figure is shown in fig. 6 (*a*) and (*d*); in (*a*) one of the

cubic faces is parallel to the plane of the paper, and in (d) an octahedral face is nearly parallel to the same plane. It possesses faces which are alternately squares and hexagons, and it will be seen that in the following photographs the faces of the pits always consist of all or part of one of these figures. Fig. 6 (b), (e), are drawings of the pits which (a) and (d) respectively would make if made to penetrate a certain distance into the plane of the paper, and may be compared to the etched pits seen in the photographs. In some cases the octahedral faces were very strongly developed,

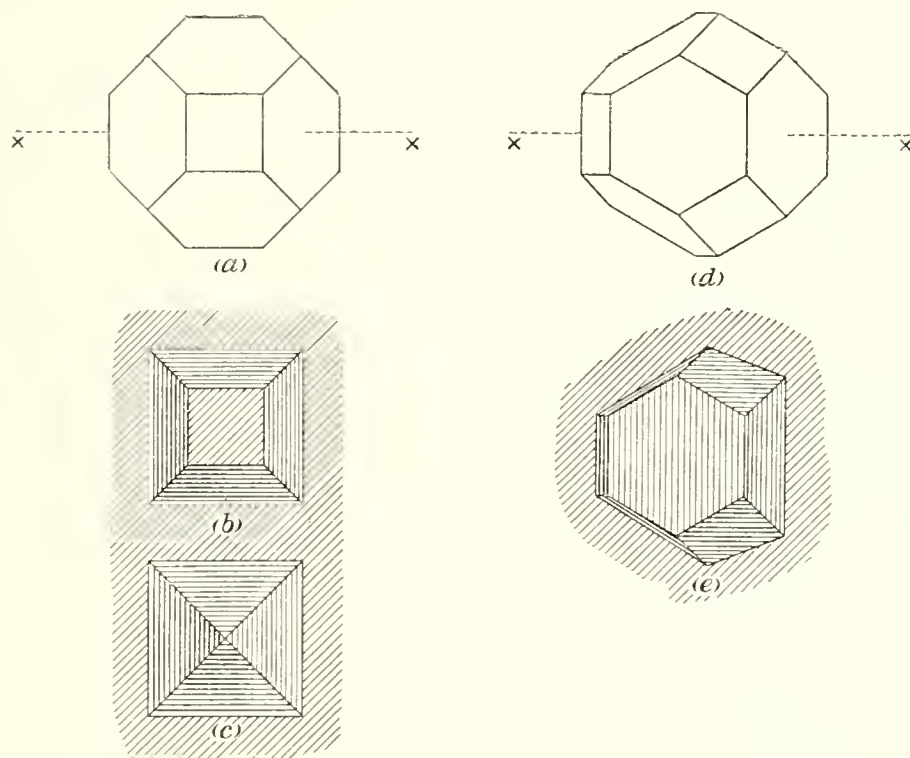


Fig. 6.

and the cubical faces were practically non-existent. The figure then developed into the regular octahedron. Fig. 6 (c) shows the type of pit we should expect in such a case when a cubic face is parallel to the surface, and may be compared with the pits seen in figs. 9 and 14. The relation of these pits to the directions of the skeletal crystal formation was found by scratching lines to show these directions upon the surface before etching, taking care to make the scratches deep enough not to be eaten away. It was then seen that where a square face of the pits was parallel to the surface of the specimen, the sides of the square made angles of 45° with the direction of the ridges. That is to say, the sides of the cube from which the cubo-octahedron was formed were parallel to these directions. Hence the pits are correctly placed as regards the crystalline axes, if we assume that the ridges are parallel to the cubic axes of the crystal. From the manner in which these ridges are formed by the crystal growing by successive branches at right angles to one another, as is illustrated in fig. 3, such an assumption seems not improbable. (See Plates 3 and 4.)

When examined under the microscope, the pits at first sight appear to be projections from the surface, but by means of the focussing screw it is readily shown that they are not projections, but true pits.

A similar appearance is also seen in the micro-photographs and is apt to be rather misleading as to the nature of these geometrical figures, but in all cases they were found to be, as stated above, pits descending into the surface.

The single crystals, when separated one from another, possess all the usual plastic properties of an ordinary sample of lead composed of numerous small crystals united together. They show no signs of sudden cleavage or parting and may be bent double without breaking, and may be hammered out into any shape.

The first observations made on the effects of strain on single crystals were in connection with the formation of slip-lines. The specimens were cut to about 2 inches to 3 inches long by $\frac{1}{2}$ inch wide by $\frac{1}{8}$ inch thick, and were generally taken from a single crystal; if, however, a sufficiently large specimen could not be obtained from a single crystal, the piece was cut so that the central portion, where the effects of strain were studied, consisted of a crystal extending right across the specimen from side to side, as well as through from front to back. After the specimens were cut out they were carefully etched until a complete system of geometrical pits had formed upon their surfaces; they were then washed until all the acid was removed, and were quickly dried. The surface remained bright for a sufficient time to enable observations to be made and photographs to be taken, but it gradually became tarnished if left exposed to the air.

The specimens were strained in tension in the small machine described by EWING and ROSENHAIN. This could be fixed to the stage of the microscope, so that a portion of the specimen was kept under observation during the whole process. As the stress was applied the specimen gradually elongated, this elongation being due to the numerous small slips along the gliding planes of the crystal. These small slips are first visible under the microscope as either bright or dark lines upon the surface, according to the illumination of the specimen, just as EWING and ROSENHAIN first observed them; but when the amount of strain becomes large the actual steps formed can be clearly seen. A point of chief interest in the experiments was the relation of the slip-lines to the etched pits upon the surface, that is to say, to the crystalline axes. Figs. 7 and 8 are photographs taken of surfaces after straining in tension. In fig. 7, which is at a magnification of 45 diameters, the etched pits had been very slowly produced and are of large size; they are not quite contiguous, but portions of the original surface remain between them. It will be seen that the slip-lines on one face of a pit are parallel to one edge of that face. Again, the illumination is such that a partially formed hexagonal face shows bright, and it will be seen that slip has occurred along planes parallel to this face, so that the small steps also appear bright. In fig. 8 (Plate 3) the pits were of considerably smaller size, and a higher power (100 diameters) was necessary to show them clearly. As before, the

slip-lines follow the sides of the pits in such a manner that the plane of slip is parallel to a hexagonal face; in fig. 9 this face and the slip-lines show up bright. Owing to the difference of level in the surface due to the pits, all of it could not be brought into focus together—this accounts for the somewhat blurred patches in the photographs. From these and other observations it may be concluded that lead tends to slip along planes perpendicular to the octahedral axes of the crystals, and there would, therefore, be at least four possible directions in which slip could occur. Four systems of slip-lines have already been noted in strained lead by EWING and ROSENHAIN.

When a single crystal is strained in tension, the slip along the gliding planes does not take place to an equal extent right across the specimen. The surface, originally fairly plane, becomes slightly undulated as the stress is applied, the undulations running diagonally across the specimen at an angle of about 45° to the direction of pull. The general character of the slip-lines can be best examined upon a plane unetched surface not broken up by pits. Such a surface exists on the upper side of the crystal, namely, the side in contact with the air when cast as described above and not afterwards etched. If we examine the slip-lines formed upon it when strained in tension, we find that the undulations are caused by the slip-lines being larger and more numerous along certain areas. When the specimen is re-etched after straining, the undulations are still visible as diagonal bands, which appear either slightly brighter or darker than the surrounding material, according to the illumination used.

Fig. 9 shows part of such an etched surface at a magnification of 45 diameters. A band of pits, more brightly illuminated than the rest, runs across the centre of the figure, and similar, though less distinct, bands can also be traced running parallel to it. The following points can be noticed: (1) that there is no distinct boundary between the light and dark portions, the change from one to the other being gradual; (2) the bands do not exhibit the same illumination; (3) the sides of the etched pits upon the bright parts are not quite parallel to those upon the darker ones.

Such slight differences of orientation occurring after a single crystal has been strained are evidently not due to a re-crystallisation. Professor EWING has suggested to the author that it is quite possible to account for them if we consider that the strain is not homogeneous. It has been stated above that the number of slip-lines was not the same all over the strained crystal, but that certain parts showed more signs of slip than others. From this we would gather that when once slip occurs in a certain part it tends to go on there rather than in other parts of the specimen. Hence these parts get more or less distorted from their original shape, while other parts of the same crystal contiguous to them either have not changed at all, or have done so in a less degree. It may readily occur that the contiguous portions, which were originally in parallel orientation, become relatively inclined through the distortion of the material between—the material between behaving as a

strained wedge—and hence differences in orientation such as are seen in fig. 9 may arise within a single crystal.

In cutting out the single crystals from the casting, a certain amount of local strain was given at the edges by the action of the saw, and it was found, upon etching, that where such strain had occurred the specimen no longer exhibited a uniform orientation, but that numerous small areas of different orientations had appeared. Such a breaking of the crystals has already been mentioned as appearing upon the sides of the casting shown in fig. 1, where it had been sawn from the surrounding metal. Some specimens also had been accidentally strained by bending, and in these cases a similar breaking-up was visible over the strained area when the specimen was re-etched. Further experiments were therefore made to obtain more definite results as to the production of comparatively small crystals by strain.

A specimen was carefully cut with a sharp knife from the centre of a single crystal in such a manner that as little strain as possible was given. After etching it was found that the orientation was uniform all through. It was now bent nearly double between the fingers, straightened again, and re-etched. Great changes were now visible over the strained area, the orientation no longer being uniform, but broken up into numerous small areas, each with a different orientation. The greatest change had occurred on what had been the concave side when the specimen was bent, that is to say, where the metal had been subjected to compression. Figs. 10 and 11 show the compression and tension sides respectively of a specimen originally uniformly oriented throughout (that is to say, originally a single crystal) after it had been strained in the manner described above and then re-etched. It will be seen that on the compression side a large area in the centre has been split up into numerous small patches of different orientations, the variety of shades within this area showing in a striking manner the extent of the change. On the tension side two or three isolated patches have appeared possessing new orientations, but the amount of change there has been far less than on the compression side. It is also noticeable that the ends which were not subjected to any strain have not in any way altered. When we look more closely into the newly oriented patches, it is seen that a great number of straight-line boundaries exist between them, and this fact is still more striking when we use a higher magnification. It is then seen that in some cases the boundaries are quite irregular and have been more or less eaten out into channels during the etching process, but that in others the boundaries are straight and sharp, one orientation changing quite abruptly into the other with no such channel between them. In the latter cases it was also apparent that there was some distinct geometrical relation existing between the pits on either side of the boundary, an edge of the pits on either side being always parallel to the boundary between. In many cases there were two parts joining in a straight line, but surrounded by an irregular boundary. Such is the case in the newly oriented patches seen in fig. 11. Each of the patches consists of two such parts, and the straight line between them

can be contrasted with the curved irregular outline surrounding them. Similar cases can also be seen in fig. 10, though surrounded by new orientations and not, as in fig. 11, by the original one. In other cases there would be two or more sets of parallel straight-line boundaries, the two orientations occurring alternately; and in others a patch showing one orientation would be surrounded on more than one side by straight lines, these being always parallel to an edge of the pits on either side. Figs. 12 and 13 illustrate these cases; all are taken at a magnification of 45 diameters. In fig. 12 we see two portions at the bottom, one bright and one dark, joining in a straight line, but surrounded by an irregular boundary; while in the centre we have a crystal with a band with parallel sides running across it, the parts on either side of the band being in similar orientation. In fig. 13 (Plate 4) we have a patch including various portions of a different orientation, though similar to one another. Some of the boundaries in this case appear irregular, but a rather higher magnification shows that these apparently irregular boundaries are made up of numerous short straight lines. This photograph is also interesting as the etching had not been carried so far as in most of the other cases and the etched pits are not contiguous. It hence gives some idea as to the manner in which they are formed.

Such straight boundaries are a strong indication that the portions on either side are connected by a twin relation, and the fact is further brought out when we examine the geometrical relation of the pits on either side. It has been said above that these pits, which are portions of a cubo-octahedron, are placed in a correct direction to the crystalline axes, and are in fact representations in miniature of what the external form of the crystal would probably be if allowed to assume its proper shape. Take the case shown in fig. 14: we have two main orientations in the picture, one having pits with a large hexagonal face nearly parallel to, and the other having pits with a cubic face parallel to, the surface. In the latter case, the cubic faces of the pits have, for the most part, been etched away, and the pits are practically octahedra. If we draw the complete form of the latter, we shall obtain the figure shown in fig. 6*a*, the dotted line being at right angles to the boundary between the two parts in fig. 14. Now, if we give this figure a turn of 180° about the octahedral axis marked *xx*, that is to say, bring it into a position in twin orientation to its original one, it will appear as shown in fig. 6*d*. Then, on comparing the etched pits on either side of the boundary in fig. 14 with 6*a* and 6*d*, the twin relation which exists between them is seen. It was found in all cases that the pits on either side of a straight boundary agreed with figures drawn in such a way. The method is of course merely a rough one, and no accurate results could be obtained from it, but it should be quite possible with a suitable goniometer to actually measure the crystallographic angles of the pits on either side. Such measurement of etched pits in lead have already been made by Professor MIERS, but the work is rather beyond the scope of the present paper.

The tension side of a specimen such as that described above was of course sub-

jected to a certain amount of compression when the specimen was re-straightened after being bent. A trial was, therefore, made to see if any splitting up of the orientation could be produced by tension alone. A crystal was strained in tension in the small machine mentioned on p. 230, the stress being applied until local contraction and fracture occurred. The two halves were then re-etched and examined, and it was found that the greater part of the specimen was unchanged, but near the fracture, where, due to the local drawing out and transverse contraction, the strain must have been specially severe, a splitting up had occurred similar to that in the bent specimen described above. That is to say, in place of the original uniform orientation covering its whole surface, numerous patches with various orientations appeared.

EWING and ROSENHAIN have already drawn attention to the progressive growth of crystals which occurs after a specimen of lead or other metal has been severely strained, especially when the specimen is moderately warmed. On the suggestion of Professor EWING, the author next tried the effect of cooking these large strained crystals, that is to say, subjecting them to prolonged exposure to moderate temperatures. The first specimen tried was one similar to that shown in figs. 10 and 11. It had been strained by bending between the fingers and then re-straightening. After etching it was found that a certain area in the centre had re-crystallised, the specimen showing a similar appearance to figs. 10 and 11. It was then cooked for some hours at a temperature of about 100°C . and then re-etched. After this treatment the area re-crystallised was far greater than before. This was especially striking upon the tension side, where, as in fig. 11, only a few isolated patches had existed before cooking; afterwards, however, a large area was found to have changed, the patches with new orientations being quite contiguous, though of rather larger individual size than upon the compression side. That such a growth of new orientations was in some way due to the straining of the specimen was clearly proved by roasting an unstrained specimen. In this no change was produced, but after straining and again cooking the specimen was found upon re-etching to have to a great extent re-crystallised. It was also found that it was not necessary for the specimen to show any signs of re-crystallisation before annealing, but that so long as a certain amount of strain had been given, whether upon re-etching after this strain the orientation showed any signs of alteration or not, yet after cooking at 100°C . the re-crystallisation either continued or was started. Specimens were strained both by bending and tension, and in both cases a further re-crystallisation occurred after heating. It has been mentioned above that, when a specimen had been strained in tension, only a small area near the fracture showed any signs of alteration when re-etched, but it was found that if such a strained specimen was afterwards heated, the whole orientation could be changed. In studying this effect it was indeed found to be far more convenient to strain the specimens in tension, as the amount of strain to which they were subjected could be more easily regulated.

The method of carrying out these experiments was as follows:—A specimen about 2 inches long and $\frac{1}{2}$ inch broad by $\frac{3}{16}$ inch thick, was cut from a single crystal and given a certain amount of tensional strain in the small machine mentioned on p. 230, re-etched and photographed. It was then cooked for a short time by exposure to a certain temperature in a small asbestos-lined wooden oven, heated by an electric incandescent light. After this cooking the specimen was re-etched, first in a 20 per cent. solution of nitric acid to remove all former pits, and then in a 5 per cent. solution to produce a new set. A second photograph was then taken, and further processes of cooking, re-etching, and photographing were gone through until no further changes were visible. The photographs were taken with a 4-inch Ross lens, fitted to the front of the camera, diffused daylight being used as an illuminant. The specimen was kept immersed in a weak solution of nitric acid whilst being photographed, in order to preserve a clean surface. Care was taken to get the illumination the same throughout the series, in order that the same patches of new orientation could be recognised in the different photographs. In all the following illustrations (figs. 18 to 32) the direction of the pull was parallel to the long edges of the specimen, that is to say from top to bottom of the page.

Fig. 15 shows a specimen after straining in tension and re-etching, magnified to five diameters. The two lines seen running across had been scratched with a sharp steel point, and were originally 1 centim. apart, the straining being carried on until they were 1.5 centim. apart; the other marks were for the sake of identification. The only change visible before annealing was a small amount of re-crystallisation along the scratches, where the material was, of course, subjected to fairly severe local strain, and in two places between the lines, where it will be seen the new orientations appear as two small dark patches. Fig. 16 shows the same specimen at the same magnification after cooking for 5 minutes at 60° C. and re-etching. It will be seen that great changes have taken place; the former patches of new orientation have greatly extended, and others have appeared in various parts of the specimen. The greatest change is at the bottom left-hand corner, where a large area has re-crystallised. Figs. 17, 18, and 19 show the same specimen magnified five diameters after further successive cookings of 5 minutes', 10 minutes', and 20 minutes' duration respectively, at 60° C. The re-crystallisation has continued until practically the whole of the original orientation is altered. In fig. 18 (Plate 5), the scratches have been almost entirely etched away, and with them the small local changes which they had produced. The stage at fig. 19 seems to be a final one, and further cooking produced no further changes. In cases such as this, where the material had been subjected to a uniform strain throughout, the patches of new orientation go right through the specimen, both sides showing a very similar pattern. In this specimen the final formation consisted for the most part of a few large differently oriented areas penetrating right through, each to a large extent split up by twin orientations in bands and patches, as can be seen in fig. 19. It will be noticed that the area of the

specimen gets gradually less in each photograph; this was due to the etching, which had to be fairly severe in order to entirely eat away the old pits and exhibit the new structure, and which was sufficient, when repeated several times, to produce a considerable change in the size of the specimen.

If we examine this series it will be seen that when once a patch with a new orientation is formed, and is surrounded by other new ones, it remains practically unaltered when subjected to further annealing. There is a slight change of outline in some cases, but this can be accounted for if we consider that the boundaries need not be at right angles to the surface, and will, therefore, change slightly as the etching solution eats down to successive depths. The structural change is entirely confined to the original orientation, which is gradually split up into numerous different ones, and when once these are formed they persist.

The slight differences of orientation which appear after straining, when the specimen is first etched, appear to some extent upon these photographs as bright bands across the original orientation. They are, however, far more clearly illustrated in the next series. Fig. 20 shows the crystal after straining in tension and re-etching magnified 5 diameters. It exhibited no signs of re-crystallisation before annealing, but the "strain bands" (as they may be called) were particularly well developed and can be clearly seen in the photograph. Fig. 21 shows the same specimen after annealing for 20 minutes at 100° C. It is evident that the strain bands have no influence on the re-crystallisation, but are swallowed up in an exactly similar manner as the original orientation. A remarkably fine development of etched pits was obtained on this specimen, and these are clearly visible even under the low magnification at which the photographs were taken.

The next experiment was to see whether any change took place in a strained crystal of lead at atmospheric temperatures. The specimen was strained in tension, and is shown (magnified to 4 diameters) in fig. 22 after this straining and re-etching. The stress had been applied until local contraction had commenced, and the strain thus produced was sufficient to cause a certain amount of re-crystallisation to be visible immediately upon re-etching. The band of patches of new orientations can be seen running diagonally across the specimen at one end. The photograph reproduced in fig. 23 was taken from the same specimen after three weeks. The specimen was kept in a small glass jar, and was simply subjected to the slightly varying temperature of the room. In fig. 23 it will be seen that re-crystallisation has continued until nearly the whole of one end has changed, the band widening out in both directions so as to fill in the right-hand bottom corner and extend further upwards.

It is clear from this series that, although a further re-crystallisation occurs at ordinary atmospheric temperature, yet it is much slower than when the temperature is slightly increased, as had been the case in the former series. It has been found in all cases that the higher the temperature at which the strained piece is kept the

quicker does re-crystallisation proceed. In some specimens which had been only slightly strained no visible change was seen until a fairly high temperature was used. One specimen was strained in tension and annealed for 20 hours at 60° C. without any visible change taking place. When, however, it was annealed for two hours at 100° C., the re-crystallisation seen in fig. 24 occurred. It is interesting to notice the large size of the patches showing new orientations in this photograph.

An important question now presented itself for solution by experiment. Is the re-crystallisation which is apparent immediately after etching in a severely-strained crystal a direct and instantaneous effect of the strain, or is it a growth which occurs during the interval of time that has elapsed between the straining and the examination? The experiments of EWING and ROSENHAIN showed that a slow progress of growth goes on at atmospheric temperature in ordinary lead after severe straining, which may result in the formation of comparatively large crystals in a severely crushed specimen after the lapse of several days or weeks, and the experiments just described show a similar slow change. In the present instance we are concerned with individual crystals in a structure of much coarser grain than was dealt with in their experiments, and with the comparatively short interval of time (some five minutes at least) which was required to prepare the specimen for examination by etching after the strain had been applied. At first it was not suspected that the re-arrangement of crystals seen after straining was other than an immediate effect of the strain, but the author has now satisfied himself that this is not the case. The re-arrangement does not occur in the act of straining like the re-arrangement (by twinning) which one can produce on straining a crystal of calcite. It occurs after the strain has taken place, during the time that elapses before the crystal is etched for re-examination, and though it requires only a short interval of time for its development, it is to be classed with the progressive growth demonstrated by EWING and ROSENHAIN and confirmed by the experiments already described in this paper.

That this was the case was first suspected from observations of the character of the re-arranged crystals. If these had been produced by successive twinning actions forming a direct result of the strain, as in the twinning of calcite, we should have expected to find the straight-line boundaries characteristic of twins, not only between the patches having new orientation, but also between these and the unchanged portion of crystal in which its original orientation was preserved. Now, although there are numerous straight-line boundaries between the patches of new orientation, demonstrating the twin relation of these patches to one another, it is remarkable that twin boundaries are not to be found between the unchanged portion of the crystal and any of the re-crystallised portions. Twins to the original orientation of the crystal would undoubtedly be formed if the action was the direct result of straining, such as has been described above, but in all cases the boundaries between any new patches and the original structure were found to be irregular in outline, and to be eaten out into channels by the etching solution; and no traceable geometrical

relation existed between the etched pits on either side of them. All these characteristics pointed to their not being twin boundaries.

One experiment directly bearing on this point consisted in examining the slip-lines which were formed as the process of straining went on. If the action was one of successive twinning, when one twin had formed the slip-lines should, as the strain proceeded, form in a new direction over its surface, and hence, as more and more strain was given, numerous systems of slip-lines running in various directions should appear, so grouped as to exhibit the twin character of the crystals produced by the earlier part of the straining. Experiments were, therefore, made to see whether the slip-lines which were formed during the application of the strain gave any such indication of a change of orientation in any part of the crystal, and thus to find out whether re-crystallisation was a direct and immediate result of the strain, or happened after the strain had been given.

As was mentioned on p. 231, a plane unetched surface, such as that obtained from the surface of the casting in contact with the air during solidification, should be used in studying the direction of the slip-lines. A specimen was, therefore, cut from an unetched single crystal, and the spiky projections were cut away from the under side. The specimen was strained by bending nearly double and re-straightening. It was known from previous experiments that after such a strain the lead always gave evidence of re-crystallisation, but, on examining the specimen under the microscope, it was found that the slip-lines extended in parallel systems all over the surface of the crystal. An area was marked by scratching upon the surface with a steel point, and was photographed. This is shown in fig. 25 (Plate 6), and although the scratches have produced a certain amount of local displacement of the slip-lines, it is obvious that these all run in directions which are uniform over the whole strained area. The surface was now etched, and the same area was again photographed (fig. 26). Numerous patches of different orientation are seen to have appeared which have no apparent relation with the slip-lines in fig. 25. It is clear, therefore, that these patches have been developed subsequent to the strain, and not in the process of straining.

In the next experiment the specimen was strained first of all in a similar manner to the above, that is to say, by bending. This developed simple uniform marking by slip-lines. The specimen was then allowed to rest for about five minutes, and was not etched. A further strain was then applied by tension. When this was applied, it was seen that upon the outlying parts the slip-lines ran in directions uniform with those already formed, but upon the middle area, which had already been severely strained, they ran in numerous directions which were parallel over certain small patches, but bore no apparent relation to the original direction. It was clear from this that over the part which was severely strained to begin with, the interval of rest had caused crystals to form differing from the original orientation, their existence being manifested by the new directions which the slip-lines assumed when

further strain was applied. Figs. 27 and 28 are photographs taken (at a magnification of 45 diameters) of a marked area on the specimen after the first and second straining respectively.

The final experiment consisted in straining a single crystal in tension until fracture occurred, and examining the slip lines. It was found that even in this case, although the surface was to some extent broken up by wavy bands, such as are described on p. 231, yet the general direction of the slip lines was constant all over the crystal, right up to the fracture. In other words, the rearrangement in structure of which the previous experiment had given evidence does not occur during the application of a strain, even when that is continued up to the limit of fracture.

From these experiments we would gather that the formation of the patches of new orientation always takes place *after* the stress has been removed and is not directly the result of a general revolution of some of the crystalline elements in the process of straining. From former experiments we have learned that in parts of the crystal where the strain has been severe the patches of new orientation appear almost immediately after the stress is removed and gradually extend from these into the remainder of the strained portion. There appears, therefore, to be no broad distinction between the change which is visible (on re-etching) almost directly after straining and that which takes place after a certain lapse of time. Such differences as are found depend on the amount of strain to which the material is subjected and the temperature at which it is kept; severe straining and a high temperature both tend to increase the subsequent rate of change of structure.

This re-crystallisation which has been shown to go on in an individual crystal must be distinguished in one important particular from that which was observed by EWING and ROSENHAIN to go on in strained specimens of lead composed of numerous crystals united together by a thin film of eutectic formed of part of the lead united with the metallic impurities. In the present case the specimens were composed of practically pure lead and the action was one of a splitting up of the originally uniformly oriented crystal into numerous differently oriented parts, the action proceeding without the aid of any eutectic. In ordinary lead EWING and ROSENHAIN found that certain crystals gradually increased in size by swallowing up their neighbours, and they have suggested that this was due to a "solution and diffusion of the pure metal constituting the crystals into the fusible and mobile eutectic forming the intercrystalline cement." It is interesting to note in this connection that in the case of a single crystal strained so as to show newly oriented parts, such parts show no inclination to grow into *one another*. When once the whole of the specimen becomes newly oriented, further cooking produces no further change. This, so far as it goes, may be regarded as in agreement with the theory of diffusion through the eutectic, as there would in the case of an originally uniformly oriented crystal be no eutectic between the newly oriented parts, and hence no such growth would be possible.

With regard to the formation of twin crystals we may, however, apply a similar

explanation in both cases. To quote from EWING and ROSENHAIN's paper: "When a metal solidifies from the liquid state it does so by the formation of skeleton crystals starting from a great number of centres, and the arms of these skeletons continue to grow until arrested by meeting with other growths. From these arms other arms again shoot out, and so on until the entire metal is solidified; but each crystalline element as it settles into place on any of these arms must assume the proper orientation to enable it to fit in, and in the process of filling space by means of such a system of many meeting and interlacing arms the formation of a twin would be almost impossible. But when the metal crystallises after severe strain it does so by the growth of skeleton arms that must often start from a cleavage plane of an actual solid crystal, and probably the new elements deposited upon such a plane would find it as easy to assume the twin orientation as the normal."

In the present case it is exceedingly probable that practically all the patches of new orientation start from a cleavage plane, and hence the formation of twin crystals would be exceedingly common, as in fact it is.

In conclusion, the author would like to express his thanks to Professor EWING for the great help and many suggestions which he has given. The research has been carried out, under his direction, in the Engineering Laboratory at Cambridge.



Fig. 1. Cast lead, etched ; no magnification.

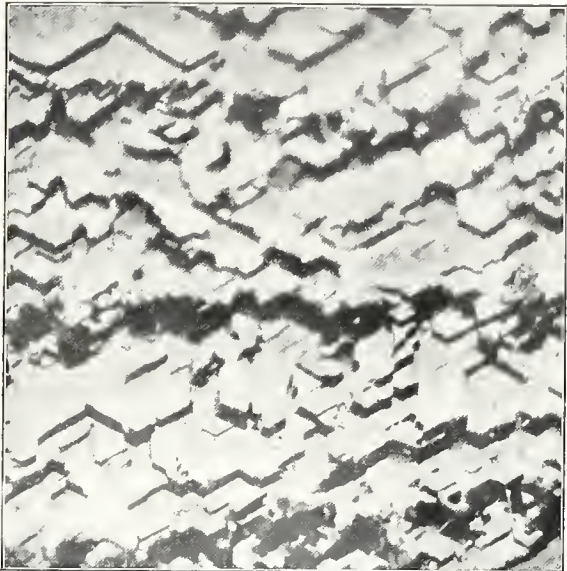


Fig. 2. Etched cast lead $\times 12$.



Fig. 4. Under side of casting ; no magnification.

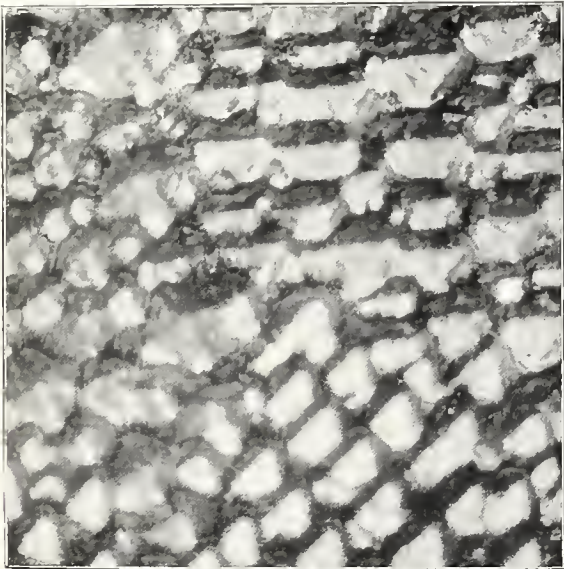


Fig. 5. Part of fig. 4 $\times 10$.



Fig. 7. Slip-lines $\times 45$.



Fig. 8. Slip-lines $\times 100$.



Fig. 9. Part of a single crystal after straining in tension and re-etching $\times 45$.



Fig. 12. Twins in strained crystal $\times 45$.



Fig. 10. Strained crystal, re-etched $\times 6$.

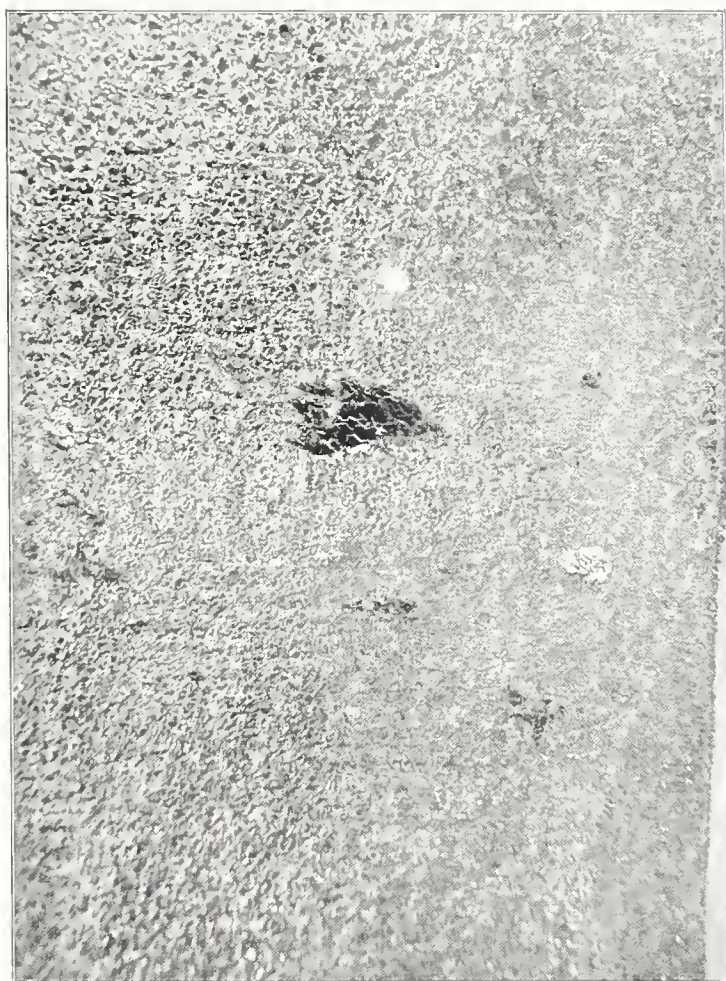


Fig. 11. Other face of same crystal $\times 6$.



Fig. 13. Twins $\times 45$.

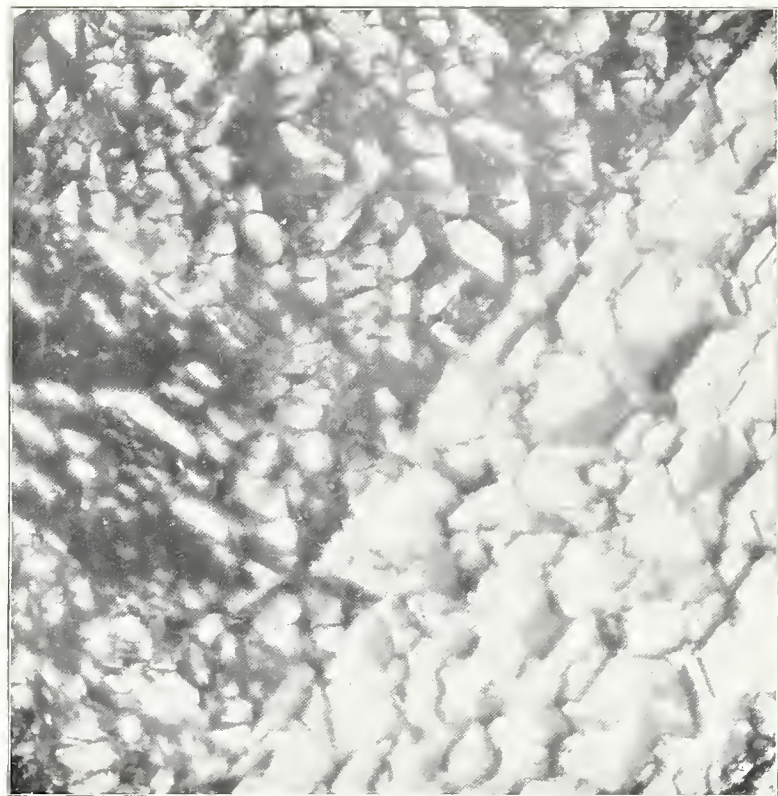


Fig. 14. Twins $\times 45$.



Fig. 15. A single crystal after straining in tension and re-etching $\times 5$.

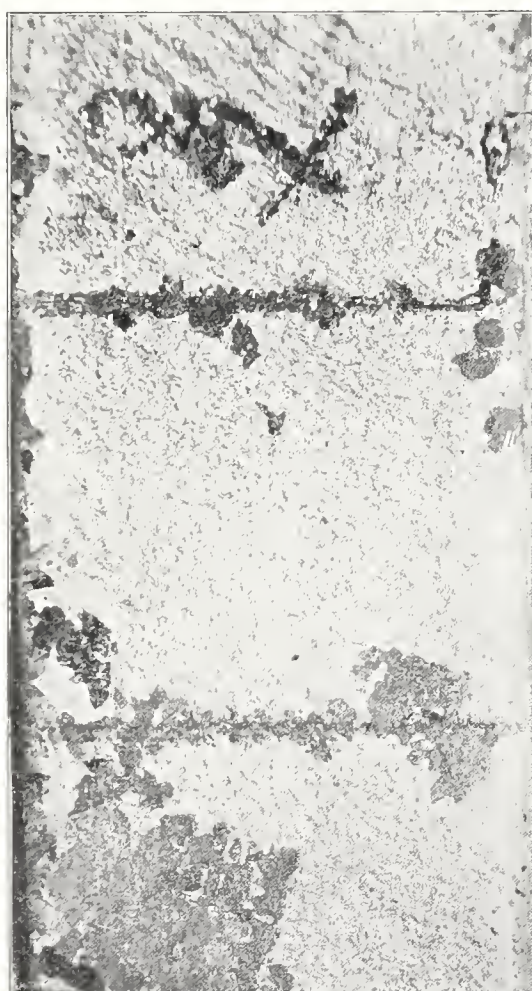


Fig. 16. Same after cooking 5 minutes at 60° C.



Fig. 17. Same after 10 minutes at 60° C.



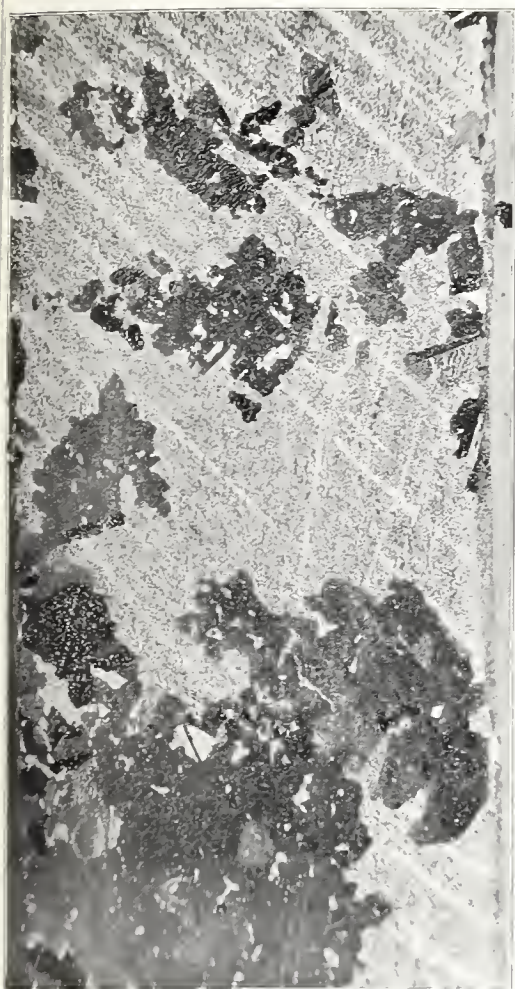


Fig. 18. Same after 20 minutes at 60° C.



Fig. 20. A single crystal after straining in tension and re-etching $\times 5$.

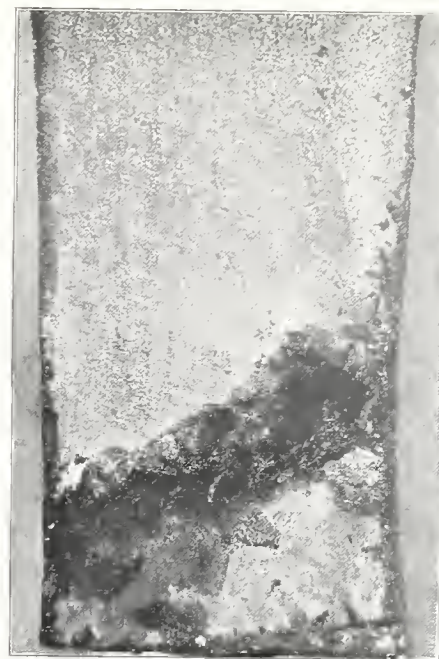


Fig. 22. A single crystal after straining in tension and re-etching $\times 4$.



Fig. 23. Same after 3 weeks at atmos. temp.



Fig. 19. Same after 40 minutes at 60° C.

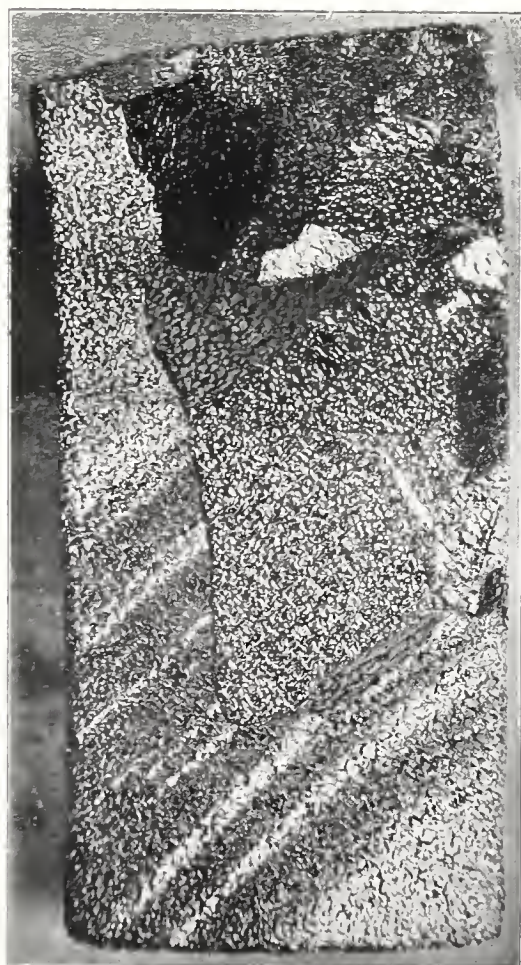


Fig. 21. Same after cooking for 20 minutes at 100° C. and re-etching.



Fig. 24. A single crystal after straining in tension, cooking 2 hours at 100° C., and re-etching.

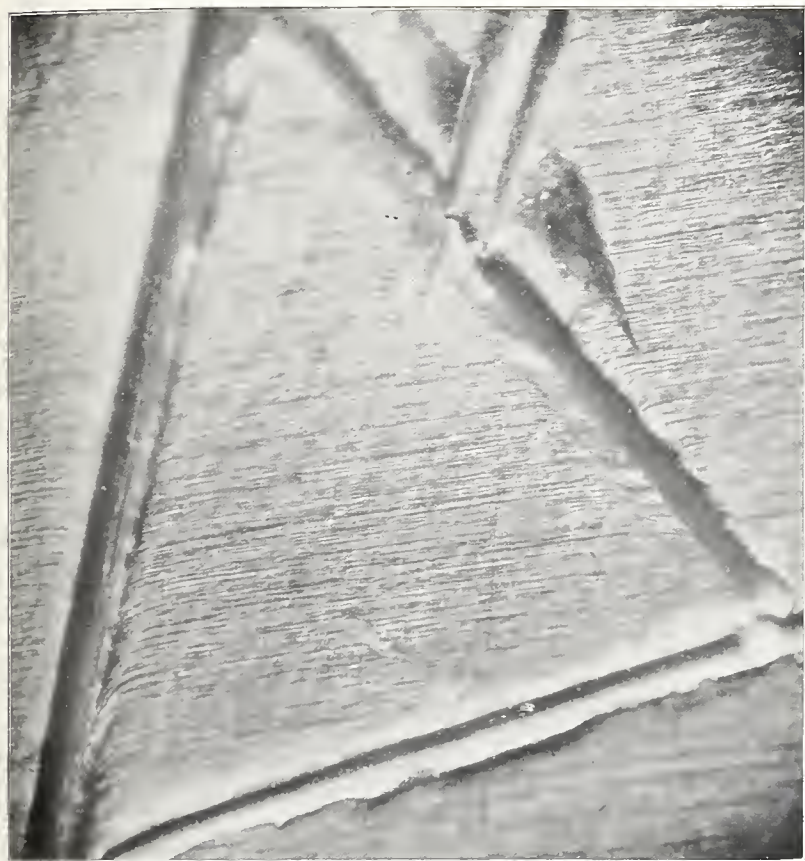


Fig. 25. Slip-lines $\times 45$.



Fig. 26. Same surface after etching.

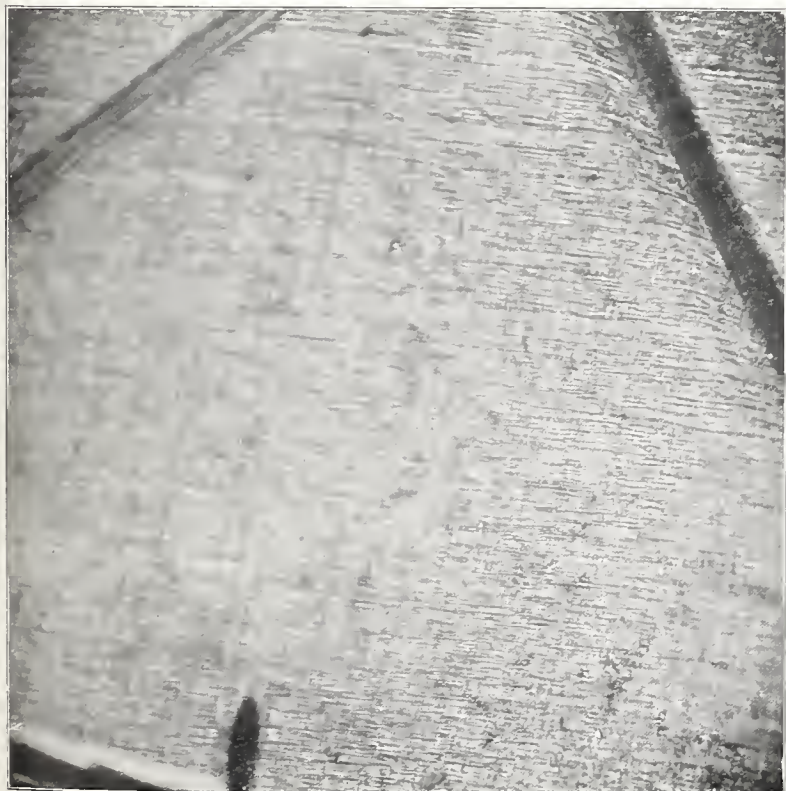


Fig. 27. Slip-lines $\times 45$.

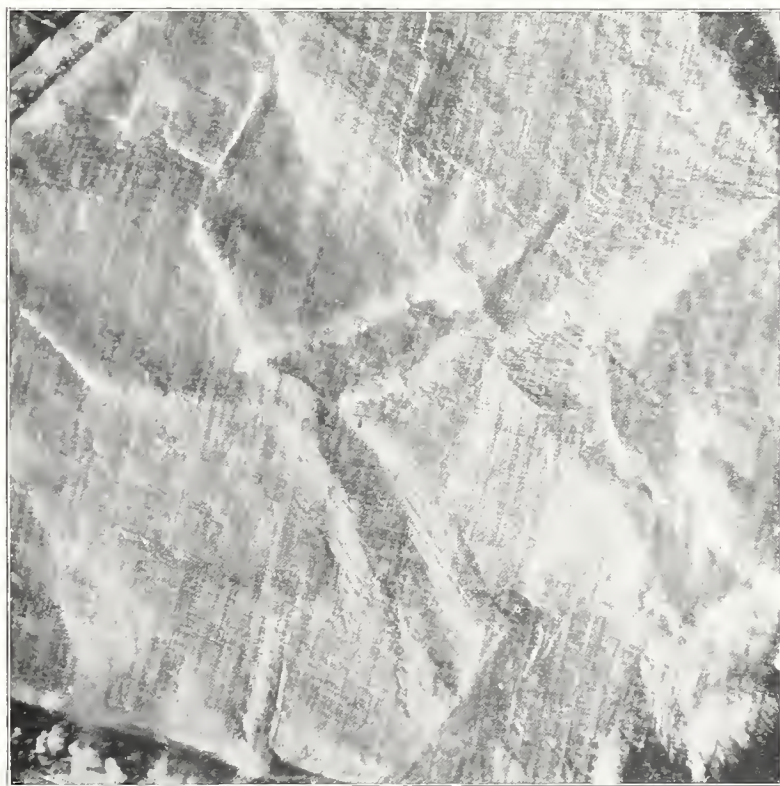


Fig. 28. Same surface after again straining.

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Fatigue of Metals produced by Reversals of Stress.

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Slip Bands of Strained Metals, Cracks produced in, through Reversal of Stress.

EWING, J. A., and HUMFREY, J. C. W.

Phil. Trans., A, vol. 200, 1902, pp. 241-250.

VI. *The Fracture of Metals under Repeated Alternations of Stress.*

By J. A. EWING, LL.D., F.R.S., *Professor of Mechanism and Applied Mechanics in the University of Cambridge*, and J. C. W. HUMFREY, B.A., *St. John's College, Cambridge*, 1851 *Exhibition Research Scholar, University College, Liverpool*.

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[PLATES 7-9.]

It is well known that metals will break down under repeated application, and especially under repeated reversal, of stresses greatly less than those that have to be applied when the “ultimate strength” of the material is tested in the ordinary way. The researches of WÖHLER have shown, for example, that iron capable of bearing about 20 tons per sq. inch of steady load will break when it is exposed to some millions of reversals of a stress of 8 or 9 tons per sq. inch, alternately in compression and extension. When the alternating stress is increased a smaller number of reversals suffices to produce rupture. On the other hand, examples such as are furnished in the balance-spring of a watch, or in a railway axle, show that very many million repetitions may be applied with impunity, provided the limit of greatest stress be kept sufficiently low. The mild steel axle of a railway carriage is exposed to many million reversals of a stress which, in some cases, approaches as high a value as 5 tons on the sq. inch, apparently with perfect impunity, for it seems probable that in the rare instances where fracture of such axles has occurred an explanation is to be found in the gradual spreading of a crack from an origin supplied by an air-bubble or other primitive defect in the material. But WÖHLER'S researches, which have been confirmed by other observers,* give evidence that a stress not very much greater than this, and far below not only the ultimate strength but even the “yield-point” of the metal, will produce what is called “fatigue” and bring about fracture when reversal of the stress is repeated many times.

The purpose of this paper is to describe experiments in which the microscope has been applied to study the nature of the process of fatigue by which breakdown occurs under repeated reversals of stress. The experiments have been made during the past year in the Engineering Laboratory at Cambridge. The metal chosen for experiment was Swedish iron, of high and very uniform quality. It had the further advantage for

* Particulars of the researches on this subject of WÖHLER, SPANGENBERG, BAKER, and BAUSCHINGER will be found in Professor UNWIN'S ‘Treatise on the Testing of Materials of Construction,’ chap. xiii.

our purpose of possessing a clearly defined and fairly large crystalline structure, well adapted when polished and etched to exhibit the characteristic lines known as "slip-lines" or "slip-bands," which appear in ordinary testing when any portion of the material has passed its limit of elasticity under strain.* We used the metal in the form of rods with a rectangular section, the dimensions being approximately 0.3 inch by 0.1 inch, and to make the structure as uniform as possible these were in all cases annealed by being kept for about two hours at a dull red heat, while enclosed in a tube filled with lime, in a muffle furnace. One of the surfaces of each rod was polished

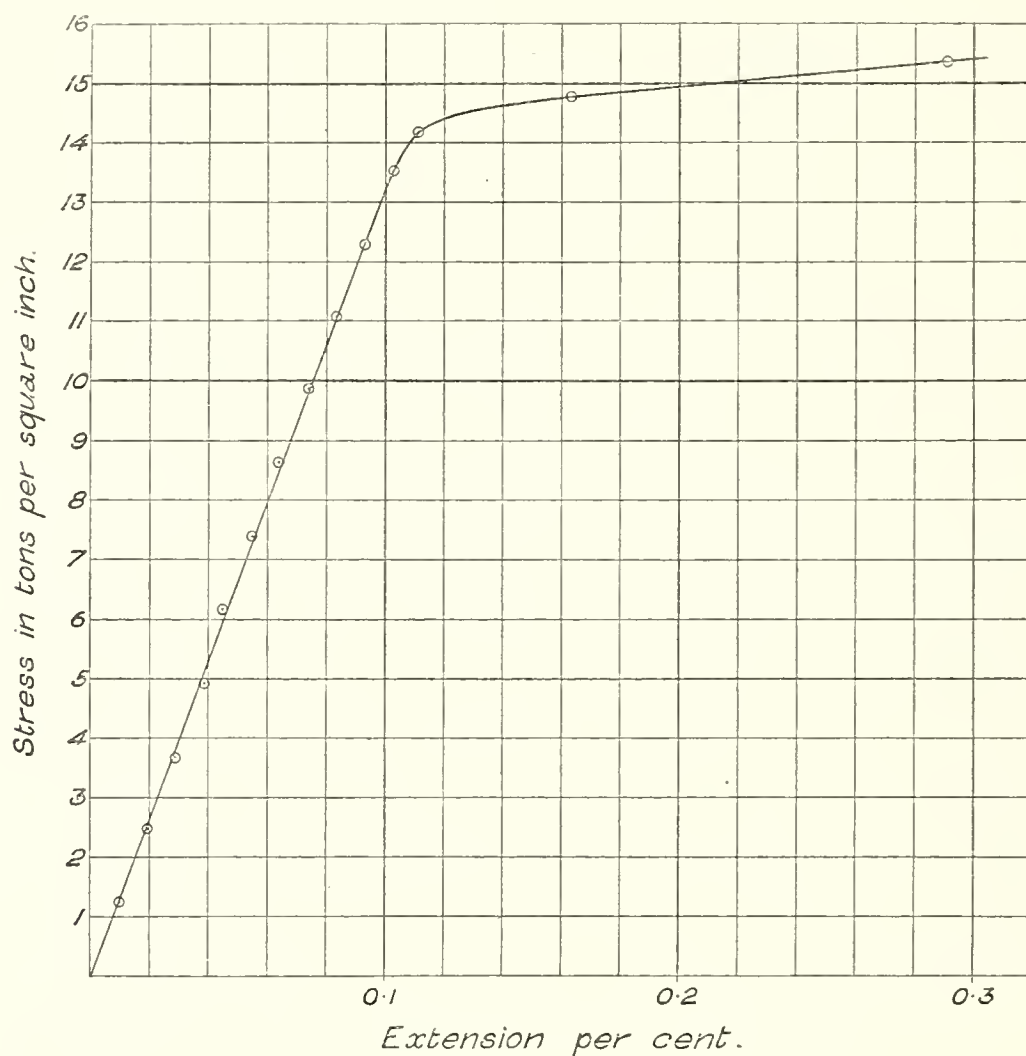


Fig. 1.

and etched, and the rod was subjected to reversals of stress by bending, so that the polished surface was alternately extended and compressed. This was done, as in WÖHLER's original experiments, by making the rod project from a revolving shaft with a load on the projecting end. As the process went on the rod was from time to time examined under the microscope, and in several cases photographs of the same crystals were taken at each stage to record the progressive effect of repeated reversals of stress.

* EWING and ROSENHAIN, "The Crystalline Structure of Metals" (Bakerian Lecture, 1899), 'Phil. Trans.,' A, vol. 193, p. 353.

A tensile test of the Swedish iron rod used in these experiments, carried out in a testing machine in the usual manner, showed a breaking strength of 23·6 tons per sq. inch (reckoned on the original area of section), with an ultimate extension of 0·8 inch in a length of 3 inches, and a contraction of area at the break amounting to 61 per cent. There was a well-marked yield-point when the stress reached the value of 14·1 tons per sq. inch. The diagram, fig. 1, shows the relation of extension to load up to this yield-point as measured by a microscope extensometer designed by one of the authors. It will be seen from this that the extension remains proportional (as nearly as can be judged) to the load, up to a stress of about 13 tons per sq. inch. The value of YOUNG'S modulus, deduced from these measurements, is 13,200 tons per sq. inch.

The apparatus for applying repeated reversals of stress is shown in fig. 2. There *a* is the specimen under observation. It projects from the end of a shaft which was

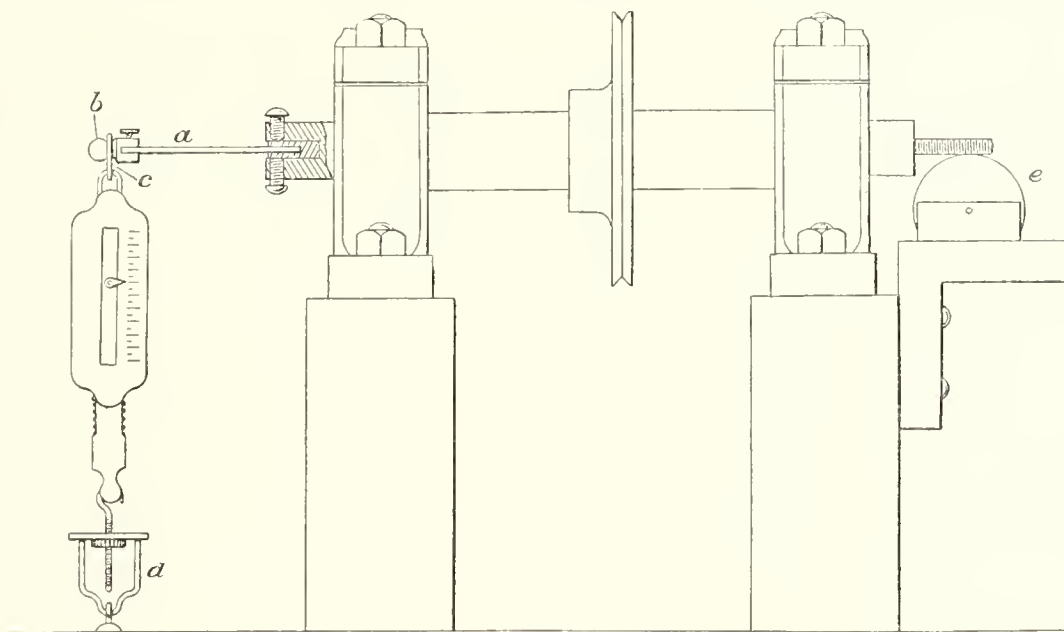


Fig. 2.

caused to rotate, by means of an electric motor, at a speed of about 400 revolutions per minute. To the outer end of the specimen a load was applied causing a bending moment. This was done by attaching a brass cap, *b*, which turned freely in a steel ring, *c*, the ring being pulled downwards with a steady force which was measured on a spring balance. The screw coupling, *d*, allowed the load to be adjusted to any desired amount. The number of reversals of stress was recorded by a revolution counter, *e*. The specimen under test was filed to a uniform rectangular section of about 0·3 inch by 0·1 inch. Part of one of the broad faces was polished and was, in general, etched by dilute nitric acid. The specimen was inserted in the grip at the end of the shaft, adjusted to run true, and the desired load was applied. After making a certain number of reversals the specimen was taken out for microscopic examination; it was then replaced for a further run, and so on, no difficulty being experienced in replacing it each time in the same position as at first.

Before the experiments were made it had been conjectured that the destructive effect of repeated alternations of stress might be ascribed to a loosening of the inter-crystalline cement rather than to damage of individual crystals. Previous experiments had shown that in fracture by ordinary progressively augmented strain the material gives way, in general, not at boundaries, but through the crystals themselves,* but it seemed possible that the effect of repeated straining might be different in this respect. By way of testing the point the experiment was made of subjecting an unetched specimen to many reversals of stress, in order to see whether the inter-crystalline boundaries became apparent as they would do if yielding took place between each crystal and its neighbours. Nothing of the kind was seen, though the boundaries in some instances could be traced through the development in different directions of slip-bands over individual neighbouring crystals. And later experiments, which will now be described, demonstrated that the mischief which is done by repeated straining occurs in quite a different way.

In experiments made with stresses ranging from 14 down to 9 tons per sq. inch it was found that fracture ultimately resulted in all cases. The course of the breakdown was as follows:—The first examination, made after a few reversals of the stress, showed slips-lines on some of the crystals, on many of them if the stress was comparatively great, but on a few only if the stress did not much exceed the lower limit named above (of 9 tons per sq. inch). At this early stage the slip-lines were quite similar in appearance to those which are seen when a simple tensile stress exceeding the elastic limit is applied. Viewed under vertical illumination they appeared as fine dark lines. After more reversals of stress additional slip-lines appeared, which had not been visible in the first instance, but the most conspicuous feature was that those which were visible before became far more distinct and showed a tendency to broaden. After many reversals they changed into comparatively wide bands with rather hazily defined edges, losing entirely the fine and sharp character which slip-lines present when they first appear. As the number of reversals increased this process of broadening continued, and some parts of the surface became almost covered with dark markings made up of groups of broadened lines. When this stage was reached it was found that some of the crystals had *cracked*. The cracks occurred along broadened slip-bands; in some instances they were first seen on a single crystal, but soon they joined up from crystal to crystal, until finally a long continuous crack was developed across the surface of the specimen. When this happened a few more reversals brought about fracture.

In this description we have provisionally named a lower limit of 9 tons per sq. inch, but the experiments give grounds for believing that an even smaller stress will produce fracture in a similar manner if the process of reversing the stress is continued sufficiently long. There is clear evidence that with 9 tons per sq. inch fracture results. But we have also observed that with 8, and even 7, tons per sq. inch slip-

* EWING and ROSENHAIN, *loc. cit.*, p. 372.

lines appear, and that after many reversals they become accentuated and broaden in the manner that has been described. There is, therefore, a strong presumption that reversals of a stress of 8, or even 7, tons would ultimately develop cracks in the same manner as they are developed by stresses of 9 tons per sq. inch and over.

The destructive process, of which the above is a brief account, is illustrated in the accompanying micro-photographs (Plates 7-9).

The first series show part of a specimen which had been subjected to a stress the maximum value of which, close to the grips, was 14.3 tons per sq. inch. In this series the magnification is 150 diameters. Fig. 3 was taken after 5000 reversals had been given. A few of the crystals exhibit signs of slip, the slip-lines being still fairly fine and sharp. Two crystals in the photograph are seen to have yielded more than the others. Figs. 4 and 5 show the same set of crystals after 40,000 and 60,000 reversals of stress respectively. In fig. 4 a good many more crystals show signs of slip, and the slip-lines which appeared in fig. 3 are far more strongly defined. In some cases it will be seen that the lines have so broadened out as to run together and form dark patches on the surface of the crystals. In fig. 5 a still further breaking up of the surface by slip has occurred. At this stage, and probably earlier, some of the slip-bands have developed into small cracks. Such a crack is seen near the top right-hand corner of the figure, first of all close to a crystalline boundary, and then in steps across the next crystal. On the latter crystal two systems of slip-bands had formed at right angles to one another and at about 45° to the direction of stress. Only one system is clearly visible in the photograph, but it will be seen that the crack runs in steps along both. Fig. 5 shows not only the development of this crack, but also a general increase in width, length, and number of the slip-bands. This was practically the final stage, so far as this portion of the surface was concerned, for the specimen broke after another 10,000 reversals along a crack outside the field of these photographs.

When viewed in this state it is not practicable to tell how many of the slip-bands have actually developed into cracks, but this is readily seen when the specimen is re-polished and re-etched. This treatment obliterates any ordinary slip-bands which are steps marking differences of surface level, but any cracks remain visible. The specimen which gave this series of photographs was accordingly re-polished sufficiently to clear away the slip-bands, and was re-etched. This left the cracks alone visible, rather accentuated indeed, for the sides of the cracks are to some extent eaten away by the acid, and hence the width of the crack is increased. Fig. 6 shows the same part of the specimen as figs. 3, 4, and 5, after re-polishing and re-etching. The slip-bands have disappeared, except where they have formed cracks. A careful comparison of this with fig. 3 will show where cracks have formed. The most conspicuous crack is at AA, and its zig-zag character as it follows the two directions of slip-bands across one of the crystals is specially noticeable.

The specimen illustrated in the next photograph, fig. 7, had been subjected to

170,000 reversals of a stress of 12·3 tons per sq. inch, at which it broke. The spot shown (magnified 150 diameters) is a little way further from the grips than the crack through which the specimen actually broke, but another severe crack is seen running across the centre of the figure. Comparing this with fig. 5, it will be at once seen that there are far fewer lines due to slip upon each individual crystal than in the former specimen. Fig. 8 shows the same spot after re-polishing and etching. Comparing this with fig. 6 it is seen that a greater proportion of the slip-lines appear in this instance as cracks after re-polishing. The maximum stress is less here than in the former example, and more than twice as many reversals were required to bring about fracture. This agrees with what has been found for all other specimens broken, viz., the lower the stress the fewer the slip-lines upon each crystal, but the greater proportion of these actually develop into cracks under the more numerous reversals to which the less severely stressed specimen is exposed.

The photographs described above are on rather too small a scale for the actual changes in the slip-lines themselves to be clearly seen, and these are better illustrated by the next series, figs. 9–12. These show with a magnification of 1000 diameters a small part of the surface of another specimen of the same iron which was subjected to reversals of 12·4 tons per sq. inch. Fig. 9 is taken after 1000 reversals. The slip-bands which have formed are very faintly visible as fine lines upon the surface of the crystal. Fig. 10 is after 2000 reversals; the slip-bands seen in fig. 9 are now more distinct, and some new ones on the right of the crystal are fairly strongly defined. Fig. 11 is after 10,000 reversals; some of the slip-bands now show a decided tendency to broaden out and those upon the right have extended further across the crystal. Fig. 12, taken after 40,000 reversals of stress, shows further broadening out and spreading of the

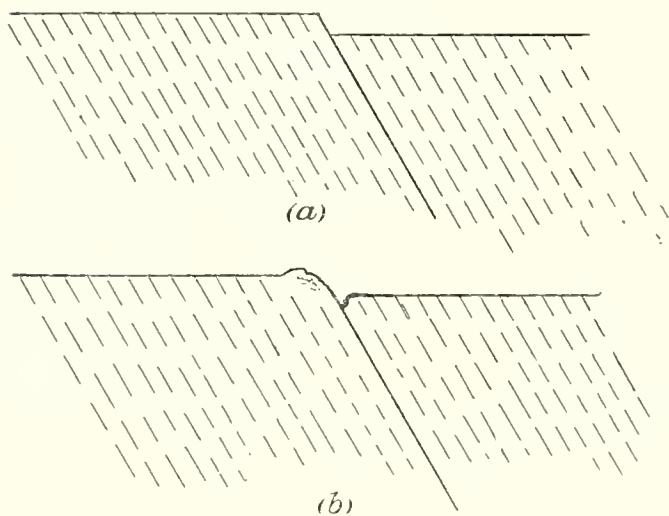


Fig. 13.

slip-bands. At this stage it could be seen by the focussing that this broadening was due to a heaving-up of the surface of the crystal in the neighbourhood of each slip-band, the markings being decidedly above the level of the other parts of the crystal. It is to be conjectured that the action is of the kind indicated in the sketch (fig. 13) where (a) represents an ordinary slip-band seen in section at right angles to its length, and (b) represents the effect of reversals of stress upon it. Very little further change took place in the particular crystal of figs.

9–12 as further reversals were applied, the specimen breaking elsewhere after 160,000 reversals. It has been noticed in this respect that when once an incipient crack begins to form across a certain set of crystals, the effect of further reversals

is mainly confined to the neighbourhood of the crack, other crystals (as was the case with that illustrated in figs. 9–12) changing but slightly.

Fig. 14 (Plate 8) shows a different part of the same specimen (after it had broken), also under a magnification of 1000 diameters. The broad band in the middle of the crystal is a crack which has developed along what was originally a line of slip. The heaving-up of the surface along the edges of the crack is well marked and may be compared with similar appearances at the edges of slip-bands in other parts of the photograph.

The stresses which are stated here are in all cases calculated from the observed load, as measured by the spring balance, acting at the end of projecting beam or “cantilever,” and they are the values which (on the ordinary theory of bending) are reached at a place close to the clamp. It was observed, however, that the destructive effects of reversals were not confined to the metal immediately adjacent to the clamps, but extended in most cases for a considerable way towards the loaded end of the specimen. The development of slip-bands, and their gradually widening and final conversion into cracks, occurred in some cases at least half-an-inch from the clamp, at a place where the fixing of the specimen could not disturb the distribution of stress in any way.

In another experiment the load was such as to produce a maximum stress, close to the clamp, of 9.2 tons per sq. inch, and 800,000 reversals were given. It was then seen that the greater number (though not all) of those crystals which closely adjoined the place where the specimen was clamped showed signs of repeated slip. Further away from the clamp the slip-lines became less numerous; but they were plainly seen on individual crystals as much as half-an-inch from the clamp. At the most distant places where slip-lines were plainly apparent the stress was only 7.3 tons per sq. inch. It was clear that a stress no greater than this was sufficient to develop slips, under many reversals, and that the lines so produced became accentuated as the process went on.

This was confirmed by another experiment in which the maximum stress, close to the clamp, was only 6.9 tons per sq. inch. After 3,000,000 reversals of this stress *one* slip-band was observed on a crystal a little way from the clamp. This is shown in fig. 15, where the slip-band is seen in the broadened condition which resulted from 3,000,000 reversals of stress. Prior to this, the same specimen had suffered 1,000,000 reversals of a stress of 5.3 tons per sq. inch, without showing the smallest sign of damage. It was only after increasing the stress to 6.9 tons per sq. inch that any action became apparent. It is an open question whether an isolated slip-band such as this would have led to fracture, if the process of reversal had been continued. At the conclusion of the experiment it was still confined to one crystal and it did not even extend all the way across that.

We have noticed that when lines indicating slip appear during reversals of a comparatively small stress, they are generally to be found in the central parts of individual crystals, not extending to the boundaries of the crystal.

It appears, then, that this material suffers no damage from repeated reversals of a stress of 5 tons per sq. inch ; but that when the stress is raised to 7 tons per sq. inch signs of fatigue are apparent after many reversals. And further, that with a stress of no more than 9 tons per sq. inch, the damage caused by reversals is so considerable that cracks are formed and the piece breaks. In all probability fracture through the formation of cracks would occur with 7 tons also, though all that is actually demonstrated for this stress is that it causes slip-bands to appear and to become accentuated in the manner which, with greater stresses, leads to the development of cracks.

It is remarkable that these actions are brought about by stresses much below what is ordinarily understood by the elastic limit of the material. A tensile test shows proportionality of strain to stress up to 12 or 13 tons per sq. inch, with no apparent defect of elasticity. The conditions under which these experiments were made seem to exclude the possibility that vibration gave rise to local augmentation of the stresses. The manner in which the slip-lines appear shows that some crystals reach a limit of elasticity sooner than others. This is no doubt to be ascribed in part to their being so oriented that the gliding planes, on which slip occurs, are inclined in the direction which is most favourable to the action of tangential stress. But besides this, it may be conjectured that in a complex structure built up of many crystals irregular in form the distribution of stress from crystal to crystal is by no means homogeneous.

Whatever the selective action of the stress is due to, the experiments demonstrate that in repeated reversals of stress certain crystals are attacked and yield by slipping, as in other cases of non-elastic strain. Then, as the reversals proceed, the surfaces on which slipping has occurred continue to be surfaces of weakness. The parts of the crystal lying on the two sides of each such surface continue to slide back and forth over one another. The effect of this repeated sliding or grinding is seen at the polished surface of the specimen by the production of a *burr*, or rough and jagged irregular edge, broadening the slip-band, and suggesting the accumulation of *débris*. Within the crystal this repeated grinding tends to destroy the cohesion of the metal across the surface of slip, and in certain cases this develops into a crack. Once the crack is formed it quickly grows in a well-known manner, by tearing at its edge, in consequence of the concentration of stress which results from lack of continuity. Engineers are familiar with the development of cracks, even in the most ductile materials, when these are initiated at air-bubbles or other flaws. The present experiments show how a crack may be formed, without any flaw to serve as nucleus, the first breach of continuity being set up through repeated grinding on a plane of slip in perfectly sound metal.

The experiments throw light on the known fact that fracture by repeated reversals or alternations of stress resembles fracture resulting from a "creeping" flaw in its abruptness and in the absence of local drawing-out or other deformation of shape.

They also help to explain why it is that a piece that has been subjected to many reversals shows no apparent loss of strength or plasticity when subjected to an ordinary tensile test. So long as the reversals have not yet reached the stage of producing cracks, it is not to be expected that such a test will detect the deterioration which has occurred. The material will still yield by slipping much as at first, and neither its plasticity nor its strength need show much change.

Interesting points suggest themselves which require further investigation. It is well known that when a plastic metal such as iron is strained sufficiently to take permanent "set," it suffers a temporary loss of elasticity, which is recoverable by lapse of time, the recovery of which, as MUIR has shown, may be enormously accelerated by warming the piece to such a temperature as $100^{\circ}\text{C}.$ * This may be ascribed to a gradual healing action which restores the resistance to sliding on the planes of slip after they have been weakened by the first severe strain. The first strain makes subsequent slipping easier, for a time, but when the material has a long enough rest recovery ensues. Probably enough a similar recovery would occur if during the application of reversals of stress long intervals of rest were allowed, and still more if during these intervals the temperature of the piece were raised. It may be conjectured that such treatment would arrest the destructive process of fatigue in its earlier stages, and give the material a new lease of life. The damage which alternating stresses produce probably depends not only on the amount of the stress and the number of alternations it suffers, but on the rapidity with which the alternations follow one another,† and on the continuity or otherwise of the alternating action.

DESCRIPTION OF PLATES.

PLATE 7.

Figs. 3-6. Fatigue of Swedish iron by reversals of a stress of 14·3 tons per sq. inch. Magnification 150 diameters.

Fig. 3 after 5,000 reversals.

Fig. 4 after 40,000 reversals.

Fig. 5 after 60,000 reversals.

Fig. 6 after 70,000 reversals, followed by re-polishing and re-etching.

* MUIR, "On the Recovery of Iron from Overstrain," 'Phil. Trans.,' A, vol. 193, p. 1.

† On this point, the experiments of OSBORNE REYNOLDS and J. H. SMITH ('Proc. Roy. Soc.,' vol. 70, 1902, p. 44) have already shown that rapid reversals are more destructive than less rapid reversals, in the sense that to make fractures result from an equal number of both, the maximum stress must be higher when the number of reversals per minute is less.

PLATES 8 AND 9.

Figs. 7-8. Fatigue of Swedish iron by reversals of a stress of 12·3 tons per sq. inch.
Magnification 150 diameters.

Fig. 7 after 170,000 reversals.

Fig. 8 same after re-polishing and re-etching.

Figs. 9-12. Fatigue of Swedish iron by reversals of a stress of 12·4 tons per sq. inch.
Magnification 1000 diameters.

Fig. 9 after 1,000 reversals.

Fig. 10 after 2,000 reversals.

Fig. 11 after 10,000 reversals.

Fig. 12 after 40,000 reversals.

Fig. 14. Another part of the same specimen after 160,000 reversals.

Fig. 15. Development of a slip-band by 3,000,000 reversals of a stress of 6·9 tons
per sq. inch.





Fig. 3. Specimen of Swedish iron after 5000 reversals of a stress of 14.3 tons per sq. inch. $\times 150$.

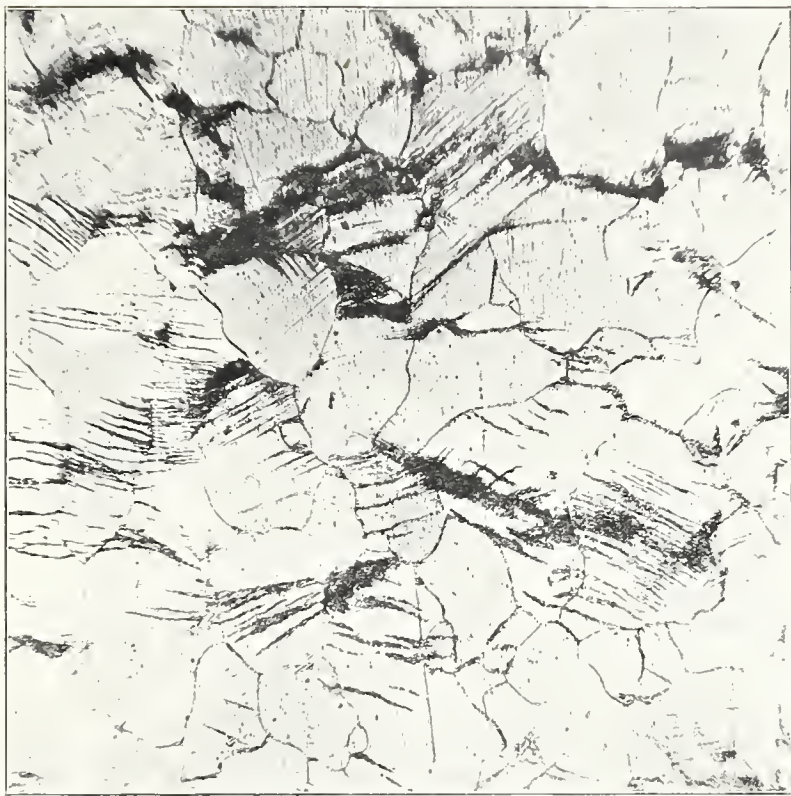


Fig. 4. Same after 40,000 reversals. $\times 150$.

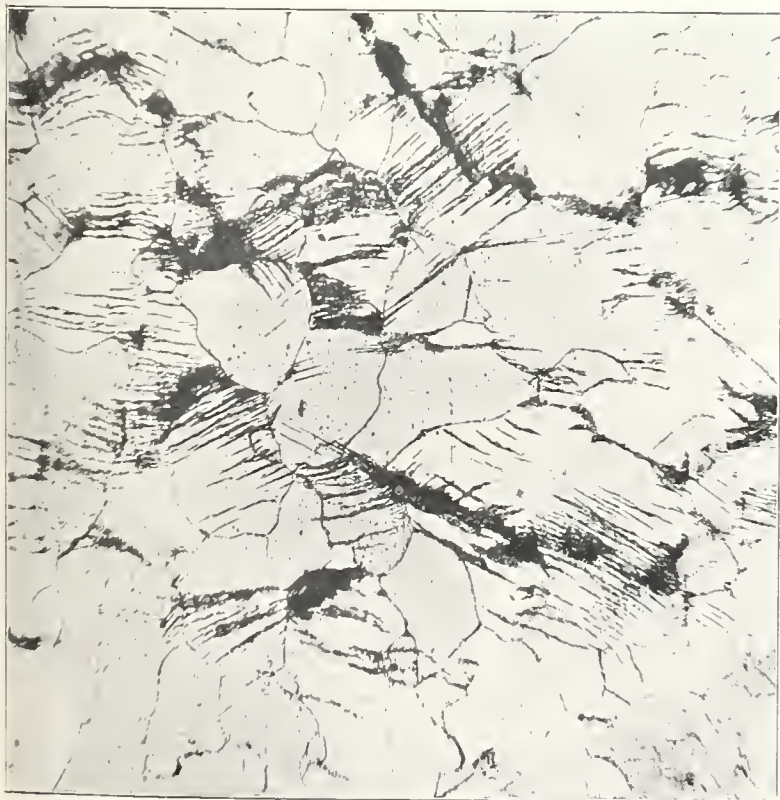


Fig. 5. Same after 60,000 reversals. $\times 150$.



Fig. 6. Same after 70,000 reversals and re-polishing and re-etching. $\times 150$.



Fig. 7. Specimen after 170,000 reversals of a stress of 12·3 tons per sq. inch. $\times 150$.



Fig. 8. Same after re-polishing and re-etching. $\times 150$.



Fig. 14. Another part of the specimen of figs. 9-12, after 160,000 reversals. $\times 1000$.



Fig. 15. Specimen after 3,000,000 reversals of a stress of 6·9 tons per sq. inch. $\times 1000$.

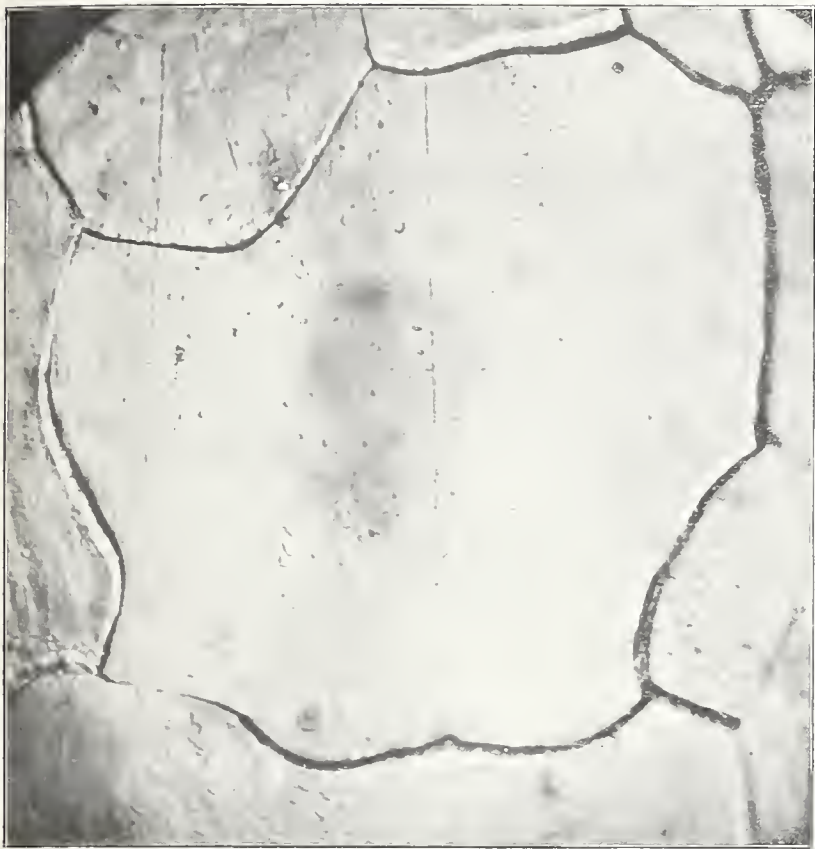


Fig. 9. Specimen after 1000 reversals of a stress of 12·4 tons per sq. inch. $\times 1000$.



Fig. 10. Same after 2000 reversals. $\times 1000$.

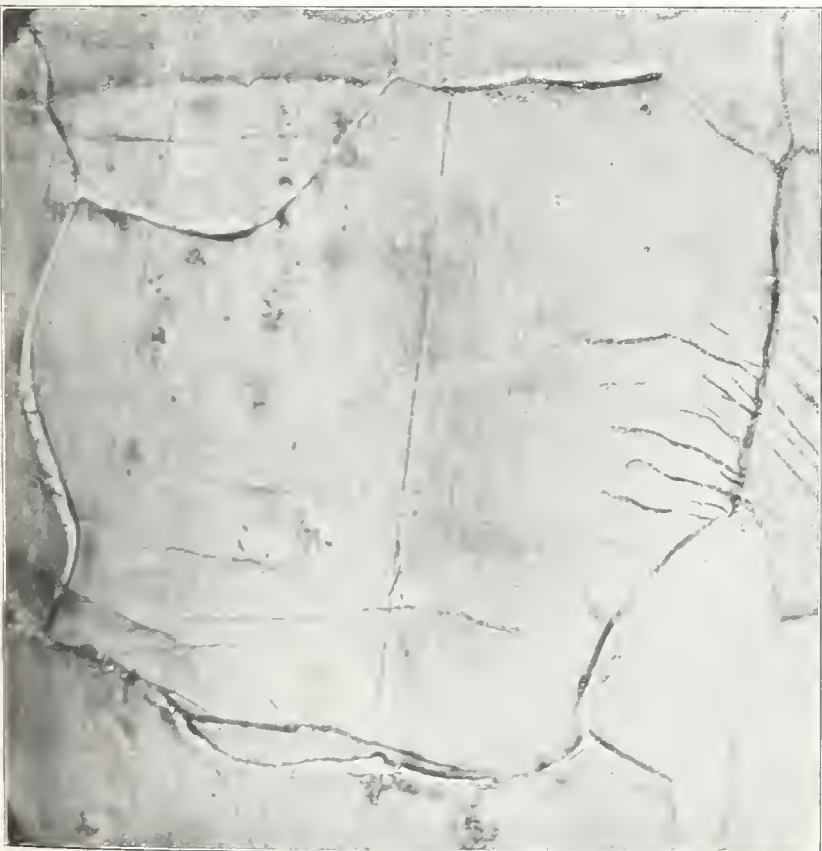


Fig. 11. Same after 10,000 reversals. $\times 1000$.

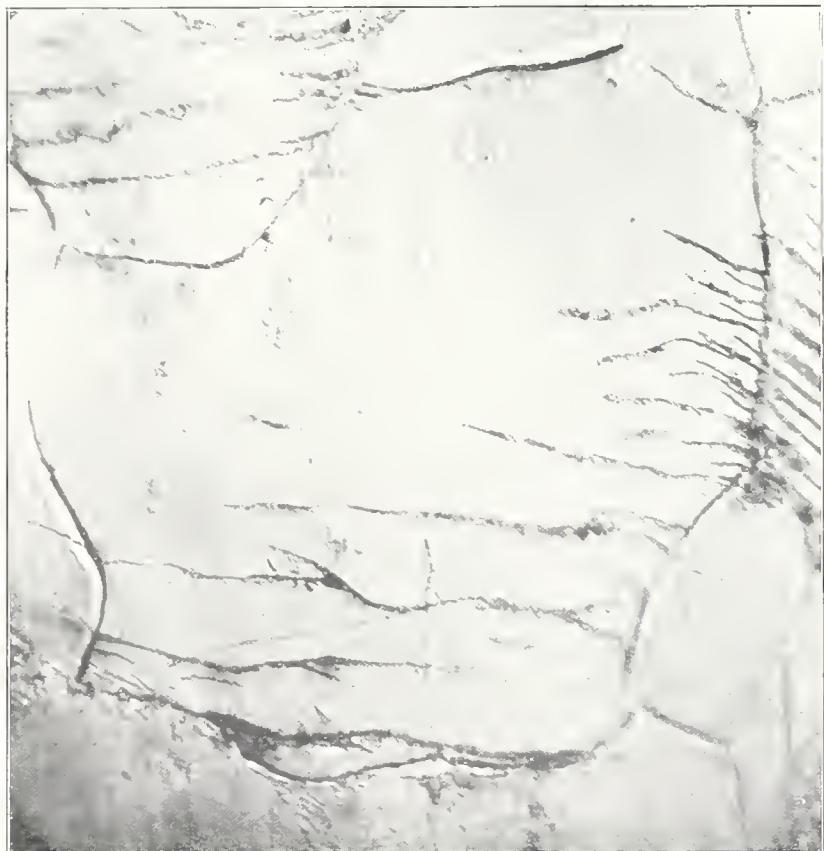


Fig. 12. Same after 40,000 reversals. $\times 1000$.

INDEX SLIP.

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of a Rotating Mass of Liquid.

Phil. Trans., A, vol. 200, 1903, pp. 251–314.

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VII. *The Stability of the Pear-Shaped Figure of Equilibrium of a Rotating Mass of Liquid.*

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INTRODUCTION.

By aid of the methods of a paper on "Ellipsoidal Harmonic Analysis" ('Phil. Trans., A, vol. 197, pp. 461-557), I here resume the subject of a previous paper ('Phil. Trans., A, vol. 198, pp. 301-331). These papers will be referred to hereafter by the abridged titles of "Harmonics" and "The Pear-shaped Figure."

At the end of the latter of these it was stated that the stability of the figure could not be proved definitely without approximation of a higher order of accuracy. After some correspondence with M. POINCARÉ during the course of my work, I made an attempt to carry out this further approximation, but found that the expression for a certain portion of the energy entirely foiled me. Meanwhile he had turned his attention to the subject, and he has shown ('Phil. Trans., A, vol. 198, pp. 333-373) by a method of the greatest ingenuity and skill how the problem may be solved. He has not, however, pursued the arduous task of converting his analytical results into numbers, so that he left the question as to the stability of the pear still unanswered.

M. POINCARÉ was so kind as to allow me to detain his manuscript on its way to the Royal Society for two or three days, and I devoted that time almost entirely to understanding the method of his attack on the key of the position—namely, the method of double layers, expounded in my own language in § 9 below. Being thus furnished with the means, I was able to resume my attempt under favourable conditions, and this paper is the result.

The substance of the analysis of this paper is, of course, essentially the same as his, but the arrangement and notation are so different that the two present but little superficial resemblance. This difference arises partly from the fact that I desired to use my own notation for the ellipsoidal harmonics, and partly because during the time that I was working at the analysis his paper was still unprinted and therefore inaccessible to me. But it is, perhaps, well that the two investigations of so complicated a subject should be nearly independent of one another.

It is rather unfortunate that I did not feel myself sufficiently expert in the use of the methods of WEIERSTRASS and SCHWARZ to evaluate the elliptic integrals after the methods suggested by M. POINCARÉ, but every exertion has been taken to insure correctness in the arithmetical results, on which the proof of stability depends. My choice of antiquated methods of computation leaves the way open for some one else to verify the conclusions by wholly independent and more elegant calculations. It is highly desirable that such a verification should be made.

As the body of this paper will hardly be studied by any one unless they should be actually working at the subject, I give a summary at the end. Even the mathematician who desires to study the subject in detail may find it advantageous to read the summary before looking at the analytical investigation.

PART I.

ANALYTICAL INVESTIGATION.

§ 1. *Method of Procedure.*

The pear-shaped figure is a deformation of the critical Jacobian ellipsoid, and to the first order of small quantities it is expressed by the third zonal harmonic with respect to the longest axis of the ellipsoid. In the higher approximation a number of other harmonic terms will arise, and the coefficients of these new terms will be of the second order of small quantities. The mass of an harmonic inequality vanishes only to the first order, and it can no longer be assumed that the centre of inertia of the pear coincides with the centre of the ellipsoid.

In order to define the pear, I describe an ellipsoid similar to and concentric with the original critical Jacobian; this new ellipsoid is taken to be sufficiently large to enclose the whole of the pear. It is clearly itself a critical Jacobian, and I adopt it as the ellipsoid of reference, and call it J . I call the region between J and the pear R . The pear may then be defined by density $+\rho$ throughout J , and density $-\rho$ throughout R .

If k is the parameter which defines J , its axes are expressed in the notation of "Harmonics" by $k\nu_0$, $k(\nu_0^2 - 1)^{\frac{1}{2}}$, $k\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$; or in the notation of the "Pear-shaped Figure" by $k/\sin\beta$, $k\cos\beta/\sin\beta$, $k\cos\gamma/\sin\beta$, where $\sin\beta = \kappa\sin\gamma$.

Now let S_i^s denote any surface harmonic, so that S_i^s is the same thing as $[\mathfrak{P}_i^s(\mu) \text{ or } \mathbf{P}_i^s(\mu)] \times [\mathfrak{C}_i^s(\phi) \text{ or } \mathbf{C}_i^s(\phi)]$. The third zonal harmonic deformation will then be eS_3 or $e\mathfrak{P}_3(\mu)\mathbf{C}_3(\phi)$, where e is of the first order of small quantities. On account of the symmetry of the figure, the new terms cannot involve the sine functions $\mathfrak{S}$ or $\mathbf{S}$, and moreover, the rank s must necessarily be even.

Suppose that the new terms are expressed by $\Sigma f_i^s S_i^s$ for all values of i from 1 to infinity, and with s equal to 0, 2, 4 . . . i or $i-1$. Then all the f 's are of order e^2 , excepting f_3 which is zero.

We have seen in "Harmonics," § 11, that if p_0 denotes the perpendicular from the centre of the ellipsoid ν_0 on to the tangent plane at μ, ϕ , the equation to a harmonic deformation of the ellipsoid is

$$\frac{k^2}{p_0^2}(\nu^2 - \nu_0^2) = 2\epsilon S_i^s.$$

Since this equation may be written in the form

$$\frac{x^2}{k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} + \frac{y^2}{k^2(\nu_0^2 - 1)} + \frac{z^2}{k^2\nu_0^2} = 1 + 2\epsilon S_i^s,$$

it is clear that if $2\epsilon S_i^s$ is a constant, say c , the surface defined is an ellipsoid similar to the surface of reference, with semi-axes augmented in the proportion of $(1 + c)^{\frac{1}{2}}$ to unity.

I now replace the variable ν by a new one, namely,

$$\tau = -\frac{k^2}{2\rho_0^2} (\nu^2 - \nu_0^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The negative sign is taken because the points to be specified will lie inside J .

Then $\tau = c$, a constant, defines an interior ellipsoid similar to and concentric with J . The equation to the pear may now be written

$$\tau = c - eS_3 - \sum_1^{\infty} f_i^s S_i^s.$$

The only condition which has been imposed on c is that it shall be great enough to make τ always positive.

In order to solve our problem it is necessary to determine the energy lost in the process of concentration from a condition of infinite dispersion into the final configuration. This involves the use of the formula for the gravity of J , inclusive of rotation. It is well known that this formula is simple for the inside of J and more complicated for the outside. Since the whole region R lies inside J there is no necessity in the present case to use the more complicated formula.

The final expression for the lost energy cannot involve the size of J , the exterior ellipsoid of reference, and therefore the arbitrary constant c must ultimately disappear. It is therefore legitimate to make c zero from the beginning.

It is clear that we might with equal justice have discussed the problem by means of an ellipsoid which should lie entirely inside the pear, the region between the pear and the ellipsoid would then have been filled with positive density, and the formula for external gravity would have been needed. The same argument as before would then have justified our putting the constant c equal to zero.

We thus arrive at the same conclusion as does M. POINCARÉ, namely, that it is immaterial whether the formula for external or internal gravity be used.

I now revert to my first hypothesis of the enveloping ellipsoid, but put c equal to zero from the first. In order, however, to afford clearness to our conceptions, I shall continue to discuss the problem as though c were not zero and as though J enclosed the whole pear. With this explanation, we may write the equation to the pear in the form

$$\tau = -eS_3 - \sum_1^{\infty} f_i^s S_i^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

§ 2. *The Lost Energy of the System.*

If the negative density in R is transported along tubes formed by a family of orthogonal curves and deposited as surface density on J , we may refer to such a

condensation as $-C$. I do not suppose the condensation actually effected, but imagine the surface of J to be coated with equal and opposite condensations $+C$ and $-C$.

The system of masses forming the pear may then be considered as being as follows:—

Density $+\rho$ throughout J , say $+J$.

Negative condensation on J , say $-C$.

Positive condensation $+C$ on J and negative volume density $-\rho$ throughout R . This last forms a double system of zero mass, say D , and $D = C - R$.

Let V_j , V_r be the potentials of $+J$ and $+R$, and V_{j-r} the potential of the pear.

An element of volume being written dv , let $\int_j dv$, $\int_r dv$, $\int_{j-r} dv$ denote integrations throughout J , R and the pear respectively.

Let d be the distance along the z axis from the centre of the ellipsoid as origin to the centre of inertia of the pear; let ω be the angular velocity of the critical Jacobian about the axis z , so that $\omega^2/2\pi\rho = 1.4200$; and let $\omega^2 + \delta\omega^2$ be the square of the angular velocity of the pear. Lastly, let M be the mass of the pear.

Then the lost energy E is given by

$$E = \frac{1}{2} \int_{j-r} V_{j-r} \rho dv + \frac{1}{2} (\omega^2 + \delta\omega^2) \int_{j-r} [y^2 + (z - d)^2] \rho dv.$$

Now $\int_{j-r} z \rho dv = Md$, so that $\int_{j-r} (-2zd + d^2) \rho dv = -Md^2$.

Again, since

$$V_{j-r} = V_j - V_r, \quad \int_{j-r} = \int_j - \int_r, \quad \int_j V_r \rho dv = \int_r V_j \rho dv,$$

we have

$$\frac{1}{2} \int_{j-r} V_{j-r} \rho dv = \frac{1}{2} \int_j V_j \rho dv - \int_r V_j \rho dv + \frac{1}{2} \int_r V_r \rho dv.$$

Also

$$\begin{aligned} \frac{1}{2} (\omega^2 + \delta\omega^2) \int_{j-r} [y^2 + (z - d)^2] \rho dv &= \frac{1}{2} \omega^2 \int_j (y^2 + z^2) \rho dv - \frac{1}{2} \omega^2 \int_r (y^2 + z^2) \rho dv \\ &\quad + \frac{1}{2} \delta\omega^2 \int_{j-r} (y^2 + z^2) \rho dv - \frac{1}{2} (\omega^2 + \delta\omega^2) Md^2. \end{aligned}$$

Hence

$$\begin{aligned} E &= \frac{1}{2} \int_j [V_j + \omega^2 (y^2 + z^2)] \rho dv - \int_r [V_j + \frac{1}{2} \omega^2 (y^2 + z^2)] \rho dv + \frac{1}{2} \int_r V_r \rho dv \\ &\quad + \frac{1}{2} \delta\omega^2 \int_{j-r} (y^2 + z^2) \rho dv - \frac{1}{2} (\omega^2 + \delta\omega^2) Md^2. \end{aligned}$$

As the several terms will be considered separately, it will be convenient to have an

abridged notation to specify them. I may denote the lost energy of J , inclusive of rotation, by $\frac{1}{2}JJ$; the mutual lost energy of J and of the region R , considered as filled with positive density, by JR ; the lost energy of the region R by $\frac{1}{2}RR$.

The moment of inertia of the pear is A , and it is equal to $A_j - A_r$, the moment of inertia of J less that of R .

Then

$$E = \frac{1}{2}JJ - JR + \frac{1}{2}RR + \frac{1}{2}(A_j - A_r)\delta\omega^2 - \frac{1}{2}(\omega^2 + \delta\omega^2)Md^2,$$

where

$$\frac{1}{2}JJ = \frac{1}{2} \int_j [V_j + \omega^2(y^2 + z^2)] \rho dv,$$

$$JR = \int_r [V_j + \frac{1}{2}\omega^2(y^2 + z^2)] \rho dv,$$

$$A_j = \int_j (y^2 + z^2) \rho dv, \quad A_r = \int_r (y^2 + z^2) \rho dv,$$

and

$$\frac{1}{2}RR = \int_r V_r \rho dv.$$

If $\frac{1}{2}DD$ denotes the lost energy of the double system described above, we clearly have

$$\frac{1}{2}RR = \frac{1}{2}(C - R)(C - R) + CR - \frac{1}{2}CC = \frac{1}{2}DD + CR - \frac{1}{2}CC.$$

We require to evaluate E to the fourth order; now d is at least of the second order and d^2 of the fourth order; hence $d^2 \cdot \delta\omega^2$ is at least of the fifth order and negligible.

Hence, finally, to the required degree of approximation

$$E = \frac{1}{2}JJ - JR + CR - \frac{1}{2}CC + \frac{1}{2}DD + \frac{1}{2}(A_j - A_r)\delta\omega^2 - \frac{1}{2}Md^2\omega^2. \quad (3).$$

It will appear below that d is not even of the second order, so that the last term will, in fact, entirely disappear, although we cannot see at the present stage that this will be so.

§ 3. *Expression for the Element of Volume.*

The parameter β of "Harmonics" is connected with κ of the "Pear-shaped Figure" by the equations

$$\frac{1 - \beta}{1 + \beta} = \kappa^2, \quad \beta = \frac{1 - \kappa^2}{1 + \kappa^2} = \frac{\kappa'^2}{1 + \kappa^2}, \quad \frac{2\beta}{1 - \beta} = \frac{\kappa'^2}{\kappa^2}, \quad \frac{2\beta}{1 + \beta} = \kappa'^2.$$

There will, I think, be no confusion if I also use β in a second sense, defining it by the equations

$$\sin \beta = \kappa \sin \gamma, \quad \cos^2 \beta = 1 - \kappa^2 \sin^2 \gamma.$$

It has already been remarked above that the squares of the semi-axes of J are

$$k^2 \nu_0^2 = \frac{k^2}{\sin^2 \beta}, \quad k^2 (\nu_0^2 - 1) = \frac{k^2 \cos^2 \beta}{\sin^2 \beta}, \quad k^2 \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right) = \frac{k^2 \cos^2 \gamma}{\sin^2 \beta}.$$

The mass of J is then $\frac{4}{3}\pi\rho k^3 \frac{\cos \beta \cos \gamma}{\sin^3 \beta}$.

I now take the mass M of the pear to be

$$M = \frac{4}{3}\pi\rho k_0^3 \frac{\cos \beta \cos \gamma}{\sin^3 \beta}.$$

Thus k_0 is a constant which specifies the volume of liquid in the pear, and the mass of J is $M (k/k_0)^3$.

It will be convenient to introduce certain new symbols, namely,

$$\begin{aligned} \Delta^2 &= 1 - \kappa^2 \sin^2 \theta, & \Gamma^2 &= 1 - \kappa'^2 \cos^2 \phi, \\ \Delta_1^2 &= 1 - \kappa^2 \sin^2 \gamma \sin^2 \theta, & \Gamma_1^2 &= \cos^2 \gamma + \kappa'^2 \sin^2 \gamma \cos^2 \phi, \end{aligned}$$

where $\sin \theta$ is the μ of "Harmonics."

The roots of the fundamental cubic were ν^2 , μ^2 , and $\frac{1 - \beta \cos 2\phi}{1 - \beta}$, and in the new notation they are ν^2 , $\sin^2 \theta$, $\frac{1 - \kappa'^2 \cos^2 \phi}{\kappa^2}$ or $\frac{\Gamma^2}{\kappa^2}$.

Since $\nu_0^2 = \frac{1}{\sin^2 \beta}$, we now have

$$\nu_0^2 - \mu^2 = \frac{\Delta_1^2}{\sin^2 \beta}, \quad \nu_0^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta} = \frac{\Gamma_1^2}{\sin^2 \beta}.$$

The expression for p_0 , the perpendicular from the centre on to the tangent plane at θ, ϕ , is given in (49) of "Harmonics," namely,

$$\frac{p_0^2}{k^2} = \frac{\nu_0^2 (\nu_0^2 - 1) \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)}{(\nu_0^2 - \mu^2) \left(\nu_0^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta} \right)} = \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \beta} \frac{1}{\Delta_1^2 \Gamma_1^2} \dots \dots \dots (4).$$

Also by (50) of "Harmonics" the element of surface $d\sigma$ of the ellipsoid is given by

$$\frac{p_0 d\sigma}{d\theta d\phi} = k^3 \nu_0 (\nu_0^2 - 1)^{\frac{1}{2}} \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} \frac{\frac{1 - \beta \cos 2\phi}{1 - \beta} - \mu^2}{\left(\frac{1 - \beta \cos 2\phi}{1 - \beta} \right)^{\frac{1}{2}} \left(\frac{1 + \beta}{1 - \beta} - \mu^2 \right)}.$$

Passing to the new notation this may be written

$$\frac{p_0 d\sigma}{d\theta d\phi} = \frac{3M}{4\pi\rho} \left(\frac{k}{k_0} \right)^3 \frac{1 - \kappa^2 \sin^2 \theta - \kappa'^2 \cos^2 \phi}{\Delta \Gamma} = \frac{3M}{4\pi\rho} \left(\frac{k}{k_0} \right)^3 \cdot \frac{\Delta_1^2 \Gamma_1^2}{\Delta \Gamma \sin^2 \gamma} \left(\frac{1}{\Gamma_1^2} - \frac{1}{\Delta_1^2} \right).$$

The new independent variable τ is to replace ν ; it was defined in (1) by

$$\tau = \frac{k^2}{2\rho_0^2} (\nu_0^2 - \nu^2),$$

and in accordance with (2) the equation to the surface of the pear is

$$\tau = -eS_3 - \sum_1^{\infty} f_i^s S_i^s.$$

From (4)

$$\nu^2 = \nu_0^2 - \frac{2\rho_0^2}{k^2} \tau = \frac{1}{\sin^2 \beta} \left(1 - \frac{2\tau \cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \right).$$

For brevity I now write

$$\tau_1 = \frac{2\tau \cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2}, \text{ so that}$$

$$\nu^2 = \frac{1}{\sin^2 \beta} (1 - \tau_1), \quad \nu^2 - 1 = \frac{\cos^2 \beta}{\sin^2 \beta} (1 - \tau_1 \sec^2 \beta), \quad \nu^2 - \frac{1 + \beta}{1 - \beta} = \frac{\cos^2 \gamma}{\sin^2 \beta} (1 - \tau_1 \sec^2 \gamma),$$

$$\nu^2 - \mu^2 = \frac{\Delta_1^2}{\sin^2 \beta} \left(1 - \frac{\tau_1}{\Delta_1^2} \right), \quad \nu^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta} = \frac{\Gamma_1^2}{\sin^2 \beta} \left(1 - \frac{\tau_1}{\Gamma_1^2} \right),$$

$$\frac{1 - \beta \cos 2\phi}{1 - \beta} - \mu^2 = \frac{1 - \kappa^2 \sin^2 \theta - \kappa'^2 \cos^2 \phi}{\kappa^2} = \frac{\Delta_1^2 \Gamma_1^2}{\sin^2 \beta} \left(\frac{1}{\Gamma_1^2} - \frac{1}{\Delta_1^2} \right).$$

Therefore

$$\frac{(\nu^2 - \mu^2) \left(\nu^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta} \right)}{\nu (\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}}} = \frac{\Delta_1^2 \Gamma_1^2}{\sin \beta \cos \beta \cos \gamma} \cdot \frac{\left(1 - \frac{\tau_1}{\Delta_1^2} \right) \left(1 - \frac{\tau_1}{\Gamma_1^2} \right)}{(1 - \tau_1)^{\frac{1}{2}} (1 - \tau_1 \sec^2 \beta)^{\frac{1}{2}} (1 - \tau_1 \sec^2 \gamma)^{\frac{1}{2}}}.$$

If we write

$$G = \frac{1}{2} (1 + \sec^2 \beta + \sec^2 \gamma)$$

$$H = \frac{3}{8} (1 + \sec^4 \beta + \sec^4 \gamma) + \frac{1}{4} (\sec^2 \beta + \sec^2 \gamma + \sec^2 \beta \sec^2 \gamma),$$

this expression, when expanded as far as τ_1^2 , becomes

$$\frac{\Delta_1^2 \Gamma_1^2}{\sin \beta \cos \beta \cos \gamma} \left[1 - \tau_1 \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) - \tau_1^2 \left(-\frac{1}{\Delta_1^2 \Gamma_1^2} + G \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) - H \right) \right].$$

The arcs of the three orthogonal curves were denoted dn , dm , df in "Harmonics," where dn was the outward normal. Since in the present case we are measuring τ inwards, the element of volume dv must be taken as $-dn \, dm \, df$.

The equations (50) of "Harmonics" give

$$\frac{dv}{-\cos\theta v dv d\theta d\phi} = k^3 \frac{(\nu^2 - \mu^2) \left(\nu^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta} \right) \left(\frac{1 - \beta \cos 2\phi}{1 - \beta} - \mu^2 \right)}{\nu (\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} \cos\theta \left(\frac{1 + \beta}{1 - \beta} - \mu^2 \right)^{\frac{1}{2}} \left(\frac{1 - \beta \cos 2\phi}{1 - \beta} \right)^{\frac{1}{2}}}.$$

But

$$-v dv = \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \beta} \cdot \frac{d\tau}{\Delta_1^2 \Gamma_1^2}, \text{ and therefore}$$

$$\begin{aligned} \frac{dv}{d\tau d\theta d\phi} = k^3 \frac{\cos \beta \cos \gamma}{\sin^3 \beta \sin^2 \gamma} \frac{\Delta_1^2 \Gamma_1^2}{\Delta \Gamma} \left(\frac{1}{\Gamma_1^2} - \frac{1}{\Delta_1^2} \right) & \left[1 - \tau_1 \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) \right. \\ & \left. - \tau_1^2 \left(-\frac{1}{\Delta_1^2 \Gamma_1^2} + G \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) - H \right) \right]. \end{aligned}$$

On comparing this with the expression for $p_0 d\sigma$, we see that

$$\frac{dv}{d\tau} = p_0 d\sigma \left[1 - \tau_1 \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) - \tau_1^2 \left(-\frac{1}{\Delta_1^2 \Gamma_1^2} + G \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) - H \right) \right] \quad (5).$$

Another form, which will be more generally useful, is found by substituting for τ_1 its value; it is

$$\begin{aligned} \frac{dv}{d\tau d\theta d\phi} = \frac{3M}{4\pi\rho} \left(\frac{k}{k_0} \right)^3 & \left\{ \frac{\Delta_1^2 - \Gamma_1^2}{\sin^2 \gamma} - 2\tau \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \gamma} \left[\frac{1}{\Gamma_1^4} - \frac{1}{\Delta_1^4} - G \left(\frac{1}{\Gamma_1^2} - \frac{1}{\Delta_1^2} \right) \right] \right. \\ & \left. - 4\tau^2 \frac{\cos^4 \beta \cos^4 \gamma}{\sin^2 \gamma} \left[-\left(\frac{1}{\Gamma_1^6 \Delta_1^4} - \frac{1}{\Gamma_1^4 \Delta_1^6} \right) + G \left(\frac{1}{\Gamma_1^6 \Delta_1^2} - \frac{1}{\Gamma_1^2 \Delta_1^6} \right) - H \left(\frac{1}{\Gamma_1^4 \Delta_1^2} - \frac{1}{\Gamma_1^2 \Delta_1^4} \right) \right] \right\} \frac{1}{\Delta \Gamma}. \end{aligned}$$

In order to express this more succinctly let

$$\left. \begin{aligned} \Phi &= \frac{6}{\pi} \frac{\Delta_1^2 - \Gamma_1^2}{\Delta \Gamma \sin^2 \gamma}, \\ \Psi &= \frac{6}{\pi} \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \gamma} \left[\frac{1}{\Gamma_1^4} - \frac{1}{\Delta_1^4} - G \left(\frac{1}{\Gamma_1^2} - \frac{1}{\Delta_1^2} \right) \right] \frac{1}{\Delta \Gamma}, \\ \Omega &= \frac{6}{\pi} \frac{\cos^4 \beta \cos^4 \gamma}{\sin^2 \gamma} \left[-\left(\frac{1}{\Gamma_1^6 \Delta_1^4} - \frac{1}{\Gamma_1^4 \Delta_1^6} \right) + G \left(\frac{1}{\Gamma_1^6 \Delta_1^2} - \frac{1}{\Gamma_1^2 \Delta_1^6} \right) \right. \\ & \quad \left. - H \left(\frac{1}{\Gamma_1^4 \Delta_1^2} - \frac{1}{\Gamma_1^2 \Delta_1^4} \right) \right] \frac{1}{\Delta \Gamma}. \end{aligned} \right\} \quad (6)$$

We note that

$$\left. \begin{aligned} \frac{p_0 d\sigma}{d\theta d\phi} &= \frac{1}{8} \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 \Phi, \\ \Psi &= \cos^2 \beta \cos^2 \gamma \left(\frac{1}{\Gamma_1^2} + \frac{1}{\Delta_1^2} - G \right) \frac{\Phi}{\Delta_1^2 \Gamma_1^2}. \end{aligned} \right\} \quad (6).$$

Then

$$\frac{dv}{d\tau d\theta d\phi} = \frac{1}{8} \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 [\Phi - 2\tau\Psi - 4\tau^2\Omega] \quad (6).$$

The surface $\tau = \text{constant}$ is an ellipsoid similar to J with squares of semi-axes reduced in the proportion $1 - 2\tau$ to unity. Therefore the volume enclosed between the two ellipsoids is

$$\int dv = \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 [1 - (1 - 2\tau)^{\frac{3}{2}}] = \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 [3\tau - \frac{3}{2}\tau^2 - \frac{1}{2}\tau^3].$$

But taking the limits of θ and ϕ as $\frac{1}{2}\pi$ to 0, so that we integrate through one octant and multiply the result by 8, we have another expression for the same thing, namely,

$$\int dv = \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 \iiint [\Phi\tau - \Psi\tau^2 - \frac{4}{3}\Omega\tau^3] d\theta d\phi.$$

Therefore equating coefficients of powers of τ in the two expressions,

$$\iiint \Phi d\theta d\phi = 3, \quad \iiint \Psi d\theta d\phi = \frac{3}{2}, \quad \iiint \Omega d\theta d\phi = \frac{3}{8} \quad . \quad . \quad . \quad (7).$$

The first of these will be of use hereafter, and all three afford formulæ of verification in the numerical work.

§ 4. *Determination of k ; Definition of Symbols for Integrals.*

The pear being defined by $\tau = -eS_3 - \sum_1^{\infty} f_i^s S_i^s$, with all the f 's of order e^2 , excepting f_3 which is zero, we have at the surface of the pear to the fourth order

$$\tau^2 = e^2 (S_3)^2 + 2\sum e^2 f_i^s S_3 S_i^s + (\sum f_i^s S_i^s)^2,$$

$$\tau^3 = -e^3 (S_3)^3 - 3\sum e^2 f_i^s (S_3)^2 S_i^s,$$

$$\tau^4 = e^4 (S_3)^4.$$

In all the integrations which follow, and especially in the present instance in the determination of the volume of the region R , it is important to note that Φ, Ψ, Ω are even functions of the angular co-ordinates, and that therefore the integral of any odd function of those co-ordinates multiplied by any of these functions will vanish. When the odd functions are omitted we may integrate throughout the octant defined by the limits $\frac{1}{2}\pi$ to 0 for θ and ϕ , and multiply the result by 8.

Then, only retaining terms as far as e^3 , we may in finding the volume R take

$$\tau = -\sum f_i^s S_i^s, \quad i \text{ only even,}$$

$$\tau^2 = e^2 (S_3)^2 + 2\sum e^2 f_i^s S_3 S_i^s, \quad i \text{ only odd,}$$

$$\tau^3 = 0.$$

To the cubes of small quantities we have, therefore,

$$\int_r dv = \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 \iint [-\Phi \Sigma f_i^s S_i^s - \Psi (e^2 (S_3)^2 + 2\Sigma e^2 f_i^s S_3 S_i^s)] d\theta d\phi.$$

The first term vanishes because S_i^s is a surface harmonic and $\Phi d\theta d\phi$ is proportional to $\rho_0 d\sigma$.

Thus we are left with

$$\int_r dv = -\frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 \iint \Psi [e^2 (S_3)^2 + 2\Sigma e^2 f_i^s S_3 S_i^s] d\theta d\phi.$$

I now introduce symbols for certain integrals, and in order to bring all the definitions together I also define several others which will only occur later.

Let

$$\left. \begin{aligned} \phi_i^s &= \iint \Phi (S_i^s)^2 d\theta d\phi \\ \omega_i^s &= \iint \Psi (S_3)^2 S_i^s d\theta d\phi \\ \rho_i^s &= \frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \iint \frac{\Phi}{\Delta_1^2 \Gamma_1^2} (S_3)^2 S_i^s d\theta d\phi \end{aligned} \right\} \dots \dots \dots (8).$$

All these integrals vanish unless i is even. For immediate use I also introduce

$$\psi_i^s = \iint \Psi S_3 S_i^s d\theta d\phi.$$

The ψ integrals vanish unless i is odd, but it will appear later that they are not actually required.

I further write

$$\left. \begin{aligned} \sigma_2 &= \iint \Psi (S_3)^2 d\theta d\phi, & \zeta_4 &= \iint \Omega (S_3)^4 d\theta d\phi, \\ \sigma_4 &= \frac{6}{\pi} \frac{\cos^3 \beta \cos^3 \gamma \sin \beta}{\sin^2 \gamma} \iint \left[\frac{5}{2} \left(\frac{1}{\Delta_1^2 \Gamma_1^6} - \frac{1}{\Delta_1^6 \Gamma_1^2} \right) - 3G \left(\frac{1}{\Delta_1^2 \Gamma_1^4} - \frac{1}{\Delta_1^4 \Gamma_1^2} \right) \right] \frac{(S_3)^4}{\Delta \Gamma} d\theta d\phi \end{aligned} \right\} (8).$$

With this notation we have at once to cubes of small quantities,

$$\int_r \rho dv = -M \left(\frac{k}{k_0} \right)^3 [e^2 \sigma_2 + 2\Sigma e f_i^s \psi_i^s] \dots \dots \dots (9).$$

But before using this I will obtain another integral to the fourth order. It is

$$\begin{aligned} \int_r \tau \rho dv &= M \left(\frac{k}{k_0} \right)^3 \iint \{ \frac{1}{2} \Phi [e^2 (S_3)^2 + 2\Sigma e^2 f_i^s S_3 S_i^s + (\Sigma f_i^s S_i^s)^2] \\ &\quad + \frac{2}{3} \Psi [e^3 (S_3)^3 + 3\Sigma e^3 f_i^s (S_3)^2 S_i^s] - \Omega e^4 (S_3)^4 \} d\theta d\phi. \end{aligned}$$

Omitting terms which vanish, amongst which are integrals of the type $\Phi S_i^s S_j^s$, we have

$$\int_r \tau \rho \, dv = \frac{1}{2} M \left(\frac{k}{k_0} \right)^3 \{ e^2 \phi_3 + \Sigma (f_i^s)^2 \phi_i^s + 4 \Sigma e^2 f_i^s \omega_i^s - 2 e^4 \zeta_4 \}. \quad (10).$$

Returning now to the determination of the mass of $+R$, and observing that the mass of the pear is equal to that of $J - R$, we have

$$M = M \left(\frac{k}{k_0} \right)^3 [1 + e^2 \sigma_2 + 2 \Sigma e f_i^s \psi_i^s].$$

Therefore

$$\left(\frac{k_0}{k} \right)^3 = 1 + e^2 \sigma_2 + 2 \Sigma e f_i^s \psi_i^s + e^4 \delta.$$

A term $e^4 \delta$ of the fourth order has been introduced, but it will appear that it is unnecessary to evaluate it.

There will be frequent occasion to express k^5 in terms of k_0^5 . Now

$$\left(\frac{k}{k_0} \right)^5 = 1 - \frac{5}{3} [e^2 \sigma_2 + 2 \Sigma e f_i^s \psi_i^s + e^4 \delta - \frac{8}{6} e^4 (\sigma_2)^2].$$

But this will only be needed explicitly as far as e^2 , and to that order

$$\left(\frac{k}{k_0} \right)^5 = 1 - \frac{5}{3} e^2 \sigma_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

It is, however, necessary to determine $\frac{3}{2} \left(\frac{k}{k_0} \right)^2 - \frac{3}{5} \left(\frac{k}{k_0} \right)^5$ to the fourth order.

Now

$$\frac{3}{2} \left(\frac{k}{k_0} \right)^2 = \frac{3}{2} \{ 1 - \frac{2}{3} [e^2 \sigma_2 + 2 \Sigma e f_i^s \psi_i^s + e^4 \delta - \frac{5}{6} e^4 (\sigma_2)^2] \},$$

$$\frac{3}{5} \left(\frac{k}{k_0} \right)^5 = \frac{3}{5} \{ 1 - \frac{5}{3} [e^2 \sigma_2 + 2 \Sigma e f_i^s \psi_i^s + e^4 \delta - \frac{8}{6} e^4 (\sigma_2)^2] \}.$$

Hence to the fourth order

$$\frac{3}{2} \left(\frac{k}{k_0} \right)^2 - \frac{3}{5} \left(\frac{k}{k_0} \right)^5 = \frac{9}{10} - \frac{1}{2} e^4 (\sigma_2)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

It will be observed that the ψ integrals and δ have both disappeared.

5. The Energies $\frac{1}{2}JJ$ and JR .

If a_1, b_1, c_1 are the semi-axes of a Jacobian ellipsoid of mass M_1 and angular velocity ω , its lost energy, inclusive of rotation, is

$$\frac{3}{10} M_1^2 \left[\Psi + \frac{b_1^2 + c_1^2}{3 M_1} \omega^2 \right],$$

where Ψ is the usual auxiliary function.

The equations to be satisfied by the ellipsoid afford expressions for $\omega^2 b_1^2$ and $\omega^2 c_1^2$ in terms of differentials of Ψ . If these expressions are added together, ω^2 may be eliminated, and the expression becomes

$$\frac{9}{20} M_1^2 \left[\Psi + a_1 \frac{d\Psi}{da_1} \right].$$

In reverting to the notation adopted here, I remark that $\mathfrak{P}_i^s$, $\mathfrak{Q}_i^s$ will be used to denote those functions when the variable is ν_0 , and the variable will only be inserted explicitly when it has any other value.

In the present case M_1 , the mass of the Jacobian ellipsoid, is $M(k/k_0)^3$, and it was shown in "the Pear-shaped Figure" that

$$\Psi = \frac{2}{k} \mathfrak{P}_0 \mathfrak{Q}_0, \quad a_1 \frac{d\Psi}{da_1} = -\frac{2}{k} \mathbf{P}_1^1 \mathbf{Q}_1^1.$$

Hence

$$\frac{1}{2} J J = \frac{9}{10} \frac{M^2}{k_0} \left(\frac{k}{k_0} \right)^5 [\mathfrak{P}_0 \mathfrak{Q}_0 - \mathbf{P}_1^1 \mathbf{Q}_1^1] \dots \dots \dots (12).$$

It was shown in the same paper that the internal potential of the Jacobian inclusive of rotation, is

$$\frac{3}{4} M_1 \left\{ \Psi + a_1 \frac{d\Psi}{da_1} \left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} \right) \right\}.$$

Therefore in the present case

$$V_j + \frac{1}{2} \omega^2 (y^2 + z^2) = \frac{3}{2} \frac{M}{k_0} \left(\frac{k}{k_0} \right)^3 \left\{ \mathfrak{P}_0 \mathfrak{Q}_0 - \frac{1}{k^2} \mathbf{P}_1^1 \mathbf{Q}_1^1 \sin^2 \beta (x^2 \sec^2 \gamma + y^2 \sec^2 \beta + z^2) \right\}.$$

But the equation to an inequality on the ellipsoid defined by τ is in our new notation

$$\sin^2 \beta (x^2 \sec^2 \gamma + y^2 \sec^2 \beta + z^2) = k^2 (1 - 2\tau);$$

therefore

$$V_j + \frac{1}{2} \omega^2 (y^2 + z^2) = \frac{3}{2} \frac{M}{k_0} \left(\frac{k}{k_0} \right)^3 \{ (\mathfrak{P}_0 \mathfrak{Q}_0 - \mathbf{P}_1^1 \mathbf{Q}_1^1) + 2\tau \mathbf{P}_1^1 \mathbf{Q}_1^1 \}.$$

Let us divide this potential into two parts, say U' , U'' , of which the first is constant and the second a constant multiplied by τ . Also let $(JR)'$, $(JR)''$ be the two corresponding portions of the energy JR .

In order to find $(JR)'$ we have simply to multiply U' by the mass of R considered as consisting of positive density. The volume of R is the excess of the volume of J above that of the pear; hence the mass of R is $M \left[\left(\frac{k}{k_0} \right)^3 - 1 \right]$.

Therefore

$$(JR)' = \frac{3}{2} \frac{M^2}{k_0} \left[\left(\frac{k}{k_0} \right)^5 - \left(\frac{k}{k_0} \right)^2 \right] (\mathfrak{P}_0 \mathfrak{Q}_0 - \mathbf{P}_1^1 \mathbf{Q}_1^1).$$

Subtracting this from $\frac{1}{2}JJ$ as given in (12),

$$\frac{1}{2}JJ - (JR)' = \frac{M^2}{k_0} (\mathfrak{P}_0 \mathfrak{Q}_0 - \mathbf{P}_1^1 \mathbf{Q}_1^1) \left[\frac{3}{2} \left(\frac{k}{k_0} \right)^2 - \frac{3}{5} \left(\frac{k}{k_0} \right)^5 \right].$$

But the latter factor was found in (11) as equal to $\frac{9}{16} - \frac{1}{2}e^4(\sigma_2)^2$. The term $\frac{9}{16}$ only contributes a constant to the whole energy and may therefore be dropped.

Accordingly

$$\frac{1}{2}JJ - (JR)' = \frac{3}{2} \frac{M^2}{k_0} \left\{ -(\mathfrak{P}_0 \mathfrak{Q}_0 - \mathbf{P}_1^1 \mathbf{Q}_1^1) \frac{1}{3} e^4 (\sigma_2)^2 \right\} \quad . \quad . \quad (13).$$

For the other portion $(JR)''$ we have

$$U'' = 3 \frac{M}{k_0} \left(\frac{k}{k_0} \right)^2 \tau \mathbf{P}_1^1 \mathbf{Q}_1^1.$$

Then by means of (10)

$$\begin{aligned} (JR)'' &= \int_r U'' \rho dv = 3 \frac{M}{k_0} \left(\frac{k}{k_0} \right)^2 \mathbf{P}_1^1 \mathbf{Q}_1^1 \int_r \tau \rho dv \\ &= \frac{3}{2} \frac{M^2}{k_0} \left(\frac{k}{k_0} \right)^5 \mathbf{P}_1^1 \mathbf{Q}_1^1 \{ e^2 \phi_3 + \Sigma (f_i^s)^2 \phi_i^s + 4 \Sigma e^2 f_i^s \omega_i^s - 2e^4 \zeta_4 \} \quad . \quad (14). \end{aligned}$$

In the terms of the fourth order we may put $(k/k_0)^5$ equal to unity. Therefore combining (13) and (14)

$$\begin{aligned} \frac{1}{2}JJ - JR &= -\frac{3}{2} \frac{M^2}{k_0} \left(\frac{k}{k_0} \right)^5 e^2 \mathbf{P}_1^1 \mathbf{Q}_1^1 \phi_3 + \frac{3}{2} \frac{M^2}{k_0} \left\{ \frac{1}{3} e^4 [-(\mathfrak{P}_0 \mathfrak{Q}_0 - \mathbf{P}_1^1 \mathbf{Q}_1^1) (\sigma_2)^2 + 6 \mathbf{P}_1^1 \mathbf{Q}_1^1 \zeta_4] \right. \\ &\quad \left. - 4 \mathbf{P}_1^1 \mathbf{Q}_1^1 \Sigma e^2 f_i^s \omega_i^s - \mathbf{P}_1^1 \mathbf{Q}_1^1 \Sigma (f_i^s)^2 \phi_i^s \right\} \quad (15). \end{aligned}$$

§ 6. *Surface Density of Concentration C; Energy CR.*

The region R being filled with positive volume density ρ , is concentrated along orthogonal tubes on to J , and there gives surface density δ .

To the first order, by (5),

$$\frac{dv}{d\tau} = p_0 d\sigma \left[1 - 2\tau \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) \right].$$

Integrating with respect to τ from the pear to J , we have as far as squares of small quantities

$$\delta = -p_0 \rho \left[e S_3 + \Sigma f_i^s S_i + e^2 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) (S_3)^2 \right].$$

It is now necessary to express δ/p_0 in surface harmonics. The first two terms are already in the required form; for the remainder let

$$\sum_0^\infty \eta_i^s S_i^s = \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) (S_3)^2.$$

Multiplying both sides by $S_i^s \Phi d\theta d\phi$ and integrating, we have

$$\begin{aligned} \eta_i^s \phi_i^s &= \cos^2 \beta \cos^2 \gamma \iint \frac{\Phi}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - G \right) (S_3)^2 S_i^s d\theta d\phi \\ &= \iint \Psi (S_3)^2 S_i^s d\theta d\phi = \omega_i^s. \end{aligned}$$

Therefore $\eta_i^s = \omega_i^s / \phi_i^s$, and vanishes unless i is even.

When $i = 0$, $\eta_0 = \frac{\omega_0}{\phi_0}$; and since by (7) $\phi_0 = 3$, and $\omega_0 = \iint \Psi (S_3)^2 d\theta d\phi = \sigma_2$, we have $\eta_0 = \frac{1}{3}\sigma_2$.

Hence we have

$$\delta = -p_0 \rho \left[e S_3 + \frac{1}{3} e^2 \sigma_2 + \sum_1^\infty \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) S_i^s \right].$$

This is expressed in surface harmonics, the middle term being of order zero.

By (51) of "Harmonics" the internal potential of δ is

$$V_c = -3 \frac{M}{k_0} \left(\frac{k}{k_0} \right)^2 \left\{ e \mathfrak{P}_3(\nu) \mathfrak{Q}_3(\nu_0) S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{P}_0 \mathfrak{Q}_0 + \sum \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{P}_i^s(\nu) \mathfrak{Q}_i^s(\nu_0) S_i^s \right\}.$$

We have $\mathfrak{P}_i^s(\nu) = \mathfrak{P}_i^s - \frac{\nu_0^2 - \nu^2}{2\nu_0} \frac{d\mathfrak{P}_i^s}{d\nu_0} = \mathfrak{P}_i^s - \tau \frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \cdot \frac{1}{\Delta_1^2 \Gamma_1^2} \frac{d\mathfrak{P}_i^s}{d\nu_0}$. But before proceeding to use this I will introduce a new abridgment, and let

$$\left. \begin{aligned} \mathfrak{A}_i^s &= \mathfrak{P}_i^s \mathfrak{Q}_i^s \\ \mathfrak{B}_i^s &= \mathfrak{Q}_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0} \end{aligned} \right\} \dots \dots \dots (16).$$

Then $\mathfrak{P}_i^s(\nu) \mathfrak{Q}_i^s(\nu_0) = \mathfrak{A}_i^s - \tau \frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \frac{\mathfrak{B}_i^s}{\Delta_1^2 \Gamma_1^2}$, and

$$\begin{aligned} V_c &= -3 \frac{M}{k_0} \left(\frac{k}{k_0} \right)^2 \left\{ e \mathfrak{A}_3 S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{A}_0 + \sum \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{A}_i^s S_i^s \right\} \\ &\quad + 3 \frac{M}{k_0} \left(\frac{k}{k_0} \right)^2 \tau \frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \left\{ e \mathfrak{B}_3 \frac{S_3}{\Delta_1^2 \Gamma_1^2} + \sum \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{B}_i^s \frac{S_i^s}{\Delta_1^2 \Gamma_1^2} \right\}. \end{aligned}$$

In order to find the energy CR we multiply V_c by the element of mass

$$\rho dv = \frac{1}{8} M \left(\frac{k}{k_0} \right)^3 [\Phi - 2\tau\Psi] d\tau d\theta d\phi,$$

and integrate throughout R .

Now

$$\begin{aligned} \frac{V_c \rho dv}{d\tau d\theta d\phi} = & -\frac{3}{8} \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \left\{ e\mathfrak{A}_2 \Phi S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{A}_0 \Phi + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{A}_i^s \Phi S_i^s \right\} \\ & + \frac{6}{8} \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \tau \left\{ e\mathfrak{A}_3 \Psi S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{A}_0 \Psi + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{A}_i^s \Psi S_i^s \right\} \\ & + \frac{3}{8} \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \tau \frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \left\{ e\mathfrak{B}_3 \frac{S_3}{\Delta_1^2 \Gamma_1^2} + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{B}_i^s \frac{S_i^s}{\Delta_1^2 \Gamma_1^2} \right\}. \end{aligned}$$

Let us integrate these three lines separately.

First integral

$$\begin{aligned} &= 3 \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \iint \left\{ e\mathfrak{A}_3 \Phi S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{A}_0 \Phi + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{A}_i^s \Phi S_i^s \right\} \{ eS_3 + \Sigma f_i^s S_i^s \} d\theta d\phi \\ &= 3 \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \left\{ e^2 \mathfrak{A}_3 \phi_3 + \Sigma \left[e^2 f_i^s \frac{\omega_i^s}{\phi_i^s} + (f_i^s)^2 \right] \mathfrak{A}_i^s \phi_i^s \right\}. \end{aligned}$$

Second integral

$$\begin{aligned} &= 3 \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \iint \left\{ e\mathfrak{A}_3 \Psi S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{A}_0 \Psi + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{A}_i^s \Psi S_i^s \right\} \\ &\quad \{ e^2 (S_3)^2 + 2\Sigma e f_i^s S_3 S_i^s \} d\theta d\phi \\ &= 3 \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \left\{ \frac{1}{3} e^4 \mathfrak{A}_0 (\sigma_2)^2 + \Sigma \left[e^4 \frac{(\omega_i^s)^2}{\phi_i^s} + e^2 f_i^s \omega_i^s \right] \mathfrak{A}_i^s + 2\Sigma e^2 f_i^s \mathfrak{A}_3 \omega_i^s \right\}. \end{aligned}$$

Third integral

$$\begin{aligned} &= \frac{3}{2} \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \iint \left\{ e\mathfrak{B}_3 \frac{S_3}{\Delta_1^2 \Gamma_1^2} + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{B}_i^s \frac{S_i^s}{\Delta_1^2 \Gamma_1^2} \right\} \\ &\quad \{ e^2 (S_3)^2 + 2\Sigma e f_i^s S_3 S_i^s \} d\theta d\phi \\ &= \frac{3}{2} \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 \left\{ \Sigma \left(e^4 \frac{\omega_i^s}{\phi_i^s} + e^2 f_i^s \right) \rho_i^s \mathfrak{B}_i^s + 2\Sigma e^2 f_i^s \mathfrak{B}_3 \rho_i^s \right\}. \end{aligned}$$

All the terms, excepting the first of the first integral, are of the fourth order, and in them we may put $(k/k_0)^5$ equal to unity.

Therefore

$$\begin{aligned} CR = & 3 \frac{M^2}{k_0} \left(\frac{k}{k_0}\right)^5 e^2 \mathfrak{A}_3 \phi_3 \\ & + \frac{3}{2} \frac{M^2}{k_0} \left\{ e^4 \left[\frac{2}{3} \mathfrak{A}_0 (\sigma_2)^2 + \Sigma (2\mathfrak{A}_i^s \omega_i^s + \mathfrak{B}_i^s \rho_i^s) \frac{\omega_i^s}{\phi_i^s} \right] \right. \\ & \left. + \Sigma e^2 f_i^s [4 (\mathfrak{A}_i^s + \mathfrak{A}_3) \omega_i^s + (\mathfrak{B}_i^s + 2\mathfrak{B}_3) \rho_i^s] + 2\Sigma (f_i^s)^2 \mathfrak{A}_i^s \phi_i^s \right\}. \quad (17). \end{aligned}$$

§ 7. *The Energy $\frac{1}{2}CC$; Result for $\frac{1}{2}JJ - JR + CR - \frac{1}{2}CC$.*

From the last section it appears that the potential of C at the surface, where $\tau = 0$, is

$$V_0 = -3 \frac{M}{k_0} \left(\frac{k}{k_0} \right)^2 \left\{ e \mathfrak{A}_3 S_3 + \frac{1}{3} e^2 \sigma_2 \mathfrak{A}_0 + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) \mathfrak{A}_i^s S_i^s \right\}.$$

For the mass of an element of the surface density we have

$$p_0 \delta d\sigma = -\frac{1}{8} M \left(\frac{k}{k_0} \right)^3 \Phi \left\{ e S_3 + \frac{1}{3} e^2 \sigma_2 + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right) S_i^s \right\} d\theta d\phi.$$

These are to be multiplied together and half the product is to be integrated. Then bearing in mind that $\iint \Phi d\theta d\phi = 3$, we have

$$\frac{1}{2}CC = \frac{3}{2} \frac{M}{k_0} \left(\frac{k}{k_0} \right)^5 \left\{ e^2 \mathfrak{A}_3 \phi_3 + \frac{1}{3} e^4 (\sigma_2)^2 \mathfrak{A}_0 + \Sigma \left(e^2 \frac{\omega_i^s}{\phi_i^s} + f_i^s \right)^2 \mathfrak{A}_i^s \phi_i^s \right\}.$$

In the terms of the fourth order we put $(k/k_0)^5$ equal to unity; thus

$$\frac{1}{2}CC = \frac{3}{2} \frac{M}{k_0} \left(\frac{k}{k_0} \right)^5 e^2 \mathfrak{A}_3 \phi_3 + \frac{3}{2} \frac{M}{k_0} \left\{ e^4 \left[\frac{1}{3} (\sigma_2)^2 \mathfrak{A}_0 + \Sigma \mathfrak{A}_i^s \frac{(\omega_i^s)^2}{\phi_i^s} \right] + 2 \Sigma e^2 f_i^s \omega_i^s \mathfrak{A}_i^s + \Sigma (f_i^s)^2 \mathfrak{A}_i^s \phi_i^s \right\}.$$

Combining this with (17)

$$\begin{aligned} CR - \frac{1}{2}CC = \frac{3}{2} \frac{M}{k_0} \left(\frac{k}{k_0} \right)^5 e^2 \mathfrak{A}_3 \phi_3 + \frac{3}{2} \frac{M}{k_0} \left\{ e^4 \left[\frac{1}{3} \mathfrak{A}_0 (\sigma_2)^2 + \Sigma (\mathfrak{A}_i^s \omega_i^s + \mathfrak{B}_i^s \rho_i^s) \frac{\omega_i^s}{\phi_i^s} \right] \right. \\ \left. + \Sigma e^2 f_i^s [2(\mathfrak{A}_i^s + 2\mathfrak{A}_3) \omega_i^s + (\mathfrak{B}_i^s + 2\mathfrak{B}_3) \rho_i^s] + \Sigma (f_i^s)^2 \mathfrak{A}_i^s \phi_i^s \right\} \quad (18). \end{aligned}$$

We are in a position to collect together all the results obtained up to this point. Now $\frac{1}{2}JJ - JR$, as given in (15), contains $\mathbf{P}_1^1 \mathbf{Q}_1^1$, $\mathfrak{P}_0 \mathbf{Q}_0$; the latter of these is what is now written $\mathfrak{A}_0$, and since the ellipsoid is critical $\mathbf{P}_1^1 \mathbf{Q}_1^1 = \mathfrak{P}_3 \mathbf{Q}_3 = \mathfrak{A}_3$.

Collecting terms we find that the terms of the second order disappear, and that

$$\begin{aligned} \frac{1}{2}JJ - JR + CR - \frac{1}{2}CC = \frac{3}{2} \frac{M^2}{k_0} \left\{ e^4 \left[\mathfrak{A}_3 \left(\frac{1}{3} (\sigma_2)^2 + 2\zeta_4 \right) + \Sigma (\mathfrak{A}_i^s \omega_i^s + \mathfrak{B}_i^s \rho_i^s) \frac{\omega_i^s}{\phi_i^s} \right] \right. \\ \left. + \Sigma e^2 f_i^s [2\mathfrak{A}_i^s \omega_i^s + (\mathfrak{B}_i^s + 2\mathfrak{B}_3) \rho_i^s] - \Sigma (f_i^s)^2 (\mathfrak{A}_3 - \mathfrak{A}_i^s) \phi_i^s \right\} \quad (19). \end{aligned}$$

The reader will recognise that the last term involves the coefficient of stability for the deformation S_i^s . It is important to note that if S_i^s is of odd order there is no term with coefficient $e^2 f_i^s$.

§ 8. *The Term* $-\frac{1}{2}Md^2\omega^2$.

In the Jacobian ellipsoid

$$\frac{b_1^2 + c_1^2}{3M_1} \omega^2 = \Psi + \frac{3}{2}a_1 \frac{d\Psi}{da_1}.$$

In the present notation this is

$$\frac{k^2\omega^2}{3M} \left(\frac{1 + \cos^2\beta}{\sin^2\beta} \right) = \frac{2}{k} (\mathfrak{P}_0\mathfrak{Q}_0 - \frac{3}{2}\mathfrak{P}_1^1\mathfrak{Q}_1^1) = \frac{2}{k} (\mathfrak{A}_0 - \frac{3}{2}\mathfrak{A}_3).$$

Hence

$$-\frac{1}{2}Md^2\omega^2 = -\frac{1}{2}(Md)^2 \frac{\omega^2}{M} = -\frac{3\sin^2\beta}{1 + \cos^2\beta} (\mathfrak{A}_0 - \frac{3}{2}\mathfrak{A}_3) \frac{(Md)^2}{k^3}.$$

I now make the following definition

$$S_1 = \sin\theta (1 - \kappa'^2 \cos^2\phi)^{\frac{1}{2}},$$

so that

$$z = kS_1.$$

Then

$$\begin{aligned} Md &= \int_{j-r} z\rho dv = \int_j z\rho dv - \int_r z\rho dv = -\int_r z\rho dv \\ &= -M \left(\frac{k}{k_0} \right)^3 k \iiint [\Phi - 2\tau\Psi] S_1 d\tau d\theta d\phi \\ &= M \left(\frac{k}{k_0} \right)^3 k \iint [\Phi (eS_3 + \Sigma f_i^s S_i^s) + \Psi e^2 (S_3)^2] S_1 d\theta d\phi \\ &= M \left(\frac{k}{k_0} \right)^3 k f_1 \phi_1. \end{aligned}$$

Therefore to the required order

$$-\frac{1}{2}Md^2\omega^2 = -\frac{3M^2}{k_0} \frac{\sin^2\beta}{1 + \cos^2\beta} (\mathfrak{A}_0 - \frac{3}{2}\mathfrak{A}_3) (f_1\phi_1)^2 \quad . \quad . \quad . \quad (20).$$

We again note that this term in the energy does not introduce any term with a coefficient $e^2 f_1$. Hence thus far the whole energy for harmonic deformations of odd order is of the form $Le^4 + M(f_i^s)^2$.

§ 9. *Double Layers.*

It remains to determine the value of $\frac{1}{2}DD$ in the energy, and for this purpose we must consider double layers, according to the ingenious method devised by M. POINCARÉ.

Let a closed surface S be intersected at every point by a member of a family of

curves, and let α be the angle between the curve and the outward normal at any point. At every point of S measure along the curve an infinitesimal arc τ , and let τ be a function of the two co-ordinates which determine position on S . The extremities of these arcs define a second surface S' , and every element of area $d\sigma$ of S has its corresponding element $d\sigma'$ on S' . Suppose that S is coated with surface density δ , and that S' is coated with surface density $-\delta'$, where $\delta d\sigma = \delta' d\sigma'$. The system SS' may then be called a double layer, and its total mass is zero. We are to discuss the potential of such a system.

Let $U(+)$ and $U(-)$ be the external and internal potentials of density δ on S , and U_0 their common value at a point P of S . At P take a system of rectangular axes, n being along the outward normal, and s and t mutually at right angles in the tangent plane.

In the neighbourhood of P

$$U(+) = U_0 + n \frac{dU}{dn}(+) + s \frac{dU}{ds}(+) + t \frac{dU}{dt}(+) \dots$$

$$U(-) = U_0 + n \frac{dU}{dn}(-) + s \frac{dU}{ds}(-) + t \frac{dU}{dt}(-) \dots$$

In the first of these n is necessarily positive, in the second negative.

Now $\frac{dU}{ds}(+) = \frac{dU}{ds}(-) = \frac{dU}{ds}$; and the like holds for the differentials with respect to t .

Also by POISSON'S equation

$$\frac{dU}{dn}(-) - \frac{dU}{dn}(+) = 4\pi\delta.$$

Let PP' be one of the family of curves whereby the double layer is defined, and let P' lie on S' , so that PP' is τ . By the definition of α the normal elevation of S' above S is $\tau \cos \alpha$.

Let v, v' be the potentials of the double layer at P and at P' .

The potential of S' at P' differs infinitely little in magnitude, but is of the opposite sign from that of S at P ; it is therefore $-U_0$. The point P' lies on the positive side of S at a point whose co-ordinates may be taken to be

$$n = \tau \cos \alpha, \quad s = \tau \sin \alpha, \quad t = 0.$$

Therefore the potential of S at P' is

$$U_0 + \tau \cos \alpha \frac{dU}{dn}(+) + \tau \sin \alpha \frac{dU}{ds}.$$

Therefore

$$v' = \tau \cos \alpha \frac{dU}{dn}(+) + \tau \sin \alpha \frac{dU}{ds}.$$

Again the potential of S at P is U_0 , and since P lies on the negative side of S' and has co-ordinates relatively to the n, s, t axes at P' given by

$$n = -\tau \cos \alpha, \quad s = -\tau \sin \alpha, \quad t = 0;$$

since further the density on S' is negative, we have

$$v = \tau \cos \alpha \frac{dU}{dn}(-) + \tau \sin \alpha \frac{dU}{ds}.$$

Therefore

$$v - v' = \tau \cos \alpha \left[\frac{dU}{dn}(-) - \frac{dU}{dn}(+) \right] = 4\pi\tau\delta \cos \alpha.$$

The differential with respect to n of the potential of S falls abruptly by $4\pi\delta$ as we cross S normally from the negative to the positive side; and the differential of the potential of S' rises abruptly by the same amount as we pass on across S' . It follows that dv/dn on the inside of S is continuous with its value on the outside of S' .

The surface S to which this theorem is to be applied is a slightly deformed ellipsoid, and the curves are the intersection of the two quadrics confocal with the ellipsoid which is deformed. The curves start normally to the ellipsoid, and where they meet S the angle α will be proportional to the deformation whereby S is derived from the ellipsoid. It follows that $\cos \alpha$ will only differ from unity by a term proportional to the square of the deformation, and as it is only necessary to retain terms of the order of the first power of the deformation, we may treat $\cos \alpha$ as unity.

We thus have the result

$$v - v' = 4\pi\tau\delta.$$

Suppose the curve PP' produced both ways, and that M_0, M_1 are two points on it either both on the same side or on opposite sides of the double layer.

Let M_0M_1 be equal to ζ , let ζ be measured in the same direction as n , and let ζ be a small quantity whose first power is to be retained in the results.

Let v_0, v_1 be the potential of the double layer at M_0 and M_1 respectively.

When ζ does not cut the layer we have

$$v_0 - v_1 = -\zeta \frac{dv}{dn},$$

and when it does cut the layer

$$v_0 - v_1 = 4\pi\tau\delta - \zeta \frac{dv}{dn}.$$

In the application which I shall make of this result the surface S' will actually be inside S . Then v_0 will denote the potential at any point not lying in the infinitely small space between S and S' , and v_1 is the potential at a point more towards the inside of the ellipsoid by a distance ζ ; δ is the surface density on the external surface

S and τ is measured inwards. If then we still choose to measure n outwards, as I shall do, our formula becomes

$$v_0 - v_1 - \zeta \frac{dv}{dn} = 4\pi\tau\delta \text{ or } 0,$$

according as ζ does or does not cut the double layer.

It may be well to remark that v being proportional to $\tau\delta$, $\zeta dv/dn$ is small compared with $4\pi\tau\delta$. It is also important to notice that the term $4\pi\tau\delta$ is independent of the form of the surface, and that dv/dn will be the same to the first order of small quantities for a slightly deformed ellipsoid as for the ellipsoid itself.

We have now to apply these results to our problem.

The position of a point in the region R may be defined by the distance measured inwards from J along one of the curves orthogonal to J . The surface of the pear as defined in this way is given by ϵ , a function of θ and ϕ . Any point on a curve may then be defined by $s\epsilon$, where s is a proper fraction. If s is the same at every point the surface s is a deformed ellipsoid; $s = 1$ gives the pear and $s = 0$ the ellipsoid J .

If $d\sigma$ is an element of area of J , the corresponding element on the surface s will be $(1 - \lambda\epsilon)s d\sigma$. The value of λ will be determined hereafter, and it is only necessary to remark that it is positive because the areas must decrease as we travel inwards.

Let s and $s + ds$ be two adjacent surfaces; then the mass of negative density enclosed between them in the tube of which $(1 - \lambda\epsilon)s d\sigma$ and $(1 - \lambda\epsilon)(s + ds) d\sigma$ are the ends is $-\rho\epsilon(1 - \lambda\epsilon)s d\sigma ds$. If this element of mass be regarded as surface density on s , that surface density is clearly $-\rho\epsilon ds$. If the same element of mass were carried along the orthogonal tube and deposited as surface density on J , that surface density would be $-\rho\epsilon(1 - \lambda\epsilon)s$. The sum for all values of s of all such transportals would constitute the condensation $-C$ already considered.

The double system D consists of the volume density $-\rho$ in R , and the positive condensation $+C$ on J , the total mass being zero.

Let z , a proper fraction, define a surface between J and the pear. Consider one of the orthogonal curves, and let V_0 be the potential of D at the point P where the curve leaves J and V_z the potential at the point Q where it cuts z . Then I require to find $V_0 - V_z$.

Since s denotes a surface intermediate between J and the pear, $\frac{d}{ds}(V_0 - V_z) ds$ is the excess of the potential at P above that at Q of surface density $-\rho\epsilon ds$ on s and surface density $+\rho\epsilon(1 - \lambda\epsilon)s$ on J . Such a system is a double layer, but there is a finite distance between the two surfaces, and the form of $\frac{d}{ds}(V_0 - V_z)$ will clearly be different according as z is greater or less than s .

The arc ϵs may be equally divided by a large number of surfaces, and we may take t to define any one of them. Now we may clothe each intermediate surface t with equal and opposite surface densities $\pm\rho\epsilon[1 - \lambda\epsilon(s - t)] dt$.

The density $+\rho\epsilon[1 - \lambda\epsilon(s - t)] dt$ on t , together with $-\rho\epsilon[1 - \lambda\epsilon(s - t - dt)] dt$ on $t + dt$, constitute an infinitesimal double layer; and since the positive density on each t surface may be coupled with the negative density on the next interior surface, the finite double layer may be built up from a number of infinitesimal double layers. Hence $\frac{d^2}{ds dt} (V_0 - V_z) dt dt$ is the excess of the potential at P above that at Q of an infinitesimal double layer of thickness ϵdt , and with surface density $\rho\epsilon[1 - \lambda\epsilon(s - t)] dt$ on its exterior surface.

We may now apply the result $v_0 - v_1 - \zeta \frac{dr}{dn} = 4\pi\delta\tau$ or 0, according as ζ does or does not cut the double layer, and it is clear that

$$\frac{d^2}{ds dt} \left[V_0 - V_z - \epsilon z \frac{dV}{dn} \right] = 4\pi\rho\epsilon^2 [1 - \lambda\epsilon(s - t)] \text{ or } 0,$$

according as z is greater or less than t .

In the next place, we must integrate this from $t = s$ to $t = 0$, and the result will have two forms.

First, suppose $z > s$; then for all the values of t , $z > t$, and the first alternative holds good. Therefore

$$\frac{d}{ds} \left(V_0 - V_z - \epsilon z \frac{dV}{dn} \right) = 4\pi\rho\epsilon^2 \left[s - \frac{1}{2}\lambda\epsilon s^2 \right].$$

Secondly, suppose $z < s$; then from $t = s$ to $t = z$, $z < t$ and the second alternative holds, while from $t = z$ to $t = 0$, $z > t$ and the first holds. Therefore

$$\frac{d}{ds} \left(V_0 - V_z - \epsilon z \frac{dV}{dn} \right) = 4\pi\rho\epsilon^2 \left[z - \lambda\epsilon \left(sz - \frac{1}{2}z^2 \right) \right].$$

We have now to integrate again from $s = 1$ to $s = 0$.

From $s = 1$ to $s = z$, $z < s$ and the second form is applicable; from $s = z$ to $s = 0$, $z > s$ and the first form applies.

Therefore

$$\begin{aligned} V_0 - V_z - \epsilon z \frac{dV}{dn} &= 4\pi\rho\epsilon^2 \int_z^1 \left[z - \lambda\epsilon \left(sz - \frac{1}{2}z^2 \right) \right] ds + 4\pi\rho\epsilon^2 \int_0^z \left[s - \frac{1}{2}\lambda\epsilon s^2 \right] ds \\ &= 4\pi\rho\epsilon^2 \left\{ z(1 - z) - \lambda\epsilon \left[\frac{1}{2}z(1 - z^2) - \frac{1}{2}z^2(1 - z) \right] + \frac{1}{2}z^2 - \frac{1}{6}\lambda\epsilon z^3 \right\} \\ &= 2\pi\rho\epsilon^2 \left\{ 2z - z^2 - \lambda\epsilon \left(z - z^2 + \frac{1}{3}z^3 \right) \right\}. \end{aligned}$$

Finally, we have to multiply $-\frac{1}{2}(V_0 - V_z)$ by an element of negative mass at the point defined by z and integrate throughout R . The physical meaning of this integral will be considered subsequently.

We have already seen that such an element of mass is given by

$$-\rho dv = -\rho\epsilon(1 - \lambda\epsilon z) d\sigma dz$$

and the limits of integration are $z = 1$ to $z = 0$.

Therefore

$$\begin{aligned} \frac{1}{2} \int (V_0 - V_z) \rho dv \\ = \pi\rho^2 \iint \epsilon^3 (1 - \lambda\epsilon z) \{2z - z^2 - \lambda\epsilon(z - z^2 + \frac{1}{3}z^3)\} dz d\sigma + \frac{1}{2}\rho \iint \epsilon^2 z (1 - \lambda\epsilon z) \frac{dV}{dn} dz d\sigma. \end{aligned}$$

In this expression we neglect terms of the order ϵ^5 and note that $\epsilon^3 z^2 \frac{dV}{dn}$ is of that order.

Thus

$$\begin{aligned} \frac{1}{2} \int (V_0 - V_z) \rho dv &= \pi\rho^2 \iint \epsilon^3 [2z - z^2 - \lambda\epsilon(z - z^2 + \frac{2}{3}z^3)] dz d\sigma + \frac{1}{2}\rho \iint \epsilon^2 z \frac{dV}{dn} dz d\sigma \quad (z = 1 \text{ to } 0), \\ &= \frac{2}{3}\pi\rho^2 \int \epsilon^3 (1 - \lambda\epsilon) d\sigma + \frac{1}{4}\rho \int \epsilon^2 \frac{dV}{dn} d\sigma, \end{aligned}$$

the integrals being taken all over the surface of the ellipsoid.

We must now consider the meaning of the integral $\frac{1}{2} \int (V_0 - V_z) \rho dv$.

Let P be a point on J and Q a point in R on the same orthogonal curve.

Let $-U$ be the potential at Q of the density $-\rho$ throughout R , and $-U_0$ its value at P .

Let δ be the surface density of the positive concentration on J , W its potential at Q , and W_0 its value at P .

The lost energy of the double system consisting of $-\rho$ throughout R , and δ on J is

$$\frac{1}{2} \int U\rho dv + \frac{1}{2} \int W_0 \delta d\sigma - \frac{1}{2} \int U_0 \delta d\sigma - \frac{1}{2} \int W\rho dv.$$

This is equal to

$$\frac{1}{2} \int (U - W) \rho dv - \frac{1}{2} \int (U_0 - W_0) \delta d\sigma.$$

Consider the triple integral $\iiint (U_0 - W_0) \rho dv$. Here $dv = \epsilon(1 - \lambda\epsilon s) d\sigma ds$; also $U_0 - W_0$ is not a function of s , and the limits of s are 1 to zero. Therefore

$$\iiint (U_0 - W_0) \rho dv = \iint (U_0 - W_0) \left[\int_0^1 \epsilon(1 - \lambda\epsilon s) \rho ds \right] d\sigma.$$

But $\int_0^1 \epsilon(1 - \lambda\epsilon s) \rho ds$ is equal to δ the surface density of concentration. Therefore

$$\iint [U_0 - W_0] \delta d\sigma = \iiint (U_0 - W_0) \rho dv.$$

We may now revert to the GAUSSIAN notation with single integral sign, and we see that the lost energy of the system is

$$\frac{1}{2} \int [(W_0 - U_0) - (W - U)] \rho \, dv.$$

But $W - U$ is the potential of the double system at Q , and is therefore V_z ; and $W_0 - U_0$ is the potential of the double system at P , and is therefore V_0 .

Accordingly the lost energy

$$\begin{aligned} \frac{1}{2} DD &= \frac{1}{2} \int (V_0 - V_z) \rho \, dv \\ &= \frac{2}{3} \pi \rho^2 \int (\epsilon^3 - \lambda \epsilon^4) \, d\sigma + \frac{1}{4} \rho \int \epsilon^2 \frac{dV}{dn} \, d\sigma. \quad \dots \quad (21). \end{aligned}$$

§ 10. Determination of ϵ and λ .

ϵ is the arc of the orthogonal curve from J to the pear.

The arc of outward normal is connected with p and our variable τ by the equation

$$-dn = -\frac{k^2}{p} \nu d\nu = \frac{p_0^2}{p} d\tau.$$

It follows that

$$\epsilon = p_0 \int \frac{p_0}{p} d\tau, \text{ integrated from } J \text{ to the pear.}$$

By (50) of "Harmonics," with the notation of § 3 of this paper

$$\frac{k}{p} = \frac{(\nu^2 - \mu^2)^{\frac{1}{2}} \left(\nu^2 - \frac{1 - \beta \cos 2\phi}{1 - \beta} \right)^{\frac{1}{2}}}{\nu (\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}}} = \frac{\sin \beta}{\cos \beta \cos \gamma} \frac{\Delta_1 \Gamma_1 \left(1 - \frac{\tau_1}{\Delta_1^2} \right)^{\frac{1}{2}} \left(1 - \frac{\tau_1}{\Gamma_1^2} \right)^{\frac{1}{2}}}{(1 - \tau_1)^{\frac{1}{2}} (1 - \tau_1 \sec^2 \beta)^{\frac{1}{2}} (1 - \tau_1 \sec^2 \gamma)^{\frac{1}{2}}}.$$

Therefore

$$\begin{aligned} \frac{p_0}{p} &= \frac{\left(1 - \frac{\tau_1}{\Delta_1^2} \right)^{\frac{1}{2}} \left(1 - \frac{\tau_1}{\Gamma_1^2} \right)^{\frac{1}{2}}}{(1 - \tau_1)^{\frac{1}{2}} (1 - \tau_1 \sec^2 \beta)^{\frac{1}{2}} (1 - \tau_1 \sec^2 \gamma)^{\frac{1}{2}}} = 1 - \frac{1}{2} \tau_1 \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right), \\ &= 1 - \tau \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right). \end{aligned}$$

Integrating this from J to the pear

$$\epsilon = -p_0 \left[eS_3 + \sum_1^{\infty} f_i^3 S_i^3 + \frac{1}{2} e^2 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right) (S_3)^2 \right]. \quad (22).$$

We have, moreover, by the formula before integration

$$-dn = p_0 \left[1 - \tau \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right) \right] d\tau.$$

Also to the order, zero $-n = p_0 \tau$.

Since $-n$ is what was denoted in § 9 by ϵs , the element of volume is $-(1 + \lambda n) dn d\sigma$, and this is equal to

$$p_0 [1 - \lambda p_0 \tau] \left[1 - \tau \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right) \right] d\sigma d\tau,$$

or

$$p_0 \left[1 - \tau \left\{ \lambda p_0 + \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right) \right\} \right] d\sigma d\tau.$$

But by (5) the element of volume is

$$p_0 \left[1 - \frac{\tau \cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{2}{\Delta_1^2} + \frac{2}{\Gamma_1^2} - 2G \right) \right].$$

Equating coefficients of τ in the two expressions we find

$$\lambda = \frac{\cos^2 \beta \cos^2 \gamma}{p_0 \Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (23).$$

$$\S 11. \text{ The Energy } \frac{2}{3} \pi \rho^2 \int \epsilon^3 (1 - \lambda \epsilon) d\sigma.$$

From (22) and (23) we have

$$\epsilon^3 = -p_0^3 \left[e^3 (S_3)^3 + 3\Sigma e^2 f_i^s (S_3)^2 S_i^s + \frac{2}{3} e^4 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right) (S_3)^4 \right],$$

$$\lambda \epsilon^4 = p_0^3 e^4 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) (S_3)^4.$$

So that

$$\epsilon^3 (1 - \lambda \epsilon) = -p_0^3 \left[e^3 (S_3)^3 + 3\Sigma e^2 f_i^s (S_3)^2 S_i^s + e^4 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left\{ \frac{5}{2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) - 3G \right\} (S_3)^4 \right].$$

Again from (6)

$$\frac{2}{3} \pi \rho^2 p_0^3 d\sigma = \frac{2}{3} \pi \rho^2 \cdot \frac{1}{8} \frac{M}{\rho} \left(\frac{k}{k_0} \right)^3 \Phi \cdot \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \beta} \frac{d\theta d\phi}{\Delta_1^2 \Gamma_1^2} = \frac{1}{16} \frac{M^2}{k_0} \left(\frac{k}{k_0} \right)^5 \cos \beta \cos \gamma \sin \beta \frac{\Phi}{\Delta_1^2 \Gamma_1^2} d\theta d\phi.$$

Therefore

$$\begin{aligned} \frac{2}{3} \pi \rho^2 \epsilon^3 (1 - \lambda \epsilon) d\sigma = & -\frac{1}{16} \frac{M^2}{k_0} \left(\frac{k}{k_0} \right)^5 \cos \beta \cos \gamma \sin \beta \left[e^3 \frac{\Phi (S_3)^3}{\Delta_1^2 \Gamma_1^2} + 3\Sigma e^2 f_i^s \frac{\Phi (S_3)^2 S_i^s}{\Delta_1^2 \Gamma_1^2} \right. \\ & \left. + e^4 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^4 \Gamma_1^4} \Phi \left\{ \frac{5}{2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} \right) - 3G \right\} \right] d\theta d\phi. \end{aligned}$$

When this is integrated we may put $(k/k_0)^5$ equal to unity. In the integral the first term vanishes, and the second term gives $-\frac{3}{2} \frac{M^2}{k_0} \frac{\sin^2 \beta}{\cos \beta \cos \gamma} \Sigma e^2 f_i^s \rho_i^s$. In the third term we substitute for Φ its value and have

$$-\frac{3}{\pi} \frac{M^2}{k_0} \frac{\cos^3 \beta \cos^3 \gamma \sin \beta}{\sin^2 \gamma} e^4 \iint \frac{1}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Gamma_1^2} - \frac{1}{\Delta_1^2} \right) \left\{ \frac{5}{2} \left(\frac{1}{\Gamma_1^2} + \frac{1}{\Delta_1^2} \right) - 3G \right\} \frac{d\theta d\phi}{\Delta \Gamma},$$

which is equal to

$$-\frac{1}{2} \frac{M^2}{k_0} e^4 \frac{6}{\pi} \frac{\cos^3 \beta \cos^3 \gamma \sin \beta}{\sin^2 \gamma} \iint \left\{ \frac{5}{2} \left(\frac{1}{\Gamma_1^6 \Delta_1^2} - \frac{1}{\Gamma_1^2 \Delta_1^6} \right) - 3G \left(\frac{1}{\Gamma_1^4 \Delta_1^2} - \frac{1}{\Gamma_1^2 \Delta_1^4} \right) \right\} \frac{d\theta d\phi}{\Delta \Gamma}.$$

By the definition (8) this is equal to $-\frac{1}{2} \frac{M^2}{k_0} e^4 \sigma_4$.

Hence the required term in the energy is

$$\frac{3}{2} \frac{M^2}{k_0} \left[-\frac{\sin^2 \beta}{\cos \beta \cos \gamma} \Sigma e^2 f_i^s \rho_i^s - \frac{1}{3} e^4 \sigma_4 \right] \quad (24).$$

§ 12. The Energy $\frac{1}{4}\rho \int \epsilon^2 \frac{dV}{dn} d\sigma$.

It is first necessary to determine dV/dn .

Suppose that the ellipsoid J is coated with surface density δ , and that a second surface is drawn inside J at an infinitesimal distance τ , and coated with negative surface density $-\delta'$, so that the two form a double layer. Then $\tau\delta$ being a function of the two angular co-ordinates on the ellipsoid may be expanded in surface harmonics; suppose then that

$$\tau\delta = \sum_0^\infty h_i^s S_i^s.$$

Consider the two functions

$$V_e = \Sigma 4\pi h_i^s (\nu_0^2 - 1)^{\frac{1}{2}} \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} \frac{d\mathfrak{P}_i^s(\nu_0)}{d\nu_0} \mathfrak{Q}_i^s(\nu) S_i^s, \text{ for external space,}$$

$$V_i = \mathfrak{P}_i^s(\nu) \frac{d\mathfrak{Q}_i^s(\nu_0)}{d\nu_0} S_i^s, \text{ for internal space.}$$

Since these functions are solid harmonics, the matter of which V_e and V_i are the potentials is entirely confined to the surface of the ellipsoid, and since they are not continuous with one another, the ellipsoid must be a double layer.

Now
$$\mathfrak{Q}_i^s(\nu) = \mathfrak{P}_i^s(\nu) \int_\nu^\infty \frac{dv}{[\mathfrak{P}_i^s(v)]^2 (v^2 - 1)^{\frac{1}{2}} \left(v^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}}},$$

and therefore

$$\mathfrak{Q}_i^s(\nu_0) \frac{d\mathfrak{P}_i^s(\nu_0)}{d\nu_0} - \mathfrak{P}_i^s(\nu_0) \frac{d\mathfrak{Q}_i^s(\nu_0)}{d\nu_0} = \frac{1}{(\nu_0 - 1)^{\frac{1}{2}} \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}}}.$$

Hence at the surface of the ellipsoid

$$V_e - V_i = \Sigma 4\pi h_i^s S_i^s = 4\pi\tau\delta.$$

But this is the law found in § 9 for the change of potential in crossing a double layer, and hence V_e , V_i are the external and internal potentials of the double layer $\tau\delta$.

Since
$$\frac{d}{dn} = \frac{p}{k^2} \frac{d}{\nu d\nu},$$

$$\frac{dV_e}{dn} = \frac{dV_i}{dn} = \frac{dV}{dn} = \sum_0^\infty \frac{4\pi p_0}{k^2 \nu_0} (\nu_0^2 - 1)^{\frac{1}{2}} \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} h_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0} \frac{d\mathfrak{Q}_i^s}{d\nu_0} S_i^s. \quad (25).$$

This result will hold good to the first order of small quantities if the surface be a slightly deformed ellipsoid, such as was the surface defined by t in § 9.

In the elementary double layer t the density was $\rho\epsilon[1 - \lambda\epsilon(s - t)]dt$, and the thickness was ϵdt , so that the thickness multiplied by the density was $\rho\epsilon^2[1 - \lambda\epsilon(s - t)]dt dt$. Since, however, we only need this to the first order, we may take it as $\rho\epsilon^2 dt dt$. It will now be convenient to change the meaning of h_i^s to some extent, and to write

$$\epsilon^2 = \sum_0^\infty h_i^s S_i^s.$$

Thus for the elementary double layer we have

$$\tau\delta = \rho dt dt \sum_0^\infty h_i^s S_i^s.$$

It follows that in applying the formula (25) to determine $\frac{dV}{dn}$ for the double system D , we may say that

$$\frac{d^2}{ds dt} \frac{dV}{dn} = \sum_0^\infty \frac{4\pi p_0 \rho}{k^2 \nu_0} (\nu_0^2 - 1)^{\frac{1}{2}} \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} h_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0} \frac{d\mathfrak{Q}_i^s}{d\nu_0} S_i^s.$$

Since the right-hand side does not contain t , we have only to consider the integral

$$\int_0^1 \int_0^s ds dt = \int_0^1 s ds = \frac{1}{2}.$$

Thus, for the system D ,

$$\frac{dV}{dn} = \sum_0^\infty \frac{2\pi p_0 \rho}{k^2 \nu_0} (\nu_0^2 - 1)^{\frac{1}{2}} \left(\nu_0^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} h_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0} \frac{d\mathfrak{Q}_i^s}{d\nu_0} S_i^s. \quad (26).$$

This result may also be obtained as follows:—To the first order we may concentrate the negative density in the region R on a surface bisecting that region. We may then consider the positive concentration C on J , and the negative concentration on the bisecting surface as an infinitesimal double layer of thickness $\frac{1}{2}\epsilon$. We have seen that the surface density $+C$ is $-ppeS_3$, and that $\epsilon = -peS_3$ (in both cases to the first order only). Thus the density δ of $+C$ is $\rho\epsilon$, and the thickness τ of our layer is $\frac{1}{2}\epsilon$; the product therefore $\tau\delta$ is $\frac{1}{2}\rho\epsilon^2$.

Hence $\tau\delta = \frac{1}{2}\rho\epsilon^2 = \frac{1}{2}\rho \sum_0^\infty h_i^s S_i^s$, and thus we arrive at the same result as before.

I now introduce an abridged notation analogous to that used previously, and write

$$\mathfrak{D}_i^s = \frac{d\mathfrak{P}_i^s}{dv_0} \frac{d\mathfrak{Q}_i^s}{dv_0}.$$

We then have by (26) on the last page

$$\frac{dV}{dn} = \sum_0^\infty \frac{2\pi p_0 \rho}{k^2} \frac{\cos \beta \cos \gamma}{\sin \beta} h_i^s \mathfrak{D}_i^s S_i^s \quad \dots \quad (26),$$

where

$$\epsilon^2 = \sum_0^\infty h_i^s S_i^s.$$

By (22) to the first order

$$\epsilon^2 = c^2 p_0^2 (S_3)^2 = c^2 k^2 \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \beta} \frac{(S_3)^2}{\Delta_1^2 \Gamma_1^2}.$$

Assume then

$$\frac{\cos^2 \beta \cos^2 \gamma}{\sin \beta} \frac{(S_3)^2}{\Delta_1^2 \Gamma_1^2} = \sum_0^\infty \zeta_i^s S_i^s.$$

Multiplying by ΦS_i^s and integrating, we have

$$\rho_i^s = \zeta_i^s \phi_i^s.$$

Hence

$$\epsilon^2 = \frac{c^2 k^2}{\sin \beta} \sum_0^\infty \frac{\rho_i^s}{\phi_i^s} S_i^s, \text{ and therefore } h_i^s = \frac{c^2 k^2}{\sin \beta} \frac{\rho_i^s}{\phi_i^s}.$$

Substituting in (26)

$$\begin{aligned} \frac{dV}{dn} &= \sum_0^\infty 2\pi p_0 \rho c^2 \frac{\cos \beta \cos \gamma}{\sin^2 \beta} \frac{\rho_i^s}{\phi_i^s} \mathfrak{D}_i^s S_i^s, \\ &= \sum_0^\infty \frac{3}{2} \frac{M}{k^3} c^2 p_0 \sin \beta \frac{\rho_i^s}{\phi_i^s} \mathfrak{D}_i^s S_i^s. \end{aligned}$$

Now

$$\epsilon^2 = \frac{c^2 k^2}{\sin \beta} \sum_0^\infty \frac{\rho_i^s}{\phi_i^s} S_i^s, \text{ and}$$

$$\epsilon^2 \frac{dV}{dn} = \frac{3}{2} \frac{M}{k_0} c^4 p_0 \left(\sum_0^\infty \frac{\rho_i^s}{\phi_i^s} \mathfrak{D}_i^s S_i^s \right) \left(\sum_0^\infty \frac{\rho_i^s}{\phi_i^s} S_i^s \right).$$

Since on integration the terms involving products of unlike harmonics will disappear, we have, as far as material,

$$\epsilon^2 \frac{dV}{dn} = \frac{3}{2} \frac{M}{k_0} e^4 p_0 \sum_0^\infty \left(\frac{\rho_i^s}{\phi_i^s} \right)^2 \mathfrak{D}_i^s (S_i^s)^2.$$

Now
$$\frac{1}{4} \rho p_0 d\sigma = \frac{1}{32} M \left(\frac{k}{k_0} \right)^3 \Phi d\theta d\phi.$$

Since the term which is being determined is of the fourth order in e , we may put $k/k_0 = 1$, and we have

$$\begin{aligned} \frac{1}{4} \rho \int \epsilon^2 \frac{dV}{dn} d\sigma &= \frac{3}{8} \frac{M^2}{k_0} e^4 \sum_0^\infty \iint \left(\frac{\rho_i^s}{\phi_i^s} \right)^2 \mathfrak{D}_i^s \Phi (S_i^s)^2 d\theta d\phi \\ &= \frac{3}{8} \frac{M^2}{k_0} e^4 \sum_0^\infty \frac{(\rho_i^s)^2}{\phi_i^s} \mathfrak{D}_i^s \dots \dots \dots (27). \end{aligned}$$

Since $\mathfrak{P}_0(\nu) = 1$, $\frac{d}{d\nu} \mathfrak{P}_0(\nu) = 0$ and $\mathfrak{D}_0 = 0$, the term in Σ corresponding to $i = 0$ vanishes.

§ 13. *Terms in the Energy Depending on the Moment of Inertia.*

We have to determine A_r , the moment of inertia of the region R considered as filled with positive density.

In order to obtain this result, we must express $y^2 + z^2$ in terms of surface harmonics. This was done in § 12 of "Harmonics," but as a different definition of S_2 and S_2^2 was adopted there from that which I shall use here, it is easier to proceed *ab initio*.

Let
$$D^2 = 1 - \kappa^2 \kappa'^2, \text{ and}$$

$$(q_0)^2 = \frac{1}{3} (1 + \kappa^2 - D), \quad (q_2)^2 = \frac{1}{3} (1 + \kappa^2 + D).$$

For both the suffixes 0 and 2, we have $q^2 + q'^2 = 1$, and

$$\kappa^2 = q^2 \frac{2 - 3q^2}{1 - 2q^2}, \quad \kappa'^2 = q'^2 \frac{1 - 3q'^2}{1 - 2q'^2}, \quad \kappa^2 - q^2 = q'^2 - \kappa'^2 = \frac{q^2 q'^2}{1 - 2q^2}.$$

In accordance with equation (10) of "The Pear-shaped Figure" I define the harmonics as follows:—

$$S_2 = (\kappa^2 \sin^2 \theta - q_0^2) (q_0'^2 - \kappa'^2 \cos^2 \phi),$$

$$S_2^2 = (\kappa^2 \sin^2 \theta - q_2^2) (q_2'^2 - \kappa'^2 \cos^2 \phi).$$

Now
$$y^2 = k^2 (\nu^2 - 1) \cos^2 \theta \sin^2 \phi, \quad z^2 = k^2 \nu^2 \sin^2 \theta (1 - \kappa'^2 \cos^2 \phi),$$

and,
$$\nu^2 = \frac{1 - \tau_1}{\sin^2 \beta}, \text{ where } \tau_1 = \frac{2\tau \cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2}.$$

Thus

$$\begin{aligned} \frac{\sin^2 \beta}{h^2} (y^2 + z^2) \\ = \cos^2 \beta - \tau_1 + \sin^2 \beta \sin^2 \theta - (\cos^2 \beta - \tau_1) \cos^2 \phi + (\cos^2 \beta - \kappa'^2 - \kappa^2 \tau_1) \sin^2 \theta \cos^2 \phi. \end{aligned}$$

Let us assume, as we know to be justifiable,

$$\begin{aligned} \frac{\sin^2 \beta}{h^2} (y^2 + z^2) &= AS_2 + BS_2^2 + C \\ &= -[Aq_0'^2 + Bq_2'^2 - C] + [Aq_0'^2 + Bq_2'^2] \kappa^2 \sin^2 \theta \\ &\quad + [Aq_0'^2 + Bq_2'^2] \kappa'^2 \cos^2 \phi - [A + B] \kappa^2 \kappa'^2 \sin^2 \theta \cos^2 \phi. \end{aligned}$$

If we equate the coefficients of $\sin^2 \theta$ and $\cos^2 \phi$ in these two expressions, we have

$$Aq_0'^2 + Bq_2'^2 = \frac{\sin^2 \beta}{\kappa^2}, \quad Aq_0'^2 + Bq_2'^2 = \frac{\tau_1 - \cos^2 \beta}{\kappa'^2}.$$

The solution of these equations may be written

$$A = \frac{1}{2Dq_0'^2} \left(1 - \frac{\cos^2 \beta}{D + \kappa^2} \right) - \frac{\tau_1}{2Dq_0'^2}, \quad B = -\frac{1}{2Dq_2'^2} \left(1 + \frac{\cos^2 \beta}{D - \kappa^2} \right) + \frac{\tau_1}{2Dq_2'^2}.$$

The simplest way of finding C is to put $\sin^2 \theta = \frac{q_0'^2}{\kappa^2}$, $\cos^2 \phi = \frac{q_2'^2}{\kappa'^2}$, so that $S_2 = S_2^2 = 0$; we thus find

$$C = \frac{1}{3} (1 + \cos^2 \beta) - \frac{2}{3} \tau_1.$$

Now for brevity write

$$L = \frac{\sin \beta}{4Dq_0'^2 \cos \beta \cos \gamma} \left(1 - \frac{\cos^2 \beta}{D + \kappa^2} \right), \quad M = \frac{\sin \beta}{4Dq_2'^2 \cos \beta \cos \gamma} \left(1 + \frac{\cos^2 \beta}{D - \kappa^2} \right).$$

We then have

$$A = 2 \frac{\cos \beta \cos \gamma}{\sin \beta} L - \frac{\tau_1}{2Dq_0'^2}, \quad B = -2 \frac{\cos \beta \cos \gamma}{\sin \beta} M + \frac{\tau_1}{2Dq_2'^2}.$$

Hence, substituting for τ_1 its value,

$$\begin{aligned} \frac{y^2 + z^2}{h^2} &= 2 \frac{\cos \beta \cos \gamma}{\sin^3 \beta} (LS_2 - MS_2^2) + \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \\ &\quad - \tau \frac{\cos^2 \beta \cos^2 \gamma}{D \sin^2 \beta} \left(\frac{1}{q_0'^2} \frac{S_2}{\Delta_1^2 \Gamma_1^2} - \frac{1}{q_2'^2} \frac{S_2^2}{\Delta_1^2 \Gamma_1^2} + \frac{4}{3} \frac{D}{\Delta_1^2 \Gamma_1^2} \right). \end{aligned}$$

Now

$$\frac{\rho dv}{d\tau d\theta d\phi} = \frac{1}{8} M \left(\frac{l}{k_0} \right)^3 (\Phi - 2\tau\Psi).$$

Therefore

$$\begin{aligned} \frac{(y^2 + z^2) \rho \, dv}{d\tau \, d\theta \, d\phi} = & \frac{1}{8} M k_0^2 \left(\frac{k}{k_0} \right)^5 \left\{ 2 \frac{\cos \beta \cos \gamma}{\sin^3 \beta} (L\Phi S_2 - M\Phi S_2^2) + \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \Phi \right. \\ & - 4\tau \frac{\cos \beta \cos \gamma}{\sin^3 \beta} (L\Psi S_2 - M\Psi S_2^2) - 2\tau \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \Psi \\ & \left. - \tau \frac{\cos^2 \beta \cos^2 \gamma}{D \sin^2 \beta} \left(\frac{1}{q_0'^2} \frac{\Phi S_2}{\Delta_1^2 \Gamma_1^2} - \frac{1}{q_2'^2} \frac{\Phi S_2^2}{\Delta_1^2 \Gamma_1^2} + \frac{4}{3} D \frac{\Phi}{\Delta_1^2 \Gamma_1^2} \right) \right\}. \end{aligned}$$

When we integrate throughout the region R the limits of τ are $-eS_3 - \sum f_i^s S_i^s$ to zero.

Accordingly

$$\begin{aligned} A_r = & - M k_0^2 \left(\frac{k}{k_0} \right)^5 \iiint \left\{ \left[2 \frac{\cos \beta \cos \gamma}{\sin^3 \beta} (L\Phi S_2 - M\Phi S_2^2) + \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \Phi \right] \left[eS_3 + \sum_2^{\infty} f_i^s S_i^s \right] \right. \\ & + \left[2 \frac{\cos \beta \cos \gamma}{\sin^3 \beta} (L\Psi S_2 - M\Psi S_2^2) + \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \Psi \right] e^2 (S_3)^2 \\ & \left. + \frac{\cos^2 \beta \cos^2 \gamma}{2D \sin^2 \beta} \left[\frac{1}{q_0'^2} \frac{\Phi S_2}{\Delta_1^2 \Gamma_1^2} - \frac{1}{q_2'^2} \frac{\Phi S_2^2}{\Delta_1^2 \Gamma_1^2} + \frac{4}{3} D \frac{\Phi}{\Delta_1^2 \Gamma_1^2} \right] e^2 (S_3)^2 \right\} d\theta \, d\phi \\ = & - M k_0^2 \left(\frac{k}{k_0} \right)^5 \left\{ \frac{2 \cos \beta \cos \gamma}{\sin^3 \beta} (L f_2 \phi_2 - M f_2^2 \phi_2^2) + 2e^2 \frac{\cos \beta \cos \gamma}{\sin^3 \beta} (L \omega_2 - M \omega_2^2) + e^2 \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \sigma_2 \right. \\ & \left. + \frac{e^2}{2D \sin \beta} \left(\frac{\rho_2}{q_0'^2} - \frac{\rho_2^2}{q_2'^2} + \frac{4}{3} D \rho_0 \right) \right\}. \end{aligned}$$

The moment of inertia of the ellipsoid J is

$$A_j = \frac{1}{5} M \left(\frac{k}{k_0} \right)^3 k^2 \frac{1 + \cos^2 \beta}{\sin^2 \beta} = M k_0^2 \left[\frac{1 + \cos^2 \beta}{5 \sin^2 \beta} - e^2 \frac{1 + \cos^2 \beta}{3 \sin^2 \beta} \sigma_2 \right].$$

Also

$$M k_0^2 = \frac{M^2}{k_0} \cdot \frac{3 \sin^3 \beta}{4\pi \rho \cos \beta \cos \gamma} = \frac{3M^2}{2k_0} \cdot \frac{1}{2\pi \rho} \cdot \frac{\sin^3 \beta}{\cos \beta \cos \gamma}.$$

Lastly, to the required order we may put $(k/k_0)^5$ equal to unity in the expression for A_r .

Then

$$\begin{aligned} \frac{1}{2} (A_j - A_r) \delta \omega^2 = & \frac{3M^2}{2k_0} \frac{\delta \omega^2}{2\pi \rho} \left\{ \frac{(1 + \cos^2 \beta) \sin \beta}{10 \cos \beta \cos \gamma} + L(f_2 \phi_2 + e^2 \omega_2) - M(f_2^2 \phi_2^2 + e^2 \omega_2^2) \right. \\ & \left. + \frac{e^2}{4D \cos \beta \cos \gamma} \left(\frac{\rho_2}{q_0'^2} - \frac{\rho_2^2}{q_2'^2} + \frac{4}{3} D \rho_0 \right) \right\}. \quad (28). \end{aligned}$$

This completes the expression for the lost energy E of the system, which may now be collected from (19), (20), (24), (27), and (28).

§ 14. *The Lost Energy of the System; Solution of the Problem.*

If the several contributions to the energy be examined, it will be seen that if i , the order of harmonics in $f_i^s S_i^s$, is odd, there is no term with coefficient $e^2 f_i^s$ in E ; this follows from the fact that the ω and ρ integrals vanish for the odd harmonics. Hence, as far as concerns the odd harmonics, E involves f_i^s only in the form $(f_i^s)^2$. The condition that the pear shall be a level surface is that E shall be stationary for variations of the f 's and of e . It follows that when i is odd f_i^s is zero. We may therefore drop all the odd harmonics, inclusive of f_1 , and it is clear that the term $-\frac{1}{2}M d^2 \omega^2$ in E (given in (20)) vanishes to our order of approximation.

For the sake of brevity, I adopt a single symbol for the coefficients of the several kinds of terms in E . Therefore let

$$\begin{aligned} A_0 &= \mathfrak{A}_3 \left[\frac{1}{3} (\sigma_2)^2 + 2\zeta_4 \right] + \sum_2^\infty (\mathfrak{A}_i^s \omega_i^s + \mathfrak{B}_i^s \rho_i^s) \frac{\omega_i^s}{\phi_i^s} - \frac{1}{3} \sigma_4 + \frac{1}{4} \sum_2^\infty \frac{(\rho_i^s)^2}{\phi_i^s} \mathfrak{D}_i^s, \\ 2B_i^s &= 2\mathfrak{A}_i^s \omega_i^s + (\mathfrak{B}_i^s + 2\mathfrak{B}_3) \rho_i^s - \frac{\sin^2 \beta}{\cos \beta \cos \gamma} \rho_i^s, \\ C_i^s &= (\mathfrak{A}_3 - \mathfrak{A}_i^s) \phi_i^s, \\ \mathfrak{a} &= \frac{(1 + \cos^2 \beta) \sin \beta}{10 \cos \beta \cos \gamma}, \\ \mathfrak{b} &= L\omega_2 - M\omega_2^2 + \frac{\sin^2 \beta}{4D \cos \beta \cos \gamma} \left(\frac{\rho_2}{q_0'^2} - \frac{\rho_2^2}{q_2'^2} + \frac{4}{3} D\rho_0 \right), \\ \mathfrak{c} &= L\phi_2, \\ \mathfrak{d} &= M\phi_2^2, \text{ where } \mathfrak{A}_i^s = \mathfrak{P}_i^s \mathfrak{Q}_i^s, \mathfrak{B}_i^s = \mathfrak{Q}_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0}, \mathfrak{D}_i^s = \frac{d\mathfrak{P}_i^s}{d\nu_0} \frac{d\mathfrak{Q}_i^s}{d\nu_0}. \end{aligned}$$

With this notation

$$E = \frac{3}{2} \frac{M^2}{k_0} \left\{ A_0 e^4 + 2 \sum_2^\infty B_i^s e^2 f_i^s - \sum_2^\infty C_i^s (f_i^s)^2 + \frac{\delta \omega^2}{2\pi \rho} (\mathfrak{a} + \mathfrak{b}e^2 + \mathfrak{c}f_2 - \mathfrak{d}f_2^2) \right\}.$$

Let us now make E stationary for variations of e and f_i^s .

First, by the variation of any f_i^s excepting f_2 and f_2^2 , we have

$$f_i^s = \frac{B_i^s}{C_i^s} e^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

On eliminating all these f_i^s , we have

$$\begin{aligned} E = \frac{3}{2} \frac{M^2}{k_0} \left\{ \left(A_0 + \sum_4^\infty \frac{(B_i^s)^2}{C_i^s} \right) e^4 + 2B_2 e^2 f_2 + 2B_2^2 e^2 f_2^2 - C_2 (f_2)^2 - C_2^2 (f_2^2)^2 \right. \\ \left. + \frac{\delta \omega^2}{2\pi \rho} (\mathfrak{a} + \mathfrak{b}e^2 + \mathfrak{c}f_2 - \mathfrak{d}f_2^2) \right\}. \end{aligned}$$

By the variations of f_2 , f_2^2 , and e^2 , we have

$$B_2 e^2 - C_2 f_2 + \frac{\delta \omega^2}{4\pi\rho} \mathfrak{e} = 0, \quad B_2^2 e^2 - C_2^2 f_2^2 - \frac{\delta \omega^2}{4\pi\rho} \mathfrak{d} = 0,$$

$$\left(A_0 + \sum_i \frac{(B_i^s)^2}{C_i^s} \right) e^2 + B_2 f_2 + B_2^2 f_2^2 + \frac{\delta \omega^2}{4\pi\rho} \mathfrak{h} = 0.$$

But from the first two of these equations

$$B_2 f_2 = \frac{(B_2)^2}{C_2} e^2 + \frac{\delta \omega^2}{4\pi\rho} \frac{B_2 \mathfrak{e}}{C_2}, \quad B_2^2 f_2^2 = \frac{(B_2^2)^2}{C_2^2} e^2 - \frac{\delta \omega^2}{4\pi\rho} \frac{B_2^2 \mathfrak{d}}{C_2^2}.$$

Therefore

$$\left(A_0 + \sum_i \frac{(B_i^s)^2}{C_i^s} \right) e^2 + \frac{\delta \omega^2}{4\pi\rho} \left(\mathfrak{h} + \frac{B_2 \mathfrak{e}}{C_2} - \frac{B_2^2 \mathfrak{d}}{C_2^2} \right) = 0 \quad . \quad . \quad . \quad . \quad (30).$$

When $\delta \omega^2$ has been found, f_2 and f_2^2 are determined from

$$\left. \begin{aligned} f_2 &= \frac{B_2}{C_2} e^2 + \frac{\delta \omega^2}{4\pi\rho} \frac{\mathfrak{e}}{C_2} \\ f_2^2 &= \frac{B_2^2}{C_2^2} e^2 - \frac{\delta \omega^2}{4\pi\rho} \frac{\mathfrak{d}}{C_2^2} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (31).$$

A consideration of these formulæ shows that it is immaterial what definition is adopted for any one of the harmonics, provided, of course, that the same definition is maintained throughout.

In order to evaluate A_0 , we must eliminate $\mathfrak{D}_i^s$.

Since $\mathfrak{A}_i^s = \mathfrak{P}_i^s \mathfrak{Q}_i^s$, $\mathfrak{B}_i^s = \mathfrak{Q}_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0}$, $\mathfrak{D}_i^s = \frac{d\mathfrak{P}_i^s}{d\nu_0} \frac{d\mathfrak{Q}_i^s}{d\nu_0}$, and

$$\mathfrak{Q}_i^s \frac{d\mathfrak{P}_i^s}{d\nu_0} - \mathfrak{P}_i^s \frac{d\mathfrak{Q}_i^s}{d\nu_0} = \frac{\sin^2 \beta}{\cos \beta \cos \gamma},$$

we see that

$$\mathfrak{D}_i^s = \left(\mathfrak{B}_i^s - \frac{\sin^2 \beta}{\cos \beta \cos \gamma} \right) \frac{\mathfrak{B}_i^s}{\mathfrak{A}_i^s}.$$

Hence

$$\mathfrak{A}_i^s \frac{(\omega_i^s)^2}{\phi_i^s} + \mathfrak{B}_i^s \frac{\omega_i^s \rho_i^s}{\phi_i^s} + \frac{1}{4} \frac{\mathfrak{D}_i^s (\rho_i^s)^2}{\phi_i^s} = \frac{1}{\mathfrak{A}_i^s \phi_i^s} (\mathfrak{A}_i^s \omega_i^s + \frac{1}{2} \mathfrak{B}_i^s \rho_i^s)^2 - \frac{1}{4} \frac{\sin^2 \beta}{\cos \beta \cos \gamma} \frac{\mathfrak{B}_i^s (\rho_i^s)^2}{\mathfrak{A}_i^s \phi_i^s}. \quad (32).$$

If for brevity we denote this last expression by $[i, s]$, we have

$$\left. \begin{aligned} A_0 &= \mathfrak{A}_3 \left[\frac{1}{3} (\sigma_2)^2 + 2\zeta_4 \right] - \frac{1}{3} \sigma_4 + \sum_i [i, s] \quad . \quad . \\ B_i^s &= (\mathfrak{A}_i^s \omega_i^s + \frac{1}{2} \mathfrak{B}_i^s \rho_i^s) + \left(\mathfrak{B}_3 - \frac{\sin^2 \beta}{2 \cos \beta \cos \gamma} \right) \rho_i^s \quad . \quad . \quad . \\ C_i^s &= (\mathfrak{A}_3 - \mathfrak{A}_i^s) \phi_i^s \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned} \right\} \quad (32).$$

We have now the complete analytical expressions necessary for the solution of the problem.

PART II.

NUMERICAL CALCULATION.

§ 15. *Determination of Certain Integrals.*

The integrals ω_i^s , ρ_i^s , ϕ_i^s , depend on certain others, namely—

$$\left. \begin{aligned} \Pi_{2m}^{2n} &= \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n} \theta}{\Delta_1^{2m} \Delta} d\theta \\ \Upsilon_{2m}^{2n} &= \int_0^{\frac{1}{2}\pi} \frac{\cos^{2n} \phi}{\Gamma_1^{2m} \Gamma} d\phi \end{aligned} \right\} \dots \dots \dots (33).$$

After a large part of the work had been done, I found that these integrals tend to give the required results in the form of the difference between two large numbers, and that it would have been more advantageous to consider the integrals

$$\left. \begin{aligned} \Lambda_{2m}^{2n} &= \int_0^{\frac{1}{2}\pi} \frac{\cos^{2n} \theta}{\Delta_1^{2m} \Delta} d\theta \\ \Omega_{2m}^{2n} &= \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n} \phi}{\Gamma_1^{2m} \Gamma} d\phi \end{aligned} \right\} \dots \dots \dots (34).$$

It will be shown hereafter how the group (34) may easily be found from the group (33), and it may be mentioned that most of the results were determined in duplicate from both forms.

I proceed then to consider the Π , Υ integrals.

Since $\Delta_1^2 = 1 - \kappa^2 \sin^2 \gamma \sin^2 \theta$, $\Gamma_1^2 = \cos^2 \gamma + \kappa'^2 \sin^2 \gamma \cos^2 \phi$, $\sin^2 \beta = \kappa^2 \sin^2 \gamma$, we have

$$\left. \begin{aligned} \Pi_{2m}^{2n} &= \frac{1}{\sin^2 \beta} (\Pi_{2m}^{2n-2} - \Pi_{2m-2}^{2n-2}) \\ \Upsilon_{2m}^{2n} &= -\frac{1}{\kappa'^2} \cot^2 \gamma \Upsilon_{2m}^{2n-2} + \frac{1}{\kappa'^2} \sin^2 \gamma \Upsilon_{2m-2}^{2n-2} \end{aligned} \right\} \dots \dots \dots (35).$$

I now write

$$\left. \begin{aligned} F &= \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\Delta}, & E &= \int_0^{\frac{1}{2}\pi} \Delta d\theta \\ F' &= \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1 - \kappa'^2 \cos^2 \phi)}}, & E' &= \int_0^{\frac{1}{2}\pi} \sqrt{(1 - \kappa'^2 \cos^2 \phi)} d\phi \\ F(\gamma) &= \int_0^\gamma \frac{d\theta}{\Delta}, & E(\gamma) &= \int_0^\gamma \Delta d\theta \end{aligned} \right\} \dots \dots (36).$$

It will be found from LEGENDRE'S tables that for $\gamma = 69^\circ 49' \cdot 0$, $\kappa = \sin 73^\circ 54' \cdot 2$

$$\left. \begin{aligned} \log F &= \cdot 4317642, & \log E &= \cdot 0355145 \\ \log F' &= \cdot 2047610, & \log E' &= \cdot 1875655 \\ \log F(\gamma) &= \cdot 2117987, & \log E(\gamma) &= 9 \cdot 9856045 \end{aligned} \right\} \dots \dots (36).$$

By integration by parts

$$\left. \begin{aligned} \Pi_0^{2n} &= \frac{2(n-1)}{2n-1} \frac{1+\kappa^2}{\kappa^2} \Pi_0^{2n-2} - \frac{2n-3}{(2n-1)\kappa^2} \Pi_0^{2n-4} \\ \Upsilon_0^{2n} &= \frac{2(n-1)}{2n-1} \frac{1+\kappa'^2}{\kappa'^2} \Upsilon_0^{2n-2} - \frac{2n-3}{(2n-1)\kappa'^2} \Upsilon_0^{2n-4} \end{aligned} \right\} \dots \dots (37).$$

Now write

$$G = \frac{1}{2}(1 + \sec^2 \beta + \sec^2 \gamma), \quad H' = \frac{1}{2}(\sec^2 \beta + \sec^2 \gamma + \sec^2 \beta \sec^2 \gamma).$$

The values of β and γ are $64^\circ 23' \cdot 712$, $69^\circ 49' \cdot 0$; whence $\log G = \cdot 8679015$, $\log H' = 1 \cdot 4678555$. Also we require hereafter $\log H = 1 \cdot 7182664$ (see § 3).

By differentiation

$$\frac{d}{d\theta} \frac{\Delta \sin \theta \cos \theta}{\Delta_1^{2n}} = \frac{2n \cos^2 \beta \cos^2 \gamma}{\Delta_1^{2n+2} \Delta} - \frac{2(2n-1)G \cos^2 \beta \cos^2 \gamma}{\Delta_1^{2n} \Delta} + \frac{2(2n-2)H' \cos^2 \beta \cos^2 \gamma}{\Delta_1^{2n-2} \Delta} - \frac{2n-3}{\Delta_1^{2n-4} \Delta}.$$

Whence, by integration,

$$\Pi_{2n+2}^0 = \frac{2n-1}{n} G \Pi_{2n}^0 - \frac{2n-2}{n} H' \Pi_{2n-2}^0 + \frac{2n-3}{2n} \sec^2 \beta \sec^2 \gamma \Pi_{2n-4}^0. \quad (38).$$

On writing $\sqrt{-1} \tan \gamma$ for $\sin \gamma$, we find that exactly the same formula holds good for the Υ 's.

To apply this to the determination of Π_0^2 , Υ_0^2 , we note that

$$\Pi_{-2}^0 = \cos^2 \gamma F + \sin^2 \gamma E, \quad \Upsilon_{-2}^0 = F' - \sin^2 \gamma E' \quad \dots \dots (39).$$

$$\text{Also} \quad \Pi_0^2 = \frac{1}{\kappa^2} (F - E), \quad \Upsilon_0^2 = \frac{1}{\kappa'^2} (F' - E') \quad \dots \dots (40).$$

From the formulæ given in CAYLEY'S 'Elliptic Integrals' it appears that

$$\left. \begin{aligned} \Pi_2^0 &= F + \frac{\sin \gamma}{\cos \beta \cos \gamma} [FE(\gamma) - EF(\gamma)] \\ \Upsilon_2^0 &= F' + \frac{\sin \gamma}{\cos \beta \cos \gamma} [F'E(\gamma) - F'F(\gamma) + E'F(\gamma)] \end{aligned} \right\} \dots \dots (41).$$

Now $\Pi_0^0 = F$, Π_0^2 is given in (40), and Π_0^{2n} for $n = 2, 3, 4 \dots$ are then given successively by (37).

Again, Π_2^0 is given by (41), and the successive Π_2^{2n} are given by the general formula (37).

Again (38) and (39) give

$$\Pi_4^0 = G\Pi_2^0 - \frac{1}{2 \cos^2 \beta \cos^2 \gamma} (\cos^2 \gamma F + \sin^2 \gamma E),$$

$$\Pi_6^0 = \frac{3}{2}G\Pi_4^0 - H'\Pi_2^0 + \frac{1}{4 \cos^2 \beta \cos^2 \gamma} F;$$

and by successive applications of the formula (37) we find the successive values of Π_4^{2n} , Π_6^{2n} .

It is convenient also to have the series of Π_{-2} , Υ_{-2} integrals. These are to be found from

$$\Pi_{-2}^{2n} = \Pi_0^{2n} - \sin^2 \beta \Pi_0^{2n+2}, \quad \Upsilon_{-2}^{2n} = \kappa'^2 \sin^2 \gamma \Upsilon_0^{2n+2} + \cos^2 \gamma \Upsilon_0^{2n} \quad \dots \quad (42).$$

The Υ integrals may apparently be derived by a similar set of formulæ, but since at each step we divide by κ'^2 , a small quantity, all accuracy is rapidly dissipated. Although we may safely derive one series of Υ integrals from a preceding one, we cannot so derive a succession of series, and it becomes necessary to find new formulæ.

In order to determine the Υ integrals, consider the group of integrals

$$U_{2m}^{2n} = \int_0^{\frac{1}{2}\pi} \frac{\cos^{2n} \phi}{\Gamma_1^{2m}} d\phi.$$

If we write $\xi = \frac{\cos \gamma \tan \phi}{\cos \beta}$, $u = \frac{\cos \gamma}{\cos \beta}$, we find

$$U_{2m}^0 = \frac{1}{\cos \beta \cos^{2m-1} \gamma} \int_0^\infty \frac{(u^2 + \xi^2)^{m-1}}{(1 + \xi^2)^m} d\xi,$$

whence, by some easy integrations,

$$U_2^0 = \frac{\pi}{2 \cos \beta \cos \gamma},$$

$$U_4^0 = \frac{\pi}{4 \cos \beta \cos \gamma} (\sec^2 \beta + \sec^2 \gamma),$$

$$U_6^0 = \frac{3\pi}{16 \cos \beta \cos \gamma} [\sec^4 \beta + \sec^4 \gamma + \frac{2}{3} (\sec^2 \beta + \sec^2 \gamma + \sec^2 \beta \sec^2 \gamma) + 1].$$

On expanding $\frac{1}{\Gamma}$ in powers of κ' we see that

$$\Upsilon_{2m}^{2n} = U_{2m}^{2n} + \frac{1}{2} \kappa'^2 U_{2m}^{2n+2} + \frac{1 \cdot 3}{2 \cdot 4} \kappa'^4 U_{2m}^{2n+4} + \dots$$

When $m = 0$ the U integrals are easily determined.

The relationship between the successive U integrals is clearly

$$U_{2n}^{2n} = \frac{1}{\kappa'^2 \sin^2 \gamma} U_{2n-2}^{2n-2} - \frac{1}{\kappa'^2} \cot^2 \gamma U_{2n-2}^{2n-2}.$$

I now write for brevity

$$x = \cos \beta, \quad y = \cos \gamma, \quad z = \sin \gamma, \quad \lambda = \frac{z}{1+z}, \quad \rho = \frac{y}{x+y}.$$

It appears that we may put

$$\begin{aligned} \Upsilon_0^{2n} &= \frac{1}{2}\pi \frac{1.3 \dots 2n-1}{2.4 \dots 2n} \left\{ 1 + \frac{1}{2} \cdot \frac{2n+1}{2n+2} \kappa'^2 + \frac{1.3}{2.4} \frac{2n+1.2n+3}{2n+2.2n+4} \kappa'^4 + \dots \right\}, \\ \Upsilon_2^{2n} &= \frac{\pi z}{2xy} A_{2n} + \frac{\pi}{2(1+z)} \frac{1.3 \dots 2n-1}{2.4 \dots 2n} \left\{ a_0 + \frac{1}{2} \frac{2n+1}{2n+2} a_2 \kappa'^2 + \frac{1.3}{2.4} \frac{2n+1.2n+3}{2n+2.2n+4} a_4 \kappa'^4 + \dots \right\}, \\ \Upsilon_4^{2n} &= \frac{\pi z}{4xy} B_{2n} + \frac{\pi}{2(1+z)} \frac{1.3 \dots 2n-1}{2.4 \dots 2n} \left\{ b_0 \quad . \quad . \quad . \quad b_2 \quad . \quad . \quad . \quad . \quad b_4 \quad . \quad . \quad . \right\}, \\ \Upsilon_6^{2n} &= \frac{3\pi z}{16xy} C_{2n} + \frac{\pi}{2(1+z)} \frac{1.3 \dots 2n-1}{2.4 \dots 2n} \left\{ c_0 \quad . \quad . \quad . \quad c_2 \quad . \quad . \quad . \quad . \quad . \quad c_4 \quad . \quad . \quad . \right\}. \end{aligned}$$

By considering in detail the cases where $n = 0$, I find

$$A_0 = 1, \quad a_0 = 1;$$

$$B_0 = 1 + \frac{1}{x^2} + \frac{1}{y^2} = 1 + \frac{1}{\rho^2 x^2} (1 - 2\rho + 2\rho^2), \quad b_0 = 1 - \frac{1}{2}\lambda;$$

$$\begin{aligned} C_0 &= \frac{1}{x^4} + \frac{1}{y^4} + \frac{2}{3} \left(\frac{1}{x^2 y^2} + \frac{1}{x^2} + \frac{1}{y^2} \right) + 1 \\ &= 1 + \frac{2}{3\rho^2 x^2} (1 - 2\rho + 2\rho^2) + \frac{1}{\rho^4 x^4} (1 - 4\rho + \frac{20}{3}\rho^2 - \frac{16}{3}\rho^3 + \frac{8}{3}\rho^4), \end{aligned}$$

$$c_0 = 1 - \frac{7}{8}\lambda + \frac{1}{4}\lambda^2.$$

By some rather tedious analysis, it may be proved, by considering the manner in which each Υ is derivable from the preceding ones, that

$$\begin{aligned} A_{2n} &= \frac{\rho^2}{1-2\rho} \left(\frac{1.3 \dots 2n-3}{2.4 \dots 2n-2} + A_{2n-2} \right) + \frac{1.3 \dots 2n-3}{2.4 \dots 2n-2} \rho, \\ a_{2n} &= 1 + \lambda + \frac{2\lambda^2}{1-2\lambda} \left(1 - \frac{n}{2n-1} a_{2n-2} \right), \\ B_{2n} &= \frac{1}{1-2\rho} \left[-\rho^2 B_{2n-2} + \frac{2(1-\rho)^2}{x^2} A_{2n-2} + \frac{1.3 \dots 2n-3}{2.4 \dots 2n-2} \rho (1-\rho) \right], \\ b_{2n} &= \frac{1}{1-2\lambda} \left[(1-\lambda)^2 a_{2n} - \frac{2n\lambda^2}{2n-1} b_{2n-2} \right], \end{aligned}$$

$$C_{2n} = \frac{1}{1-2\rho} \left[-\rho^2 C_{2n-2} + \frac{1(1-\rho)^2}{3x^2} B_{2n-2} + \frac{1 \cdot 3 \dots 2n-3}{2 \cdot 4 \dots 2n-2} \rho (1-\rho) \right],$$

$$c_{2n} = \frac{1}{1-2\lambda} \left[(1-\lambda)^2 b_{2n} - \frac{2n\lambda^2}{2n-1} c_{2n-2} \right].$$

By successive applications, starting from the values for $n = 0$, I find

$$A_0 = 1, \quad A_2 = \rho, \quad A_4 = \frac{1}{2}\rho(1+\rho), \quad A_6 = \frac{1 \cdot 3}{2 \cdot 4}\rho(1+\rho+\frac{2}{3}\rho^2),$$

$$A_8 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\rho(1+\rho+\frac{4}{5}\rho^2+\frac{2}{5}\rho^3), \quad A_{10} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\rho(1+\rho+\frac{6}{7}\rho^2+\frac{4}{7}\rho^3+\frac{8}{3 \cdot 5}\rho^4),$$

$$A_{12} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}\rho(1+\rho+\frac{8}{9}\rho^2+\frac{6}{9}\rho^3+\frac{8}{2 \cdot 1}\rho^4+\frac{8}{6 \cdot 3}\rho^5),$$

$$a_0 = 1, \quad a_2 = 1+\lambda, \quad a_4 = 1+\lambda+\frac{2}{3}\lambda^2, \quad a_6 = 1+\lambda+\frac{4}{5}\lambda^2+\frac{2}{5}\lambda^3,$$

$$a_8 = 1+\lambda+\frac{6}{7}\lambda^2+\frac{4}{7}\lambda^3+\frac{8}{3 \cdot 5}\lambda^4,$$

$$a_{10} = 1+\lambda+\frac{8}{9}\lambda^2+\frac{6}{9}\lambda^3+\frac{8}{2 \cdot 1}\lambda^4+\frac{8}{6 \cdot 3}\lambda^5,$$

$$B_0 = 1 + \frac{1}{x^2} + \frac{1}{y^2}, \quad B_2 = \rho + \frac{1}{x^2}, \quad B_4 = \frac{1}{2}\rho \left[1 + \rho + \frac{1}{x^2} (1 - \frac{1}{2}\rho) \right],$$

$$B_6 = \frac{1 \cdot 3}{2 \cdot 4}\rho \left[1 + \rho + \frac{2}{3}\rho^2 + \frac{8}{3x^2} (1 + \rho - \rho^2) \right],$$

$$B_8 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\rho \left[1 + \rho + \frac{4}{5}\rho^2 + \frac{2}{5}\rho^3 + \frac{12}{5x^2} (1 + \rho + \frac{1}{3}\rho^2 - \rho^3) \right],$$

$$B_{10} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\rho \left[1 + \rho + \frac{6}{7}\rho^2 + \frac{4}{7}\rho^3 + \frac{8}{3 \cdot 5}\rho^4 + \frac{16}{7x^2} (1 + \rho + \frac{3}{5}\rho^2 - \frac{1}{5}\rho^3 - \frac{4}{5}\rho^4) \right],$$

$$B_{12} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}\rho \left[1 + \rho + \frac{8}{9}\rho^2 + \frac{6}{9}\rho^3 + \frac{8}{2 \cdot 1}\rho^4 + \frac{8}{6 \cdot 3}\rho^5 + \frac{20}{9x^2} (1 + \rho + \frac{5}{7}\rho^2 + \frac{1}{7}\rho^3 - \frac{1 \cdot 6}{3 \cdot 5}\rho^4 - \frac{4}{7}\rho^5) \right],$$

$$b_0 = 1 - \frac{1}{2}\lambda, \quad b_2 = 1 + \lambda - \lambda^2, \quad b_4 = 1 + \lambda + \frac{1}{3}\lambda^2 - \lambda^3,$$

$$b_6 = 1 + \lambda + \frac{3}{5}\lambda^2 - \frac{1}{5}\lambda^3 - \frac{4}{5}\lambda^4, \quad b_8 = 1 + \lambda + \frac{5}{7}\lambda^2 + \frac{1}{7}\lambda^3 - \frac{1 \cdot 6}{3 \cdot 5}\lambda^4 - \frac{4}{7}\lambda^5,$$

$$C_0 = \frac{1}{x^4} + \frac{1}{y^4} + \frac{2}{3} \left(\frac{1}{x^2 y^2} + \frac{1}{x^2} + \frac{1}{y^2} \right) + 1,$$

$$C_2 = \rho + \frac{2}{3x^2} + \frac{1}{3x^2 y^2} + \frac{1}{x^4},$$

$$C_4 = \frac{1}{2}\rho \left[1 + \rho + \frac{8}{3x^2} (1 - \frac{1}{2}\rho) + \frac{2}{\rho x^4} \right],$$

$$C_6 = \frac{1 \cdot 3}{2 \cdot 4}\rho \left[1 + \rho + \frac{2}{3}\rho^2 + \frac{16}{9x^2} (1 + \rho - \rho^2) + \frac{64}{9x^4} (1 - \frac{7}{8}\rho + \frac{1}{4}\rho^2) \right],$$

$$C_8 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \rho \left[1 + \rho + \frac{4}{5} \rho^2 + \frac{2}{5} \rho^3 + \frac{8}{5x^2} (1 + \rho + \frac{1}{3} \rho^2 - \rho^3) + \frac{64}{15x^4} (1 + \rho - 2\rho^2 + \frac{3}{4} \rho^3) \right],$$
$$C_{10} = \frac{1 \dots 7}{2 \dots 8} \rho \left[1 + \rho + \frac{6}{7} \rho^2 + \frac{4}{7} \rho^3 + \frac{8}{35} \rho^4 + \frac{32}{21x^2} (1 + \rho + \frac{3}{5} \rho^2 - \frac{1}{5} \rho^3 - \frac{4}{5} \rho^4) \right. \\ \left. + \frac{128}{35x^4} (1 + \rho + 0 \times \rho^2 - 2\rho^3 + \rho^4) \right],$$
$$C_{12} = ?$$
$$c_0 = 1 - \frac{7}{8} \lambda + \frac{1}{4} \lambda^2, \quad c_2 = 1 + \lambda - 2\lambda^2 + \frac{3}{4} \lambda^3, \quad c_4 = 1 + \lambda + 0 \times \lambda^2 - 2\lambda^3 + \lambda^4,$$
$$c_6 = 1 + \lambda + \frac{2}{5} \lambda^2 - \frac{4}{5} \lambda^3 - \frac{7}{5} \lambda^4 + \lambda^5, \quad c_8 = 1 + \lambda + \frac{4}{7} \lambda^2 - \frac{2}{7} \lambda^3 - \frac{3}{5} \frac{7}{5} \lambda^4 - \frac{5}{7} \lambda^5 + \frac{6}{7} \lambda^6.$$

By means of these formulæ I then formed a table of the Π , Υ integrals, corresponding to the critical Jacobian for which $\gamma = 69^\circ 49' \cdot 0$, $\kappa = \sin 73^\circ 54' \cdot 2$.

A little consideration will show that if $\Pi_{2n}^0, \Pi_{2n}^2, \Pi_{2n}^4 \dots$ are a series of Π integrals, the Λ integrals as defined in (34) are as follows :—

$$\Lambda_{2n}^0 = \Pi_{2n}^0, \quad \Lambda_{2n}^2 = -\Delta \Pi_{2n}^0, \quad \Lambda_{2n}^4 = \Delta^2 \Pi_{2n}^0, \quad \Lambda_{2n}^6 = -\Delta^3 \Pi_{2n}^0, \text{ \&c.}$$

Hence by differencing the Π integrals we find the Λ integrals, and similarly the differences of the Υ integrals give the Ω integrals.

The converse is also true, and by differencing Λ, Ω we return to Π, Υ .

In this way I obtain a series of values of the required integrals. It may be that the last decimal place is erroneous in some cases, but the results given in the following table are sufficiently accurate for our purpose.

TABLE of Logarithms of Λ and Ω Integrals.

$n.$	$\log \Lambda_{-2}^n.$	$\log \Lambda_0^n.$	$\log \Lambda_2^n.$	$\log \Lambda_4^n.$	$\log \Lambda_6^n.$
0	·1064412	·4317642	·9049643	1·4848513	2·1218296
2	9·8518424	9·9779679	·1696589	·4573383	·8543000
4	9·7422028	9·8168606	9·9187709	·0641097	·2747811
6	9·6704204	9·7230801	9·7896484	9·8771928	9·9969099
8	9·6167543	9·6573681	9·7062560	9·7667474	9·8443343
10	9·5738269	9·6068642	9·6454046	9·6909753	9·7467821
12	9·5380322	9·5658708	9·5977435	9·6339429	9·6770135

$n.$	$\log \Omega_{-2}^n.$	$\log \Omega_0^n.$	$\log \Omega_2^n.$	$\log \Omega_4^n.$	$\log \Omega_6^n.$
0	9·3901374	·2047610	1·0302812	1·8667636	2·7138342
2	9·0333614	9·8993680	·7715371	1·6492556	2·5319082
4	8·8883413	9·7729763	·6613027	1·5528790	2·4473548
6	8·7988101	9·6930495	·5898109	1·4886133	2·3893878
8	8·7344878	9·6345770	·5366725	1·4398972	2·3245748
10	8·6844351	9·5884555	·4944045	1·4005018	2·2861489
12	8·6435159	9·5503544	·4594781	1·3669609	2·2533269

§ 16. *The Integrals σ_2 , σ_4 , ζ_4 .*

In accordance with equation (14) of the "Pear-shaped Figure" the third zonal harmonic is defined by

$$S_3 = \sin \theta (\kappa^2 \sin^2 \theta - q^2) (q'^2 - \kappa'^2 \cos^2 \phi) \sqrt{(1 - \kappa'^2 \cos^2 \phi)},$$

where

$$q^2 = \frac{2}{5} [1 + \kappa^2 - (1 - \frac{7}{4}\kappa^2 + \kappa^2)^{\frac{1}{2}}], \quad q'^2 = 1 - q^2.$$

The numerical values for the critical Jacobian are

$$\kappa^2 = \cdot 9231276, \quad q^2 = \cdot 5746473.$$

Writing $p^2 = \kappa^2 - q^2$, we have

$$S_3 = (p^2 - \kappa^2 \cos^2 \theta) \sqrt{(1 - \cos^2 \theta)} (p^2 + \kappa'^2 \sin^2 \phi) \sqrt{(\kappa^2 + \kappa'^2 \sin^2 \phi)}.$$

Now let

$$\begin{aligned} \alpha &= p^4, & \beta &= 2p^2\kappa^2 + p^4, & \gamma &= \kappa^4 + 2p^2\kappa^2, & \delta &= \kappa^4, \\ \alpha' &= p^4\kappa^2, & \beta' &= 2p^2\kappa^2\kappa'^2 + p^4\kappa'^2, & \gamma' &= \kappa^2\kappa'^4 + 2p^2\kappa'^4, & \delta' &= \kappa'^6, \end{aligned}$$

and we have

$$(S_3)^2 = (\alpha - \beta \cos^2 \theta + \gamma \cos^4 \theta - \delta \cos^6 \theta) (\alpha' + \beta' \sin^2 \phi + \gamma' \sin^4 \phi + \delta' \sin^6 \phi).$$

The numerical values of the logarithms of the coefficients are

$$\begin{aligned} \log \alpha &= 9\cdot0843568, & \log \alpha' &= 9\cdot0496186, \\ \log \beta &= 9\cdot8835606, & \log \beta' &= 8\cdot7693310, \\ \log \gamma &= \cdot1748006, & \log \gamma' &= 7\cdot9810798, \\ \log \delta &= 9\cdot9305236, & \log \delta' &= 6\cdot6573112. \end{aligned}$$

Let

$$\left. \begin{aligned} f(\Lambda_{2n}) &= \alpha \Lambda_{2n}^0 - \beta \Lambda_{2n}^2 + \gamma \Lambda_{2n}^4 - \delta \Lambda_{2n}^6 \\ f(\Omega_{2n}) &= \alpha' \Omega_{2n}^0 + \beta' \Omega_{2n}^2 + \gamma' \Omega_{2n}^4 + \delta' \Omega_{2n}^6 \end{aligned} \right\} \text{for } n = 0, 1, 2.$$

The definition of σ_2 in (8) then shows that

$$\sigma_2 = \frac{6}{\pi} \frac{\cos^2 \beta \cos^2 \gamma}{\sin^2 \gamma} \{f(\Lambda_0) f(\Omega_4) - f(\Lambda_4) f(\Omega_0) - G[f(\Lambda_0) f(\Omega_2) - f(\Lambda_2) f(\Omega_0)]\}.$$

In order to find σ_4 and ζ_4 , S_3 must be raised to the fourth power, and we now define, for $n = 1, 2, 3$,

$$\begin{aligned} f(\Lambda_{2n}) &= \alpha^2 \Lambda_{2n}^0 - 2\alpha\beta \Lambda_{2n}^2 + (2\alpha\gamma + \beta^2) \Lambda_{2n}^4 - (2\alpha\delta + 2\beta\gamma) \Lambda_{2n}^6 \\ &\quad + (2\beta\delta + \gamma^2) \Lambda_{2n}^8 - 2\gamma\delta \Lambda_{2n}^{10} + \delta^2 \Lambda_{2n}^{12}, \end{aligned}$$

$$f(\Omega_{2n}) = \alpha'^2 \Omega_{2n}^0 + 2\alpha'\beta' \Omega_{2n}^2 + (2\alpha'\gamma' + \beta'^2) \Omega_{2n}^4 + (2\alpha'\delta' + 2\beta'\gamma') \Omega_{2n}^6 \\ + (2\beta'\delta' + \gamma'^2) \Omega_{2n}^8 + 2\gamma'\delta' \Omega_{2n}^{10} + \delta'^2 \Omega_{2n}^{12}.$$

From the definitions of σ_4 , ζ_4 in (8), we see that

$$\zeta_4 = \frac{6}{\pi} \frac{\cos^4 \beta \cos^4 \gamma}{\sin^2 \gamma} \{ - [f(\Lambda_4) f(\Omega_6) - f(\Lambda_6) f(\Omega_4)] + G [f(\Lambda_2) f(\Omega_6) - f(\Lambda_6) f(\Omega_2)] \\ - H [f(\Lambda_2) f(\Omega_4) - f(\Lambda_4) f(\Omega_2)] \}, \\ \sigma_4 = \frac{6}{\pi} \frac{\cos^3 \beta \cos^3 \gamma \sin \beta}{\sin^2 \gamma} \{ \frac{5}{2} [f(\Lambda_2) f(\Omega_6) - f(\Omega_6) f(\Lambda_2)] \\ - 3G [f(\Lambda_2) f(\Omega_4) - f(\Lambda_4) f(\Omega_2)] \}.$$

The computations (which were in this case actually made from the corresponding formulæ involving the Π , Υ integrals) gave

$$\sigma_2 = \cdot 0136866, \quad \zeta_4 = \cdot 00009246, \quad \sigma_4 = \cdot 00176135.$$

These have to be used in a formula which also involves $\mathfrak{A}_3$. Now $\mathfrak{A}_3$ denotes $\mathfrak{P}_3 \mathfrak{Q}_3$, or what should be the same thing, $\mathbf{P}_1^1 \mathbf{Q}_1^1$. The formulæ in the "Pear-shaped Figure" with $\gamma = 69^\circ 49' \cdot 0$, $\kappa = \sin 73^\circ 54' \cdot 2$, give

$$\mathbf{P}_1^1 \mathbf{Q}_1^1 = \cdot 351697, \quad \mathfrak{P}_3 \mathfrak{Q}_3 = \cdot 351744.$$

Thus the two functions, which should be identical in value, differ by $\cdot 000047$. I think that if I had taken $\gamma = 69^\circ 48' \cdot 997$, $\kappa = \sin 73^\circ 54' \cdot 225$ (the actual numerical solution for the critical Jacobian, although not fully stated in the "Pear-shaped Figure") this small discrepancy would have been removed. However, the difference is quite unimportant, and as $\mathfrak{A}_3$ generally means $\mathbf{P}_1^1 \mathbf{Q}_1^1$, I take the former value and put $\log \mathfrak{A}_3 = 9 \cdot 54617$.

With this value I find the required result, namely

$$\mathfrak{A}_3 [\frac{1}{3} (\sigma_2)^2 + 2\zeta_4] - \frac{1}{3} \sigma_4 = - \cdot 00050012 \quad . \quad . \quad . \quad . \quad (43).$$

§ 17. *The Integrals ω_i^s , ρ_i^s , ϕ_i^s .*

Any harmonic S_i^s , where i and s are both even, whether in the approximate form of "Harmonics" or in the rigorous form, may be written

$$S_i^s = (a - b \cos^2 \theta + c \cos^4 \theta - d \cos^6 \theta + \dots) (\alpha' + b' \sin^2 \phi + c' \sin^4 \phi + d' \sin^6 \phi + \dots).$$

Each series is, of course, terminable, the number of terms in each of the two factors being $\frac{1}{2}i + 1$.

For the determination of the ω , ρ integrals this must be multiplied by $(S_3)^2$. It

can be seen, without actually writing down the product, how the coefficients will occur; I write therefore those coefficients as follows:—

$$l_0 = a\alpha, \quad l_2 = a\beta + b\alpha, \quad l_4 = a\gamma + b\beta + c\alpha, \quad l_6 = a\delta + b\gamma + c\beta + d\alpha, \quad \&c.$$

$$m_0 = a'\alpha', \quad m_2 = a'\beta' + b'\alpha', \quad m_4 = a'\gamma' + b'\beta' + c'\alpha', \quad m_6 = a'\delta' + b'\gamma' + c'\beta' + d'\alpha', \quad \&c.$$

Next let

$$f(\Lambda_{2n}) = l_0\Lambda_{2n}^0 - l_2\Lambda_{2n}^2 + l_4\Lambda_{2n}^4 - l_6\Lambda_{2n}^6 \dots \quad \text{for } n = 0, 1, 2.$$

$$f(\Omega_{2n}) = m_0\Omega_{2n}^0 + m_2\Omega_{2n}^2 + m_4\Omega_{2n}^4 + m_6\Omega_{2n}^6 \dots$$

Then it follows from the definitions of ω_i^s , ρ_i^s in (8) that

$$\omega_i^s = \frac{6 \cos^2 \beta \cos^2 \gamma}{\pi \sin^2 \gamma} \{f(\Lambda_0)f(\Omega_4) - f(\Lambda_4)f(\Omega_0) - G[f(\Lambda_0)f(\Omega_2) - f(\Lambda_2)f(\Omega_0)]\},$$

$$\rho_i^s = \frac{6 \cos^2 \beta \cos^2 \gamma}{\pi \sin^2 \gamma \sin \beta} \{f(\Lambda_0)f(\Omega_2) - f(\Lambda_2)f(\Omega_0)\}.$$

It is, of course, necessary to reduce the two factors of S_i^s to the required forms. The harmonics of the second order are

$$S_2^s = (\kappa^2 \sin^2 \theta - q_s'^2)(q_s'^2 - \kappa'^2 \cos^2 \phi), \quad (s = 0, 2),$$

and I find $q_0'^2 = \cdot 3197540$, $q_2'^2 = \cdot 9623311$; whence we may find a , b , a' , b' for these harmonics.

For the harmonics of the fourth and sixth orders I take the formulæ of "Harmonics," and attributing to the parameter β its value $\cdot 0399726$ (or more shortly $\cdot 04$ in the terms of the sixth order), I reduce $\mathfrak{P}_i^s(\mu)$, $\mathfrak{C}_i^s(\phi)$ to the required forms and determine a , b , c , &c., a' , b' , c' , &c. The numerical values of these coefficients are given in the tables of § 20 hereafter.

It may be well to remark that ρ_0 is needed (but not ω_0), and in this case $S_0 = 1$, so that $a = a' = 1$.

It seems useless to go in detail through the tedious operations involved in carrying out this process in the several cases.

Approximate formulæ are given for the ϕ_i^s integrals in § 22 of "Harmonics." The $\int p_0 d\sigma$ of that paper is the same as $\frac{4}{3}\pi k^3 \frac{\cos \beta \cos \gamma}{\sin^3 \beta} \iint \Phi d\theta d\phi$ of the present one, and the factor there written M is $k^3 \frac{\cos \beta \cos \gamma}{\sin^3 \beta}$. Hence it follows that

$$\phi_i^s = \frac{3}{4\pi M} \int (\mathfrak{P}_i^s \mathfrak{C}_i^s)^2 p d\sigma \text{ of "Harmonics."}$$

In order to apply this to the harmonics of the second degree, it must be borne in mind that a different definition of S_2^s ($s = 0, 2$) is being used here. If $[\phi_2]$, $[\phi_2^2]$ be

the values which would be found from “Harmonics” without this correction, and if ϕ_2, ϕ_2^2 are the required values, it appears that

$$\phi_2 = [\phi_2] \frac{\kappa^4 \kappa'^4}{\alpha^2 \epsilon^2}, \qquad \phi_2^2 = [\phi_2^2] \frac{\kappa^4 \kappa'^4}{\alpha'^2 \epsilon'^2},$$

where $\alpha, \epsilon, \alpha', \epsilon'$ are the coefficients specified in § 12 of “Harmonics.”

The approximate values found in this way for all the ϕ integrals are very near to the more correct values, and might have been adopted throughout without material error. But as there was not much certainty that the approximation was a good one—and indeed for S_6 was probably bad—I also found all these integrals, excepting ϕ_6^2, ϕ_6^4 , by the method now to be described.

From (6) and (8) it appears that

$$\phi_i^s = \frac{6}{\pi \sin^2 \gamma} \iint \left(\frac{1}{\Delta_1^{-2}} - \frac{1}{\Gamma_1^{-2}} \right) \frac{(S_i^s)^2}{\Delta \Gamma} d\theta d\phi.$$

If, therefore, we write

$$f(\Lambda_{2n}) = a^2 \Lambda_{2n}^0 - 2ab \Lambda_{2n}^2 + (2ac + b^2) \Lambda_{2n}^4 - (2ad + 2bc) \Lambda_{2n}^6 + \dots$$
$$f(\Omega_{2n}) = a'^2 \Omega_{2n}^0 + 2a'b' \Omega_{2n}^2 + (2a'c' + b'^2) \Omega_{2n}^4 + (2a'd' + 2b'c') \Omega_{2n}^6 + \dots$$

for $n = -1, 0$.

we have

$$\phi_i^s = \frac{6}{\pi \sin^2 \gamma} [f(\Lambda_{-2}) f(\Omega_0) - f(\Lambda_0) f(\Omega_{-2})].$$

The following table gives the results for all the $\omega_i^s, \rho_i^s, \phi_i^s$ integrals:—

TABLE of Logarithms of ϕ, ω, ρ Integrals.

<i>i.</i>	<i>s.</i>	$\log \phi_i^s.$	Approximate $\log \phi_i^s$ from “Harmonics.”	$\log \omega_i^s + 10.$	$\log \rho_i^s + 10.$
0	0	—	—	—	7·63099
2	0	9·00516 – 10	9·00515 – 10	7·67371	7·02716
2	2	7·03970 – 10	7·03973 – 10	(–) 5·68193	(–) 5·05256
4	0	9·69080 – 10	9·69323 – 10	8·03358	7·35625
4	2	1·72664	1·72729	(–) 8·33367	(–) 7·59132
4	4	3·81541	3·81612	8·29058	7·37446
6	0	9·71219 – 10	9·69305 – 10	7·97301	7·32602
6	2	—	2·20562	(–) 8·72778	(–) 7·94094
6	4	—	5·29999	9·10094	8·13161

§ 18. *The Integrals $\mathfrak{A}_i^s$, $\mathfrak{B}_i^s$.*

Adopting the notation of the last section we have

$$\mathfrak{P}_i^s(\nu) = a + b(\nu^2 - 1) + c(\nu^2 - 1)^2 + d(\nu^2 - 1)^3 + \dots$$

Let $\nu = \frac{1}{\kappa \sin \psi}$, and $\kappa \sin \psi = \sin \chi$, so that at the surface of the ellipsoid where $\psi = \gamma$, we have $\chi = \beta$.

Then

$$\mathfrak{P}_i^s(\nu) = a + b \cot^2 \chi + c \cot^4 \chi + \dots$$

$$\mathfrak{P}_i^s(\nu_0) = a + b \cot^2 \beta + c \cot^4 \beta + \dots$$

$$\text{Now} \quad \int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}}} = \kappa \int_0^{\gamma} \frac{d\psi}{\sqrt{(1 - \kappa^2 \sin^2 \psi)}} = \kappa \int_0^{\gamma} \sec \chi \, d\psi \quad .$$

$$\text{and} \quad \mathfrak{A}_i^s = [\mathfrak{P}_i^s(\nu_0)]^2 \int_{\nu_0}^{\infty} \frac{d\nu}{(\mathfrak{P}_i^s)^2 (\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}}}.$$

$$\text{Hence } \mathfrak{A}_i^s = \kappa (a + b \cot^2 \beta + c \cot^4 \beta + \dots)^2 \int_0^{\gamma} \frac{\sec \chi \, d\psi}{[a + b \cot^2 \chi + c \cot^4 \chi + \dots]^2}.$$

We have, in § 4 of the "Pear-shaped Figure," the rigorous expression of this integral for harmonics of the second order, viz. :—

$$\mathfrak{A}_2^s = \frac{\kappa (1 - 2q_s^2)(1 - q_s^2 \sin^2 \gamma)^2}{2q_s^2 q_s'^2 \sin^4 \gamma} \left\{ \frac{F(\gamma)}{q_s^2} - \frac{E(\gamma)}{q_s^2 q_s'^2} + \frac{\sin \gamma \cos \gamma \cos \beta}{q_s'^2 (1 - q_s^2 \sin^2 \gamma)} \right\}, \quad s = 0, 2.$$

The values of q_0^2 , q_2^2 have been already given, and thus all the quantities involved are known.

The two factors of $\mathfrak{A}_i^s$ (viz., $\mathfrak{P}_i^s$ and $\mathfrak{Q}_i^s$) are given in approximate forms in "Harmonics," and therefore, if we made allowance for the different definition of $\mathfrak{Q}_i^s$ adopted in that paper, we might calculate $\mathfrak{A}_i^s$. The computations I made showed that the results obtained in that way would have been sufficiently exact, but as it was clear that the approximation to the $\mathfrak{Q}$ functions was not very close, and as the computation is tedious, it seemed better to find the $\mathfrak{A}_i^s$ by quadratures.

In order to do this I divided γ or $69^\circ 49'$ by 12, and took $5^\circ 49' \frac{1}{12}$ as the common difference, say δ . I then computed $\sec \chi$, $a + b \cot^2 \chi + c \cot^4 \chi + \dots$, and $\sec \chi \div (a + b \cot^2 \chi + c \cot^4 \chi + \dots)^2$ for values of $\psi = 0, \delta, 2\delta, \dots 12\delta$ or γ .

As a fact the first five or six values need not be computed because the early values of the functions to be integrated are practically zero. The ordinary formulæ of

quadratures are inappropriate for these integrations, because the function, say u_n , to be integrated increases so very rapidly. I therefore take an empirical and integrable function, say v_n , which is such that $v_{12} = u_{12}$, $v_{11} = u_{11}$; the quadratures may then be applied to $u_n - v_n$, and the result applied as a correction to $\int v d\psi$. In fact this correction is always very small, and we might well be satisfied to use $\int v d\psi$, which is very easy to calculate.

The empirical function v is given by

$$v = u_{12} e^{\frac{(\psi-\gamma)}{\delta} \log_e \frac{u_{12}}{u_{11}}}.$$

Then when $\psi = \gamma$, $v = u_{12}$; when $\psi = \gamma - \delta$, $v = u_{11}$; and

$$\int_0^\gamma v d\psi = \frac{u_{12} \delta}{\log_e \frac{u_{12}}{u_{11}}} \left(1 - e^{-\frac{\gamma}{\delta} \log_e \frac{u_{12}}{u_{11}}} \right).$$

In all the cases I have to consider the exponential term is negligible, and the integral is $\frac{u_{12} \delta}{\log_e \frac{u_{12}}{u_{11}}}$.

For the quadratures we have

$$v_{12} = u_{12}, \quad v_{11} = u_{11}, \quad v_{10} = u_{12} \left(\frac{u_{11}}{u_{12}} \right)^2, \quad v_9 = u_{12} \left(\frac{u_{11}}{u_{12}} \right)^3, \quad \&c.,$$

and the equidistant values of the function, to be integrated (arranged backwards), are

$$0, \quad 0, \quad u_{10} - u_{12} \left(\frac{u_{11}}{u_{12}} \right)^2, \quad u_9 - u_{12} \left(\frac{u_{11}}{u_{12}} \right)^3, \quad u_8 - u_{12} \left(\frac{u_{11}}{u_{12}} \right)^4, \quad \&c.$$

The first two are zero, the next three or four are found to be sensible, and the rest are insensible; hence the quadrature is very easy.

The $\mathfrak{B}_i^s$ integrals are found thus:—

$$\mathfrak{B}_i^s = \mathfrak{A}_i^s \frac{1}{\mathfrak{P}_i^s(\nu_0)} \frac{d}{d\nu_0} \mathfrak{P}_i^s(\nu_0) = \frac{2\mathfrak{A}_i^s}{\sin \beta} \frac{b + 2c \cot^2 \beta + 3d \cot^4 \beta + \dots}{a + b \cot^2 \beta + c \cot^4 \beta + \dots}.$$

The following table gives the $\mathfrak{A}_i^s$, $\mathfrak{B}_i^s$ integrals.

TABLE of Logarithms of the $\mathfrak{A}$, $\mathfrak{B}$ integrals.

$i.$	$s.$	$\log \mathfrak{A}_i^s + 10.$	$\log \mathfrak{B}_i^s.$
2	0	9·69312	·09295
2	2	9·33300	·40665
3	0	9·54617	·20467
4	0	9·44928	·28206
4	2	9·25987	·41239
4	4	9·06489	·44858
6	0	9·24383	·35876
6	2	9·16199	·41249
6	4	9·02369	·44195

§ 19. *Synthesis of Numerical Results; Stability of the Pear.*

In the following tables and remarks I collect together some of the results which occur in the course of the work. The final places of decimals as given have, perhaps, in many cases but little significance :—

$i.$	$s.$	(1.) $\mathfrak{A}_3 - \mathfrak{A}_i^s.$	(2.) $\log(\mathfrak{A}_3 - \mathfrak{A}_i^s) \phi_i^s = \log C_i^s.$	(3.) $\mathfrak{A}_i^s \omega_i^s + \frac{1}{2} \mathfrak{B}_i^s \rho_i^s.$	(4.) $\left(\mathfrak{B}_3 - \frac{\sin^2 \beta}{2 \cos \beta \cos \gamma} \right) \rho_i^s.$	(3) + (4). $B_i^s.$
2	0	— ·141617	(—) 8·1562926 — 10	·0029865	— ·0011976	·0017889
2	2	·136417	6·1745705 — 10	— ·0000247	·0000127	— ·0000120
4	0	·07033	8·53794 — 10	·005214	— ·002555	·002659
4	2	·16978	·95653	— ·008965	·004390	— ·004575
4	4	·23558	3·18755	·005594	— ·002664	·002930
6	0	·17638	8·95864 — 10	·004067	— ·002384	·001684
6	2	·20649	1·52052	— ·019041	·009820	— ·009221
6	4	·24609	4·69108	·032054	— ·015232	·016822

For all harmonics higher than those of the second degree $\mathfrak{A}_3 - \mathfrak{A}_i^s$ is the coefficient of stability. Since in all these cases this expression is positive, the ellipsoid is stable for all such deformations.

If $U + \delta U$ be the energy function for the pear, whose variations for constant moment of momentum are considered by M. POINCARÉ, we have in our notation

$$U + \delta U = -\frac{1}{2} \int \frac{dm_1 dm_2}{D_{12}} + \frac{1}{2} (A_j - A_r) (\omega^2 + \delta \omega^2).$$

It is easy to show from our analysis that for the deformation $f_2 S_2$,

$$\delta U = \frac{3M^2}{2k_0} (f_2)^2 \left\{ (\mathfrak{A}_3 - \mathfrak{A}_2) \phi_2 + \frac{\omega^2}{2\pi\rho} \frac{\mathfrak{r}^2}{a} \right\},$$

and that the corresponding expression with $\mathfrak{D}^2$ in place of $\mathfrak{r}^2$ holds good for the deformation $f_2^2 S_2^2$.

Forestalling the results obtained below, it may be stated that for $f_2 S_2$

$$\delta U = \frac{3M^2}{2k_0} (f_2)^2 \{ -0.01433 + 0.03959 \};$$

and for $f_2^2 S_2^2$

$$\delta U = \frac{3M^2}{2k_0} (f_2^2)^2 \{ 0.00015 + 0.00002 \}.$$

Thus in both cases δU is positive, and this shows that the Jacobian ellipsoid is also stable for the ellipsoidal deformations. The fact, that δE (the variation of my function of energy for constant angular velocity) is negative for the deformation S_2 , illustrates the truth of M. POINCARÉ'S remark ('Acta Math.,' 7, p. 365): "Si au contraire la rotation de la masse fluide était déterminée par celle d'un axe rigide (comme dans les expériences de Plateau par exemple), tout déplacement produirait une résistance passive et l'ellipsoïde de Jacobi serait toujours instable."

I have in (32) written

$$[i, s] = \frac{1}{\mathfrak{A}_i^s \phi_i^s} \left\{ (\mathfrak{A}_i^s \omega_i^s + \frac{1}{2} \mathfrak{B}_i^s \rho_i^s)^2 - \frac{\sin^2 \beta}{4 \cos \beta \cos \gamma} \mathfrak{B}_i^s (\rho_i^s)^2 \right\}.$$

The following table then gives further stages in the work:—

$i.$	$s.$	$[i, s]$	$(B_i^s)^2 / C_i^s.$	$B_i^s / C_i^s.$
2	0	·00014032	— ·00022329	— ·12482
2	2	·00000072	+ ·00000097	— ·08059
4	0	·00009937	·00020486	·07705
4	2	·00000276	·00000231	— ·000506
4	4	·00000001	·00000001	·0000019
6	0	·00002835	·00003118	·01852
6	2	·00000403	·00000256	— ·000278
6	4	·00000002	·00000001	·00000034
$\mathfrak{A}_3 \left[\frac{1}{3} (\sigma_2)^2 + 2\zeta_4 \right] - \frac{1}{3} \sigma_4 =$ $A_0 =$ $\Sigma \frac{(B_i^s)^2}{C_i^s} =$		$\Sigma [i, s] =$	$\frac{\Sigma (B_i^s)^2}{C_i^s} = + \cdot 00001861$	
		·00027558	·00024190 — ·00022329	
		— ·00050012	+ ·00024190	
		— ·00022454		
$\Sigma \frac{(B_i^s)^2}{C_i^s} =$		+ ·00001861		
Numerator.		— ·00020593		

The next step is to find

$$L = \frac{\sin \beta}{4 D q_0^2 \cos \beta \cos \gamma} \left(1 - \frac{\cos^2 \beta}{D + \kappa^2} \right), \quad M = \frac{\sin \beta}{4 D q_2^2 \cos \beta \cos \gamma} \left(1 + \frac{\cos^2 \beta}{D - \kappa^2} \right),$$

where $D^2 = 1 - \kappa^2 \kappa'^2$.

The numerical values are $\log D = 9.9840165$, $\log L = .6454565$, $\log M = .9591960$. From these we obtain $\mathfrak{c} = L\phi_2$, $\mathfrak{d} = M\phi_2^2$; whence

$\frac{B_2}{C_2} \mathfrak{c} =$	$-.055837$
$-\frac{B_2^2}{C_2^2} \mathfrak{d} =$	$.000804$
$\mathfrak{h} =$	$.031701$
Denominator =	$-.023332$

In accordance with (32) the Numerator divided by the Denominator is $-\frac{\delta\omega^2}{4\pi\rho e^2}$, and I thus find

$$\log \frac{\delta\omega^2}{4\pi\rho e^2} = (-) 7.94578.$$

It was found in § 7 of the "Pear-shaped Figure" that the angular velocity of the critical Jacobian was given by $\frac{\omega^2}{2\pi\rho} = .14200$. Accordingly, the square of the angular velocity of the pear being $\omega^2 + \delta\omega^2$, we have

$$\omega^2 + \delta\omega^2 = \omega^2 [1 - .124314e^2].$$

From the formulæ (31) I then find

$$f_2 = .15068e^2, \quad f_2^2 = .50839e^2.$$

The other f_i^s are equal to $\frac{B_i}{C_i^s} e^2$, and are given in the preceding table. From (28) and the definitions of $\mathfrak{a}$, $\mathfrak{h}$, $\mathfrak{c}$, $\mathfrak{d}$ it appears that the moment of inertia of the pear is

$$A_j - A_r = \frac{3M^2\mathfrak{a}}{2\pi\rho k_0} \left[1 + \frac{\mathfrak{h}}{\mathfrak{a}} e^2 + \frac{\mathfrak{c}}{\mathfrak{a}} f_2 - \frac{\mathfrak{d}}{\mathfrak{a}} f_2^2 \right].$$

With $\log \mathfrak{a} = 9.8559758$, I find

$$A_j - A_r = \frac{3M^2\mathfrak{a}}{2\pi\rho k_0} [1 + .131011e^2].$$

The angular velocity of the pear is

$$\sqrt{(\omega^2 + \delta\omega^2)} = \omega [1 - .062157e^2].$$

Multiplying these last two expressions together, we have the moment of momentum of the pear; it is

$$\frac{3M^2\mathfrak{a}\omega}{2\pi\rho k_0} [1 + \cdot 068854e^2].$$

It follows that, whilst the angular velocity of the pear is less than that of the critical Jacobian, the moment of momentum is greater. This result would afford a rigorous proof of the stability of the pear if the numbers were based on a complete solution of the problem. But as we have not determined an infinite series of new harmonic terms, it becomes necessary to consider how the result might differ if the hitherto uncomputed terms were added.

If ϵ denotes the uncomputed portion of the infinite series $\Sigma \left\{ [i, s] + \frac{(B_i^s)^2}{C_i^s} \right\}$, and if Δ denotes the addition to be made on that account to any of the results as already computed, we have

$$\Delta \left(\frac{\delta\omega^2}{4\pi\rho} \right) = \frac{\epsilon e^2}{\cdot 023332}, \quad \text{and} \quad \Delta \left(\frac{\delta\omega^2}{\omega^2} \right) = \frac{2\epsilon e^2}{\cdot 023332 \times \cdot 142}.$$

Whence

$$\Delta [\sqrt{(\omega^2 + \delta\omega^2)}] = \frac{1}{2}\omega\Delta \frac{\delta\omega^2}{\omega^2} = \omega [301\cdot 8346\epsilon e^2].$$

Since

$$f_2 = \frac{B_2}{C_2} e^2 + \frac{\delta\omega_2}{4\pi\rho} \frac{\mathfrak{r}}{C_2}, \quad f_2^2 = \frac{B_2^2}{C_2^2} e^2 - \frac{\delta\omega^2}{4\pi\rho} \frac{\mathfrak{d}}{C_2^2}$$

$$\Delta f_2 = \frac{\mathfrak{r}}{C_2} \Delta \frac{\delta\omega^2}{4\pi\rho} = -10^{3\cdot 1263984},$$

$$\Delta f_2^2 = -\frac{\mathfrak{d}}{C_2^2} \Delta \frac{\delta\omega^2}{4\pi\rho} = -10^{3\cdot 4563848}.$$

Then

$$\begin{aligned} \Delta (A_j - A_r) &= \frac{3M^2\mathfrak{a}}{2\pi\rho k_0} \left[\frac{\mathfrak{r}}{\mathfrak{a}} \Delta f_2 - \frac{\mathfrak{d}}{\mathfrak{a}} \Delta f_2^2 \right] = \frac{3M^2\mathfrak{a}}{2\pi\rho k_0} [-833\cdot 7892 + 39\cdot 7472] \epsilon e^2 \\ &= \frac{3M^2\mathfrak{a}}{2\pi\rho k_0} (-794\cdot 0420) \epsilon e^2. \end{aligned}$$

Therefore

$$\sqrt{(\omega^2 + \delta\omega^2)} = \omega [1 - \cdot 0621568e^2 + 301\cdot 8346\epsilon e^2]$$

$$A_j - A_r = \frac{3M^2\mathfrak{a}}{2\pi\rho k_0} [1 + \cdot 13101068e^2 - 794\cdot 0420\epsilon e^2]$$

By multiplication we find that the moment of momentum is

$$\frac{3M^2\mathfrak{a}\omega}{2\pi\rho k_0} [1 + \cdot 0688539e^2 - 492\cdot 2074\epsilon e^2].$$

The coefficient of e^2 is positive and the pear is stable, provided that

$$492.2074\epsilon < .0688539,$$

$$\text{or} \quad \epsilon < .00014.$$

Inspection of the table of numerical results shows that the zonal harmonic terms contribute by very far the larger portion of the sum. Now the sixth zonal term was

$$[6, 0] + \frac{(B_6)^2}{C_6} = .00002835 + .00003118 = .00005953.$$

This is about $\frac{3}{7}$ of the critical total .00014. The pear is then stable unless the residue of the apparently highly convergent series shall amount to $2\frac{1}{3}$ times the contribution of the sixth zonal term. Such an hypothesis appears profoundly improbable, but I have thought it expedient to make a rough determination of the contribution of the eighth zonal harmonic to the sum.

If we take κ as equal to unity, $S_8 = \mathfrak{P}_8(\mu) \mathfrak{C}_8(\phi) = P_8(\mu)$, and we easily see that the formulæ (8) reduce to

$$\omega_8 = \frac{3}{\pi} \cos^2 \gamma \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{(1 + \sin^2 \gamma \sin^2 \theta) \cos \theta}{(1 - \sin^2 \gamma \sin^2 \theta)^2} (S_3)^2 S_8 d\theta d\phi,$$

$$\rho_8 = \frac{6}{\pi} \frac{\cos^2 \gamma}{\sin \gamma} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\cos \theta}{1 - \sin^2 \gamma \sin^2 \theta} (S_3)^2 S_8 d\theta d\phi.$$

In these integrals ϕ only enters through $(S_3)^2$ or $[\mathfrak{P}_3(\mu)]^2 [\mathfrak{C}_3(\phi)]^2$.

Now

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} [\mathfrak{C}_3(\phi)]^2 d\phi &= \frac{1}{2}\pi \left[\alpha' + \frac{1}{2}\beta' + \frac{1.3}{2.4} \gamma' + \frac{1.3.5}{2.4.6} \delta' \right] \\ &= \frac{1}{2}\pi K, \quad \text{where } K = .1452. \end{aligned}$$

Hence

$$\omega_8 = \frac{3}{2} K \cos^2 \gamma \int_0^{\frac{1}{2}\pi} \frac{(1 + \sin^2 \gamma \sin^2 \theta) \cos \theta}{(1 - \sin^2 \gamma \sin^2 \theta)^2} [\mathfrak{P}_3(\mu)]^2 P_8(\mu) d\theta$$

$$\rho_8 = \frac{3K \cos^2 \gamma}{\sin \gamma} \int_0^{\frac{1}{2}\pi} \frac{\cos \theta}{1 - \sin^2 \gamma \sin^2 \theta} [\mathfrak{P}_3(\mu)]^2 P_8(\mu) d\theta.$$

In these integrals

$$P_8(\mu) = \frac{1}{128} [6435 \sin^8 \theta - 12012 \sin^6 \theta + 6930 \sin^4 \theta - 1260 \sin^2 \theta + 35]$$

$$[\mathfrak{P}_3(\mu)]^2 = \alpha - \beta \cos^2 \theta + \gamma \cos^4 \theta - \delta \cos^6 \theta,$$

where $\alpha, \beta, \gamma, \delta$ have known numerical values.

The integrations may of course be effected rigorously, but it seemed far easier to determine them by quadratures. I therefore computed the values of the functions to

be integrated for $\theta = 0, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ$, drew curves on squared paper, and counted the squares on the positive and negative sides of the axis.

In this way I find $\log \omega_8 = 6.765$, $\log \rho_8 = 5.653$.

The integral ϕ_8 is found at once from § 22 of "Harmonics" with $\beta = 0$. This gives $\phi_8 = \frac{3}{17}$, or $\log \phi_8 = 9.247$.

If $P_8(\mu)$ be expressed in terms of cosines of θ we have

$$P_8(\mu) = a - b \cos^2 \theta + c \cos^4 \theta - d \cos^6 \theta + e \cos^8 \theta,$$

where $a = 1$, $b = 18$, $c = 74.25$, $d = 107.25$, $e = 50.273$.

Then we may, as in § 18, put

$$u_\nu = P_8(\nu) = a + b \cot^2 \chi + c \cot^4 \chi + d \cot^6 \chi + e \cot^8 \chi.$$

As was done in that section, I then computed u_{12} and u_{11} , and so found the integral of the empirical function. The result gave

$$\log \mathfrak{A}_8 = 9.191; \quad \text{whence} \quad \log \mathfrak{B}_8 = .370.$$

It may be admitted that the determination of $\mathfrak{A}_8$, $\mathfrak{B}_8$ is not wholly consistent with that of the previous integrals, since I only assume κ to be unity in as far as the values of a , b , c , d , e are affected.

Applying these values as before, I find $\mathfrak{A}_3 - \mathfrak{A}_8 = .197$, $\log C_8 = 8.540$, $B_8 = .000092$, $\frac{B_8}{C_8} = .0027$, and

$$[8, 0] = .00000051, \quad \frac{(B_8)^2}{C_8} = .00000025.$$

Hence that part of ϵ (the uncomputed residue of the series) which depends on the eight zonal harmonic is only about .0000008. The contribution is so insignificant compared with the critical total .00014, that I have not thought it worth while to make estimates for the tenth and twelfth harmonics.

It may then be confidently asserted that the pear is stable.

In the course of this estimate we have also found $f_8 = \frac{B_8}{C_8} e^2 = .0027 e^2$.

§ 20. Second Approximation to the Form of the Pear.

Extracting the numerical values of the f 's from our results, we find that the inequality of the critical Jacobian ellipsoid is

$$eS_3 + e^3 [.15068 S_2 + .50839 S_2^2 + .07705 S_4 - .000506 S_4^2 + .00000019 S_4^4 \\ + .01852 S_6 - .000278 S_6^2 + .00000034 S_6^4 - ? S_6^6 + .0027 S_8 - \dots].$$

In order to give this expression a clear meaning, it is well to define the several S 's.

$$S_3 = (\kappa^2 \sin^3 \theta - q^2 \sin \theta) (q'^2 - \kappa'^2 \cos^2 \phi) \sqrt{(1 - \kappa'^2 \cos^2 \phi)},$$

where $\kappa^2 = \cdot 923128, \quad q^2 = \cdot 574647$
 $\kappa'^2 = \cdot 076872, \quad q'^2 = \cdot 425353.$

For the other harmonics we have

$$S_i^s = (a - b \cos^2 \theta + c \cos^4 \theta - d \cos^6 \theta + \dots) (a' + b' \sin^2 \phi + c' \sin^4 \phi + d' \sin^6 \phi + \dots),$$

where the values of $a, b, \&c., a', b', \&c.$ are as given in the following table :—

<i>i.</i>	<i>s.</i>	<i>a.</i>	<i>b.</i>	<i>c.</i>	<i>d.</i>	<i>e.</i>
2	0	·603374	·923128			
2	2	−·039203	·923128			
4	0	1	5·450	4·901		
4	2	−1·7988	36·006	44·805		
4	4	·0839	−7·975	95·574		
6	0	1	12·6	29·984	18·834	
6	2	−8·4	121·8	439·425	320·523	
6	4	3·78	−338·312	3680·303	4482·844	
8	0	1	18	74·25	107·25	50·273

<i>i.</i>	<i>s.</i>	<i>a'.</i>	<i>b'.</i>	<i>c'.</i>	<i>d'.</i>	<i>e'.</i>
2	0	·603374	·076872			
2	2	−·039203	·076872			
4	0	·8036	·3718	·0280		
4	2	−1·0666	1·8135	·1865		
4	4	1·0136	−8·0266	8·		
6	0	·5989	·6888	·1512	·0000	
6	2	−1·1408	1·5349	·4404	·0264	
6	4	1·0305	−7·704	6·944	·704	
8	0	1	0	0	0	0

The surface of the pear is determined by measuring a certain length along the arc of curves orthogonal to the surface of the ellipsoid. By equation (22) it appears that that length measured in the direction of the positive normal is

$$p_0 \left[eS_3 + \Sigma f_i^s S_i^s + \frac{1}{2} e^2 \frac{\cos^2 \beta \cos^2 \gamma}{\Delta_1^2 \Gamma_1^2} \left(\frac{1}{\Delta_1^2} + \frac{1}{\Gamma_1^2} - 2G \right) (S_3)^2 \right].$$

In order to construct a figure it will be convenient to adopt as unit of length c , the greatest axis of the ellipsoid which is deformed. We know that

$c = \frac{k}{\sin \beta}$, $b = k \cot \beta$, $a = \frac{k \cos \gamma}{\sin \beta}$, so that $b = c \cos \beta$, $a = c \cos \gamma$, and the mass of

the ellipsoid is $\frac{4}{3} \pi \rho c^3 \cos \beta \cos \gamma$. But since the mass of the pear is $\frac{4}{3} \pi \rho k_0^3 \frac{\cos \beta \cos \gamma}{\sin^3 \beta}$

where $k_0^3 = k^3 (1 + e^2 \sigma_2)$, it follows that it is

$$\frac{4}{3} \pi \rho c^3 \cos \beta \cos \gamma (1 + .0136866 e^2).$$

Hence the mass of the pear is a little greater than that of the ellipsoid whose deformations we shall draw, and the protuberances above the surface slightly exceed in volume the depressions below it.

We have
$$p_0 = \frac{e \cos \beta \cos \gamma}{\Delta_1 \Gamma_1} = \frac{e \cos \beta \cos \gamma}{(1 - \sin^2 \beta \sin^2 \theta)^{\frac{1}{2}} (\cos^2 \gamma + \kappa'^2 \sin^2 \gamma \cos^2 \phi)^{\frac{1}{2}}},$$

and the expression for the orthogonal arc, measured from the ellipsoid to the pear, is therefore

$$p_0 \left[eS_3 + \left(\frac{p_0 e S_3}{c} \right)^2 \left\{ \frac{1}{2(1 - \sin^2 \beta \sin^2 \theta)} + \frac{1}{2(\cos^2 \gamma + \kappa'^2 \sin^2 \theta \cos^2 \phi)} - \frac{1}{2}(1 + \sec^2 \beta + \sec^2 \gamma) \right\} + \Sigma f_i^s S_i^s \right].$$

It appears to me that it will afford a sufficient idea of the corrected form of surface if I draw two principal sections, namely, first, a section through the axis of rotation and the longest axis of the ellipsoid, and, secondly, a section at right angles to the axis of rotation. It is not worth while to consider the third section drawn through the axis of rotation and the mean axis of the ellipsoid, since it will hardly differ sensibly from the uppermost figure shown in the "Pear-shaped Figure."

For the sake of brevity I will call the first and second sections the meridian and the equator.

The three ellipsoidal co-ordinates ν , θ , ϕ of any point are connected with x , y , z by the relationships

$$\begin{aligned} x &= c \sin \gamma \cdot (\kappa^2 \nu^2 - 1)^{\frac{1}{2}} (1 - \kappa^2 \sin^2 \theta)^{\frac{1}{2}} \cos \phi, \\ y &= c \sin \gamma \cdot \kappa (\nu^2 - 1)^{\frac{1}{2}} \cos \theta \sin \phi, \\ z &= c \sin \gamma \cdot \kappa \nu \sin \theta (1 - \kappa'^2 \cos^2 \phi)^{\frac{1}{2}}. \end{aligned}$$

The equation to the surface of the ellipsoid is $\nu = \frac{1}{\kappa \sin \gamma} = \frac{1}{\sin \beta}$.

The equation to the meridian plane in rectangular co-ordinates is simply $y = 0$, that to the equator is $x = 0$.

In ellipsoidal co-ordinates the equation to the equator is simply $\phi = \frac{1}{2}\pi$, but the equation to the meridian is peculiar, for it is in part represented by $\theta = \frac{1}{2}\pi$ and in part by $\phi = 0$.

The curve $\theta = \frac{1}{2}\pi$, $\phi = 0$, which defines the limit between the two regions where the equation to the plane has different forms, is clearly the hyperbola

$$z^2 - \frac{x^2}{\kappa'^2} = c^2 \sin^2 \gamma.$$

In the region from $z = \infty$ and x small down to this hyperbola the equation is $\theta = \frac{1}{2}\pi$; and between the origin and the hyperbola it is $\phi = 0$.

If we follow the arc of the ellipse from the extremity of the c axis we begin with $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$, and θ remains constant whilst ϕ falls to zero. Then ϕ maintains a constant zero value whilst θ falls from $\frac{1}{2}\pi$ to zero.

On the side of the origin where z is negative, θ is of course negative and undergoes parallel changes.

The hyperbola $\theta = \frac{1}{2}\pi$, $\phi = 0$ cuts the ellipsoid so near to the extremities of the c axis that an adequate idea of the deformation may be derived from the two extreme values of ϕ , namely, $\frac{1}{2}\pi$ and 0. I have also thought it sufficient to compute the deformations for $\theta = 0, 30^\circ, 60^\circ, 90^\circ$. We thus obtain the following scheme of values of θ, ϕ , together with the corresponding rectangular co-ordinates (with c taken as unity), at which to compute the deformation:—

Meridian ($y=0$).		Equator ($x=0$).	
$\theta = 90^\circ$, $\phi = 90^\circ$;	$z = 1, \quad x = 0$	$\theta = 90^\circ$, $\phi = 90^\circ$;	$z = 1, \quad y = 0$
$\theta = 90^\circ$, $\phi = 0$;	$z = \cdot 961, x = \cdot 096$	$\theta = 60^\circ$, $\phi = 90^\circ$;	$z = \cdot 866, y = \cdot 216$
$\theta = 60^\circ$, $\phi = 0$;	$z = \cdot 832, x = \cdot 191$	$\theta = 30^\circ$, $\phi = 90^\circ$;	$z = \cdot 5, \quad y = \cdot 374$
$\theta = 30^\circ$, $\phi = 0$;	$z = \cdot 480, x = \cdot 303$	$\theta = 0^\circ$, $\phi = 90^\circ$;	$z = 0, \quad y = \cdot 432$
$\theta = 0$, $\phi = 0$;	$z = 0, \quad x = \cdot 345$		

It did not seem to be worth while to compute the deformations due to the eighth zonal harmonic, since it would be quite impossible to show them on a drawing of any reasonable scale.

In order to exhibit the magnitudes of the contributions of the harmonics of the several orders, I give the normal departures δn at the points $z = \pm 1, x = 0, y = 0$.

Term of first order . . .	S_3	$\pm \cdot 148227e$	
Terms of second order proportional to e^2	$(S_3)^2$		$-\cdot 010986$
	S_2^2	$\cdot 061844$	
	S_2^2		$\cdot 000751$
	S_4	$\cdot 092715$	
	S_4^2	$\cdot 000849$	
	S_4^4	$\cdot 000000$	
	S_6	$\cdot 026647$	
	S_6^2	$\cdot 002011$	
	S_6^4	$\cdot 000001$	
		$\cdot 184067$ $-\cdot 011737$	$-\cdot 011737$
		$\cdot 172330e^2$	

The following are then the results for the normal departures at the several points whose rectangular co-ordinates are specified :—

Meridian ($y=0$).

$$\begin{aligned}
 z = \pm 1, \quad x = 0, \quad \delta n &= \pm \cdot 1482e + \cdot 1723e^2. \\
 z = \pm \cdot 961, \quad x = \cdot 096, \quad \delta n &= \pm \cdot 0932e + \cdot 0858e^2. \\
 z = \pm \cdot 832, \quad x = \cdot 191, \quad \delta n &= \pm \cdot 0189e + \cdot 0103e^2. \\
 z = \pm \cdot 480, \quad x = \cdot 303, \quad \delta n &= \mp \cdot 0223e - \cdot 0033e^2. \\
 z = 0, \quad x = \cdot 345, \quad \delta n &= + \cdot 0046e^2.
 \end{aligned}$$

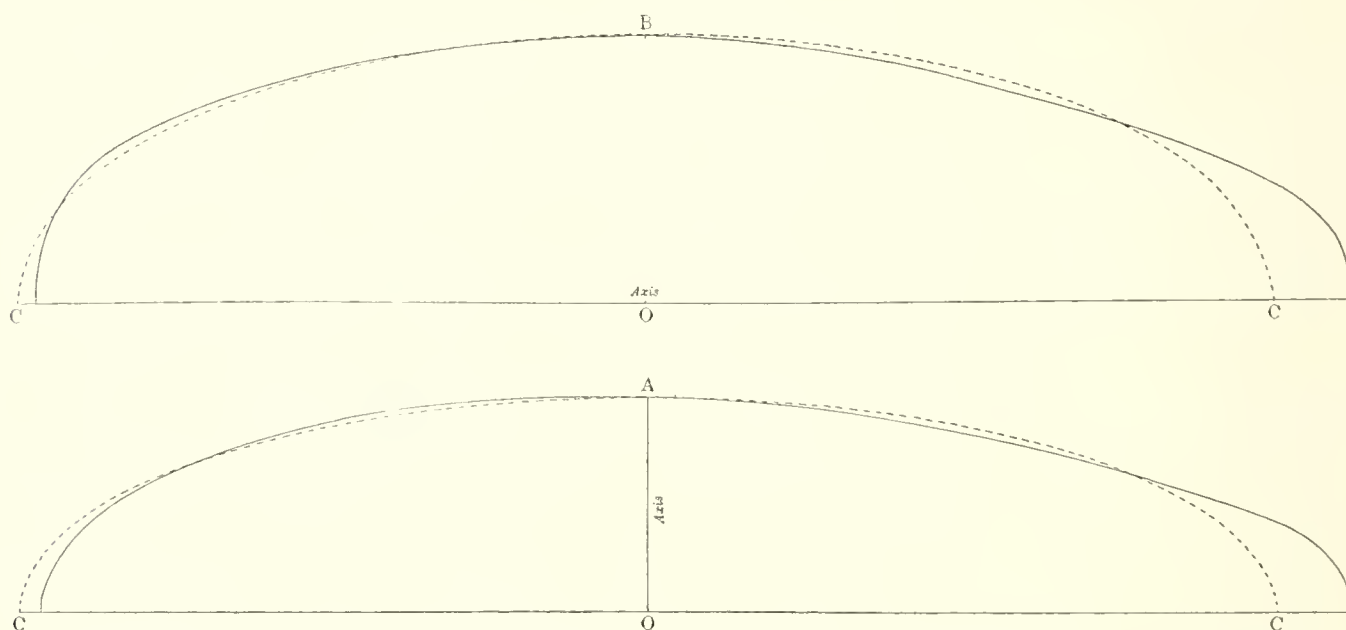
Equator ($r=0$).

$$\begin{aligned}
 z = \pm 1, \quad y = 0, \quad \delta n &= \pm \cdot 1482e + \cdot 1723e^2. \\
 z = \pm \cdot 866, \quad y = \cdot 216, \quad \delta n &= \pm \cdot 0300e + \cdot 1265e^2. \\
 z = \pm \cdot 5, \quad y = \cdot 374, \quad \delta n &= \mp \cdot 0354e - \cdot 0220e^2. \\
 z = 0, \quad y = \cdot 432, \quad \delta n &= - \cdot 0095e^2.
 \end{aligned}$$

In order to draw a figure I take $e = \frac{1}{2}$. Throughout most of the arc of the ellipsoid the approximation is probably good, but at the vertices, which are just the points of most interest, it is pretty clear that we are using a somewhat extreme value for e . The results are :—

Meridian ($y=0$).				Equator ($x=0$).			
$z =$	1,	$x = 0,$	$\delta n = + \cdot 117.$	$z =$	1,	$y = 0,$	$\delta n = + \cdot 117.$
$z =$	$\cdot 96,$	$x = \cdot 096,$	$\delta n = + \cdot 068.$	$z =$	$\cdot 866,$	$y = \cdot 216,$	$\delta n = + \cdot 047.$
$z =$	$\cdot 83,$	$x = \cdot 19,$	$\delta n = + \cdot 012.$	$z =$	$\cdot 5,$	$y = \cdot 374,$	$\delta n = - \cdot 014.$
$z =$	$\cdot 48,$	$x = \cdot 30,$	$\delta n = - \cdot 011.$	$z =$	0,	$y = \cdot 432,$	$\delta n = - \cdot 002.$
$z =$	0,	$x = \cdot 345,$	$\delta n = + \cdot 001.$	$z = -$	$\cdot 5,$	$y = \cdot 374,$	$\delta n = + \cdot 003.$
$z = -$	$\cdot 48,$	$x = \cdot 30,$	$\delta n = + \cdot 010.$	$z = -$	$\cdot 866,$	$y = \cdot 216,$	$\delta n = + \cdot 017.$
$z = -$	$\cdot 83,$	$x = \cdot 19,$	$\delta n = - \cdot 007.$	$z = -$	1,	$y = 0,$	$\delta n = - \cdot 031.$
$z = -$	$\cdot 96,$	$x = \cdot 096,$	$\delta n = - \cdot 025.$	<i>N.B.—For $z = \pm \cdot 866$, δn is in both cases positive.</i>			
$z = -$	1,	$x = 0,$	$\delta n = - \cdot 031.$				

These numbers are set out graphically in the annexed figure. It will be noticed that whereas the protuberance at the positive end of the z axis is great, the



Second approximation to Pear-shaped Figure. Upper section "equatorial," lower "meridional."

deficiency at the negative end is almost filled up. We may describe the general effect by saying that the Jacobian ellipsoid is very little changed, excepting at one end of its longest axis, where it shoots forth a protuberance.

SUMMARY.

If a mass of liquid be rotating like a rigid body with uniform angular velocity, the determination of the figure of equilibrium may be treated as a statical problem, if the mass be subjected to a rotation potential.

The energy, say W , lost in the concentration of a body from a condition of infinite dispersion is equal to the potential of the body in its final configuration at the

position of each molecule, multiplied by the mass of the molecule and summed throughout the body. In the system, as rendered statical, it is necessary to add the rotation-potential to the gravitation potential before effecting the summation. That portion, say T , of the whole lost energy which arises from the rotation-potential is simply the same thing as the kinetic energy of the mass, when the system is regarded as a dynamical one. If we replace $W + T$ by E to denote the whole lost energy of the statical system, the condition that the surface shall be in equilibrium is that the variations of E for constant angular velocity shall be stationary. E must then be a maximum or a minimum, or a maximum for some variations and a minimum for others.

It might appear at first sight that the condition for the secular stability of the figure is that E should be a maximum for all variations, and this is so if certain constraints are introduced; but in the absence of such constraints the figure may be stable although E is a minimax.

It has been shown by M. POINCARÉ that the stability must be determined from the variations, subject to constancy of angular momentum, of the total energy of the system, both kinetic and potential. The two portions of the total energy, say U , are again W and T ; but whereas E involves the lost energy W of the system under the action of the gravitation potential, U involves the potential energy which is equal to $-W$. Thus U is equal to $-W + T$.

The variation of U with constant angular momentum leads to results for the determination of the figure identical with those found from the variation of E with constant angular velocity. But there is this important difference, that to insure secular stability U must be an absolute minimum. It appears, in fact, that, in the case of the pear-shaped figure, while E is actually a maximum for all the deformations but one, it is a minimum for that one, which consists of an ellipsoidal strain of the critical Jacobian ellipsoid from which the pear-shaped figures bifurcate (§ 19).

But M. POINCARÉ has adduced another consideration which enables us to determine the stability of the pear by means of the function E , without a direct proof that U is a minimum for all variations. For he has shown that if for given angular momentum slightly less than that of the critical Jacobian ellipsoid, the only possible figure is the Jacobian, and if for slightly greater angular momentum there are two figures (namely, the Jacobian and the pear*), then exchange of stability between the two series must occur at the bifurcation. If, on the other hand, the smaller momentum corresponds with the two figures and the larger with only one, one of the two coalescent series must be stable and the other unstable. Now it has been proved that the less elongated Jacobian ellipsoids are stable, so that if the first alternative holds the stability must pass from the Jacobian series to the pear series; and if the second alternative holds the pear series must be unstable throughout. The question

* For the sake of simplicity we may speak of a single pear, instead of two similar pears in azimuths 180° apart.

of stability is then completely determined by means of the angular momentum of the pear; if it is greater than that of the critical Jacobian the pear is stable, and, if less, unstable.

It suffices then to determine the figure by means of the variations of E with constant angular velocity, and afterwards to evaluate the angular momentum.

It was proved by M. POINCARÉ, and repeated by me in my previous paper, that the first approximation to the pear-shaped figure is given by the third zonal harmonic inequality of the critical Jacobian ellipsoid—zonal with respect to its longest axis. In proceeding to the higher approximation I suppose that the amplitude of the third zonal harmonic is measured by a parameter e , which is to be regarded as a quantity of the first order. We must now also suppose the ellipsoid to be deformed by all and any other harmonics, but with amplitudes of order e^2 . In the first approximation the lost energy W is proportional to e^2 , but it now becomes necessary to determine W as far as the order e^4 . A change in the sign of e means that the figure of equilibrium is rotated in azimuth through 180° . Such a rotation cannot affect the value of the energy, and it thus becomes obvious that the odd powers of e must be absent from the expression for W . We have further to find the moment of inertia of the body as far as the terms of order e^2 , and thence to find the kinetic energy T . The function E is equal to $W + T$.

In order to attain the requisite degree of accuracy, it is convenient to regard the pear as being built up in an artificial manner. I construct an ellipsoid similar to and concentric with the critical Jacobian, and therefore itself possessing the same character. The size of this new Jacobian, which I call J , is undefined, and is subject only to the condition that it shall be large enough to enclose the whole pear. The regions between J and the pear being called R , I suppose the pear to consist of positive density throughout J and negative density throughout R (§ 1).

The lost energy of the pear consists of that of J with itself, say $\frac{1}{2}JJ$; of J with R , which is filled with negative density, say $-JR$; and of R with itself, say $\frac{1}{2}RR$. This last contribution to the energy must be broken into several portions. It was the evaluation of $\frac{1}{2}RR$ which baffled me, until M. POINCARÉ'S solution came to my help.

If we imagine the ellipsoid J to be intersected by a family of orthogonal quadrics, and if we suppose for the moment that the region R is filled with positive density, we may further imagine the matter lying inside any orthogonal tube to be transported along the tube, and to be deposited on the surface of J in the form of a concentration of positive surface density $+C$. The mass of $+C$ is equal to that of $+R$, but it is differently arranged. In the actual system R is filled with negative volume density, and we may clearly add to this two equal and opposite surface densities $+C$ and $-C$ on J .

Thus the matter lying in the region R may be regarded as consisting of negative surface density $-C$ on J , together with a double system, namely negative volume

density $-R$ in conjunction with equal and opposite surface density $+C$. This double system, say D , is therefore $C - R$. The lost energy $\frac{1}{2}RR$ may be considered as consisting of three parts; first the energy of $-C$ with itself, say $\frac{1}{2}CC$; secondly that of D with itself, say $\frac{1}{2}DD$; thirdly that of $-C$ with D . This third item is obviously equal to $-CC + CR$, and therefore $\frac{1}{2}RR$ is equal to $-\frac{1}{2}CC + CR + \frac{1}{2}DD$.

It follows that the gravitational lost energy of the pear may be written symbolically in the form

$$\frac{1}{2}JJ - JR + CR - \frac{1}{2}CC + \frac{1}{2}DD.$$

In this discussion no attention has as yet been paid to the rotation, but fortunately it happens that the introduction of this consideration actually simplifies the problem, for if we suppose $\frac{1}{2}JJ$ and JR to mean the lost energies of J with itself and with R on the supposition that the mass is rotating with the angular velocity of the critical Jacobian, the formulæ become much more tractable than would have been the case otherwise.

The inclusion of part of the angular velocity in this portion of the function E , only leaves outstanding the excess of the kinetic energy of the pear above the kinetic energy, which it would have if it rotated with the angular velocity of the critical Jacobian. If ω denotes the latter angular velocity, and $(\omega^2 + \delta\omega^2)^{\frac{1}{2}}$ the actual angular velocity of the pear; if A_j be the moment of inertia of J , and A_r that of R considered as filled with positive density, we have

$$E = \frac{1}{2}JJ - JR + CR - \frac{1}{2}CC + \frac{1}{2}DD + \frac{1}{2}(A_j - A_r)\delta\omega^2.$$

In this statement I have omitted a term which arises from the displacement of the centre of inertia from the centre of the ellipsoid; it is duly considered in the paper, but is shown to vanish to the requisite order of approximation (§§ 2, 14).

The co-ordinates of points are determined by reference to the ellipsoid J , which envelopes the whole pear, and the formula for the internal gravitation of J , inclusive of the rotation ω , is of a simple character. The size of J is indeterminate, and therefore the formulæ must involve an arbitrary constant expressive of the size of J . But the final result E cannot in any way depend on the size of the ellipsoid which is chosen as a basis for measurement, and therefore this arbitrary constant must ultimately disappear. Hence it is justifiable to treat it as zero from the beginning. It appears then that we are justified in using the formula for internal gravity throughout the investigation. If the artifice of the enveloping ellipsoid had not been adopted, it would have been necessary to take note of the fact that the pear is in part protuberant above and in part depressed below the ellipsoid of reference. M. POINCARÉ did follow this last plan, and then proceeded to prove the justifiability of using the formula for internal gravity throughout. The argument adduced above seems, however, sufficient to prove the point.

Although the constant expressive of the size of J is put equal to zero—which means that the pear is really partly protuberant above the ellipsoid—I have found that a considerable amount of mental convenience results from always discussing the subject as though the constant were not zero, so that the ellipsoid envelopes the pear, and I shall continue to do so here.

When an ellipsoid is deformed by an harmonic inequality, the volume of the deformed body is only equal to that of the ellipsoid to the first order of small quantities. In the case of the pear, all the inequalities, excepting the third zonal one, are of the second order, and as far as concerns them the volumes of J and of the pear are the same. But it is otherwise as regards the third zonal harmonic term, and the first task is to find the volume of such an inequality as far as c^2 . When this is done we can express the volume of J in terms of that of the pear, which is, of course, a constant (§§ 3, 4).

By aid of ellipsoidal harmonic analysis we may now express the first four terms of E in terms of the mass of the pear, and of certain definite integrals which depend on the shape of the critical Jacobian ellipsoid (§§ 5, 6, 7).

The energy $\frac{1}{2}DD$ presents much more difficulty, and it is especially in this that M. POINCARÉ'S insight and skill have been shown. The system D consists of a layer of negative volume density, coated on its outer surface with a layer of surface density of equal and opposite mass.

Two surfaces, infinitely near to one another, coated with equal and opposite surface densities, form together a magnetic layer or a layer of doublets. The change of potential in crossing such a layer is 4π times the magnetic moment at the point of crossing, and is independent of the form of surface. To find the difference between the potential at two points at a finite distance apart, one being on one side and the other on the other side of the layer, we have to add to the preceding difference a term equal to the force on either side of the magnetic layer multiplied by the distance between the two points. This additional term is small compared with that involving the magnetic moment, provided that the distance is small. If the magnetic layer coincided with the surface of an ellipsoid the force in question would be exactly calculable, and if it lies on the surface of a slightly deformed ellipsoid the force remains unchanged by the deformation as a first approximation.

Thus it follows that it is possible to calculate the difference of potential at two points lying on a curve orthogonal to an ellipsoid, when one point is on one side and the other on the other side of a magnetic layer residing on a deformation of the ellipsoid. Further, if the two points lie on the same side of the magnetic layer the term dependent on magnetic moment (which would represent the crossing of the layer) disappears, and only the term dependent on the force remains.

Two equal and opposite layers of matter at a finite distance apart may be built up from an infinite number of magnetic layers interposed between the two surfaces. Hence by the integration of the result for a magnetic layer we may find the change

of potential in passing from any one point to any other lying on the same orthogonal curve in the neighbourhood of a finite double layer.

Again, the system D , consisting of $-R$ and $+C$, may be built up by an infinite number of finite double layers. Hence by a second integration we may find the difference between the potential of D at any point inside R and the point lying on J where the orthogonal curve through the first point cuts the surface of J .

Finally, it may be proved that the lost energy $\frac{1}{2}DD$ is equal to half the difference of potentials just determined multiplied by the density and integrated throughout the region R . The required expression of this portion of the energy is found to consist of two parts, of which one depends on magnetic moment and the other on the force (§ 9). The reduction of this part of the energy to calculable forms is not very simple; it is carried out in §§ 11, 12.

The calculation of the moment of inertia of the pear is comparatively easy, since it only involves those harmonic inequalities of J which are expressible by harmonics of the second degree (§ 13). On multiplying the moment of inertia by $\frac{1}{2}\delta\omega^2$, we obtain the last contribution to the expression for E .

The energy function cannot involve e^2 , since the vanishing of the coefficient of that term is the condition whence the critical Jacobian was determined. If f denotes the coefficient of any harmonic inequality other than the third zonal one, the part of E independent of $\delta\omega^2$ is found to contain terms in e^2 , e^2f and $(f)^2$. The coefficient of $\delta\omega^2$ consists of a constant term, a term in e^2 and terms in f_2 and f_2^2 , where these f 's denote the coefficients of the second zonal and sectorial harmonics. This last part does not contain the coefficient of any harmonic of odd degree, and in the first part the coefficient of the term in e^2f for all such harmonics is found to vanish.

The condition for the figure of equilibrium is that the variations of E for variations of e^2 and of each f shall vanish. On differentiating E with respect to the f of any harmonic of odd degree and equating the result to zero, we see that that f must vanish. Hence it follows that the pear cannot involve any odd harmonic excepting the third zonal one. Again, the symmetry of the figure negatives the existence of any even functions involving sine-functions of the quasi-longitude measured from the prime meridian (as I may call it) of symmetry through the axis of rotation. The same consideration negatives the existence of even functions involving cosine functions of odd rank. Accordingly the only functions to be considered are the even ones of even rank, involving the cosine functions of the longitude.

The equation to zero of the variations of E for all the f 's, excepting f_2, f_2^2 , gives at once all those f 's in terms of e^2 . The equations to zero of the variations for e^2, f_2, f_2^2 give three equations for the determination of $\delta\omega^2, f_2, f_2^2$ as multiples of e^2 . We thus have the means of finding the angular velocity and all the f 's in terms of the parameter e , which measures the amount of departure of the pear from the critical Jacobian ellipsoid (§ 14).

It seems unnecessary to give here any explanation of the methods adopted for

reducing the analytical results to numbers, and it may suffice to say that the task proved to be a very laborious one.

The harmonic terms included in the computation were those of degree 2 and ranks 0 and 2, of degree 4 and ranks 0, 2, 4, and of degree 6 and ranks 0, 2, 4. The sixth sectorial harmonic was omitted because its contribution would certainly prove negligible.

The expression for $\delta\omega^2$ was found in the form of a fraction, of which the denominator is determinate and the numerator consists of the sum of an infinite series. Nine terms of this series were computed, namely, a constant term and the contribution of the eight harmonic terms specified above. I found, in fact, that it would only change the numerator by about one-twentieth part of itself, if all the harmonics excepting the zonal ones of degrees 2, 4, 6 had been dropped.

The result shows that the square of the angular velocity of the pear is less than that of the critical Jacobian ellipsoid in about the proportion to $1 - \frac{1}{8}e^2$ to 1. On the other hand the angular momentum of the pear is greater than that of the ellipsoid in about the proportion of $1 + \frac{1}{15}e^2$ to 1. If this last result were based on a rigorous summation of the infinite series, it would, in accordance with the principle explained above, absolutely prove the stability of the pear. The inclusion of the uncomputed residue of the series would undoubtedly tend in the direction of reducing the coefficient given in round numbers as $\frac{1}{15}$, and if it were to reduce it to a negative quantity, we should conclude that the pear was unstable after all. The apparently rapid convergence of the series seemed to render it almost incredible that the inclusion of the residue could bring about such a reversal of our conclusion, yet I thought it advisable to make a rough estimate of the amount of change which would arise from the contribution of the eighth zonal harmonic.

The contribution of the sixth zonal harmonic to the series above referred to was about .00006, and I find that if the contribution of the uncomputed residue should amount to .00014, the apparent stability of the pear would be just reversed. Now my estimate of the contribution of the eighth zonal harmonic to the same sum is .0000008, or only $\frac{1}{175}$ th of the critical amount.

Since the convergency of the series is obviously very rapid, it is wholly incredible that the inclusion of the uncomputed residue could materially alter, much less reverse our result. I regard it then as proved, but by something short of an absolute algebraic argument, that the pear-shaped figure is stable.

The numbers obtained in the course of the determination of the stability afford the means of giving a second approximation to the form of the pear. The result is shown graphically in the figure of § 20, where the largest value of e is adopted which seemed to secure a fair degree of approximation in the result. I originally called the figure "pear-shaped," because M. POINCARÉ's conjectural sketch in the "*Acta Mathematica*" was very like a pear. In the first approximation, shown in my former paper, the resemblance to a pear was not striking, and it needs some imagina-

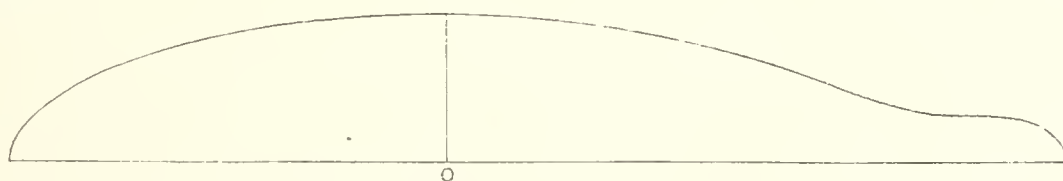
tion to recognise the pear shape in the second approximation shown here ; but a distinctive name is so convenient that we may as well continue to call it by that name.

The effects of the new terms now added are almost entirely concentrated at the two ends. All these terms, excepting a very small one arising from the second sectorial harmonic, tend to augment the protuberance at the stalk and to fill up the depression at the blunt end. It is true that there is a small term, arising from the square of the third zonal harmonic, which diminishes the protuberance and increases the depression, but this cannot be regarded as a new term, since it only represents the effect of the fundamental harmonic carried to the second order of small quantities.

The new zonal harmonics furnish by far the most important contributions. The second zonal harmonic denotes that the ellipsoid most nearly resembling the pear is longer and less broad than the Jacobian. The largest contribution of all is that due to the fourth zonal harmonic, and this may be regarded as the octave of the second zonal term. A rough estimate shows that the eighth harmonic, or the double octave of the second, is still sensible. The sixth harmonic is the octave of the fundamental third zonal harmonic, and is the last of the three important terms.

The general effect is that the protuberance at the stalk of the pear is much increased, and the depression at the other end nearly filled up. Over the greater part of the whole surface the depressions and protuberances are less conspicuous than they were. The nodal lines where the surface of the pear cuts that of the ellipsoid are entirely shifted from their former positions. It did not seem worth while to attempt to specify their new positions, because the choice of the ellipsoid to which we refer influences the result so largely. The ellipsoid on which these figures are constructed is that which is called *J* in this summary. Its volume is a little less than that of the pear, so that the protuberances are a little greater in volume than the depressions.

I think it is hardly too much to say, that in a well-developed "pear" the Jacobian ellipsoid has nearly regained its primitive figure, but subject to a small tidal distortion due to the attraction of a protuberance which it shoots forth at one end. I venture to give here a conjectural sketch of a further stage of the development.



Conjectural Sketch.

If we look at this figure and at those drawn by Mr. JEANS in his striking investigation of the parallel changes in the shape of an infinite rotating cylinder

(*supra*, p. 67), we can hardly fail to be reminded of some such phenomenon as the protrusion of a filament of protoplasm from a mass of living matter.

Notwithstanding the *caveat* which M. POINCARÉ enters as to the dangers of applying these results to heterogeneous masses and to cosmogony, I cannot restrain myself from joining him in seeing in this almost life-like process a counterpart to at least one form of the birth of double stars, planets, and satellites.

NOTE.—*Erratum* in “Ellipsoidal Harmonic Analysis,” ‘Phil. Trans.’ A, vol. 197, p. 512, line 4 from foot. The first term inside the bracket should be negative. The mistake runs on and the same correction should be made in equations (62), (63), and (64), and in line 9 on the following page.

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VIII. *On the Movements of the Flame in the Explosion of Gases.*

By HAROLD B. DIXON, M.A., F.R.S., *Professor of Chemistry in the Owens College, Manchester.*

Received 5 June,—Read 5 June, 1902.

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PART I.

HISTORICAL INTRODUCTION.

(1.) *On the Rate of Movement of the Flames, and the Pressures produced in the Explosion of Gases.*

HUMPHRY DAVY* was the first to observe the rate at which an explosion of gases was propagated in a tube, and he also made the first rough experiment on the temperature reached in an explosion. When gas from the distillation of coal (which he found more inflammable than fire-damp) was mixed with eight times its volume of air, and was fired in a glass tube 1 foot long and $\frac{1}{4}$ inch in diameter, the flame took more than a second to traverse the tube. When cyanogen mixed with twice its volume of oxygen was fired in a bent tube over water, the quantity of water displaced showed that the gases had expanded fifteen times their original bulk.†

BUNSEN,‡ in 1867, made the first careful measurement of the rate at which an explosion is propagated in gases, and he also made the first systematic researches on the pressure and temperature produced by the explosion of gases in closed vessels. His results led him to the remarkable conclusion that there was a discontinuous combustion in explosions. When electrolytic gas, or when carbonic oxide with half its volume of oxygen, is fired, only one-third of the mixture is burnt, according to BUNSEN, raising the temperature of the whole to about 3000° C. No further chemical action then occurs until the gaseous mixture falls, by cooling, below 2500° C. Then

* ‘Phil. Trans.’ 1816; ‘Collected Works,’ vol. 6, p. 26.

† ‘Phil. Trans.’ 1817; ‘Collected Works,’ vol. 6, p. 73.

‡ ‘Ann. Phys. Chem.’ vol. 131, p. 161; ‘Phil. Mag.’ vol. 34, p. 489 (1867).

a further combustion begins, and so on *per saltum*. These deductions were criticised by BERTHELOT, who pointed out that they assumed the constancy of the specific heats of steam and of carbonic acid at high temperatures.

BUNSEN also in the same paper makes the important statement that the rapidity with which a flame spreads is synchronous with the attainment of the maximum temperature and with complete combustion. Having determined the rate at which the flame is propagated in pure electrolytic gas as 34 metres per second, BUNSEN finds that the time required to ignite the gas in his pressure-tube from a central spark through a radius of 8.5 millims. is $8.5/34,000 = 1/4000$ second. *We may conclude*, he says, *that the time which the whole of the gas takes to burn completely, and therefore also to reach the maximum temperature, is not more than 1/4000 part of a second*. BUNSEN thus identifies the rate of ignition with that of complete combustion. I do not know on what grounds BUNSEN based this statement, which I have found to be nearly true of the detonation-wave itself (*l'onde explosive* of BERTHELOT), but not of the initial periods of the explosion.

In 1881 BERTHELOT* and LE CHATELIER† independently discovered the great velocity with which the flame travels in gaseous explosions. BERTHELOT showed that this velocity was a constant for each gaseous mixture, and compared the rate of the detonation-wave (*l'onde explosive*) with the mean velocity of the molecules produced by the combustion before they had lost any heat. In the Bakerian Lecture for 1893 the author showed that BERTHELOT's theory did not account for many observed rates of explosion, and put forward the view that the "explosion-wave" (now called the detonation-wave) travelled with the velocity of sound in the burning gases. Using the rates determined by the author, D. L. CHAPMAN‡ has argued that, if the detonation-wave is of a permanent type, an equation can be deduced from RIEMANN's formula by which the rates of explosion can be calculated if the specific heats are known, and *vice versâ*. The rate of the detonation-wave may therefore be utilised, according to CHAPMAN, to determine the specific heats of gases at very high temperatures.

In 1883 MALLARD and LE CHATELIER§ published their researches on the combustion of gaseous explosive mixtures. They were the first to study the movements of the flame by a photographic method, of which I will speak more fully later on. In order to measure the pressures produced in the explosion, MALLARD and LE CHATELIER at first employed a very delicate Deprez indicator connected with the explosion bomb. A spring could be screwed up against the piston so that a certain pressure was required to move it. A thin metal tongue attached to a cord and weight was held in its place by the piston-rod; if the pressure on the piston equalled

* 'Comptes Rendus,' vol. 93, p. 18.

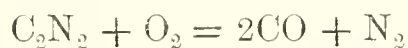
† 'Comptes Rendus,' vol. 93, p. 145.

‡ 'Phil. Mag.,' Jan., 1899, p. 90.

§ 'Annales des Mines' (8th-series), vol. 4, 1883.

the pressure of the spring the metal tongue moved; if not, it remained stationary. By successive trials the pressures produced in the bomb could thus be obtained. With this apparatus MALLARD and LE CHATELIER found that, with all rapidly exploding mixtures, very high pressures were often produced during very small intervals of time. These pressures were not always the same for the same mixture. These fugitive pressures are due, according to MALLARD and LE CHATELIER, to the compression-wave which is propagated as the inflammation spreads from layer to layer, and may become of enormous intensity in the detonation-wave itself. As their object was to measure the mean pressure in the whole mass of gas, they abandoned the delicate Deprez indicator and used a Bourdon gauge. From the curves of pressure so registered they obtained expressions for the rate of cooling of the products of combustion, and so calculated the maximum pressures and temperatures of the explosions. Their results may be summarised in the statement that the maximum temperature of explosion of moist electrolytic gas is 3350° and the mean specific heat of steam between this temperature and 0° is 16.6, dissociation being very slight, if any, between these limits; on the other hand, the mean specific heat of carbonic acid rises to 13.6 at 2000° , and above this temperature dissociation begins. The diatomic gases (O_2 , N_2 , CO , &c.) also show a rise of specific heat, though far less marked than steam or carbonic acid exhibits.

In 1885 BERTHELOT and VIEILLE* published their researches on the pressures produced in the explosion of gases. They measured the pressures by determining the maximum acceleration of a piston of known weight moved against gravity, making a correction for the cooling effect of the walls when a small explosion vessel was employed. From the maximum pressures they calculated the maximum temperatures, arriving at results for hydrogen and carbonic oxide mixtures similar to those obtained by BUNSEN and by MALLARD and LE CHATELIER. But whereas BUNSEN attributed the defect of pressure observed to the inability of two-thirds of the gases to combine at the temperature reached, the French chemists attribute this defect of pressure to the great increase of the specific heats of the gaseous products of combustion. By determining the pressure produced in the explosion of cyanogen with its own volume of oxygen—



—they calculate the maximum temperature as 4394° C., and the mean specific heat of CO and N_2 at constant volume between 0° and 4400° C. is found to be 9.6, which is just double the value found at ordinary temperatures. Similar determinations were made with cyanogen and the oxides of nitrogen. It is to be observed that BERTHELOT and VIEILLE assume that at the moment of maximum pressure the combustion is complete. In the same way BERTHELOT and VIEILLE, by determining the pressures produced in the explosion of hydrogen and carbonic oxide, obtain

* 'Annales de Chim. et Phys.' [VI.], vol. 4, p. 13 (1885).

values for the specific heats of steam and carbonic acid at high temperatures. For the purpose of comparison, the results obtained by MALLARD and LE CHATELIER, by BERTHELOT and VIEILLE, and those calculated by CHAPMAN are printed in parallel columns in the following table :—

TABLE I.—Mean Specific Heat of Diatomic Gases (N₂, &c.) at Constant Volume.

Temperature.	MALLARD and LE CHATELIER.	BERTHELOT and VIEILLE.	D. L. CHAPMAN.
0° C.	4·8	4·75	—
0°—1000°	5·4	4·75	—
0°—2000°	6·0	5·4	7·45
0°—3000°	6·6	7·0	7·56
0°—4000°	7·2	8·6	7·67
0°—5000°	7·8*	10·2	7·78
Mean Specific Heat of Steam.			
0°	5·6	—	—
0°—1000	8·9	—	—
0°—2000°	12·2	16·2	8·7
0°—3000	15·5	18·1	11·9
0°—4000°	18·7	20·0	14·3
Mean Specific Heat of CO ₂ .			
0°	6·3	6·4	—
0°—1000°	11·1	—	—
0°—2000°	13·6	19·1	—
0°—3000°	Dissociation	20·6	—

On the other hand, D. CLERK† contends that in an explosion of gases, combustion is not completed instantaneously, the products of combustion are cooling while the unburnt particles are still combining, and therefore the maximum temperature reached never coincides with the theoretical temperature calculated for instantaneous combustion. The spreading of the flame throughout the vessel, *i.e.*, the inflammation of the whole mass of gas, is not coincident with the moment of maximum temperature and pressure. On exploding mixtures of coal-gas and air, and of hydrogen and air, in a closed vessel, and registering the pressure with a RICHARDS' indicator, CLERK comes to the conclusion that neither dissociation of the products of combustion, nor a rise in the specific heats, can account for the fall of the observed pressures below those calculated. This fall must be accounted for by the gradual progress of the combustion.

Concerning the rise in the specific heat of carbonic acid there can, I think, be

* Direct experiments made by MALLARD and LE CHATELIER on the explosion of C₂N₂ + O₂ give the mean specific heat of CO and of N₂ at 4200° as 10, and at 2500° as 7·5. 'Recherches,' p. 276.

† 'Proc. Inst. Civil Engineers,' vol. 85 (1886).

little doubt that it really exists. REGNAULT and WIEDEMANN both found a distinct increase at constant pressure between 100°C . and 200°C . I have lately, in conjunction with Mr. F. W. RIXON, determined the specific heat of carbonic acid at constant volume at 100°C ., 200°C ., 300°C ., and 400°C ., and find a steady rise in the constant.* On the other hand, the facility with which carbonic acid breaks up under the influence of electric discharges indicates that, as its temperature is raised, work is done in loosening the chemical bonds—work which BERTHELOT calls “le travail de désagrégation moléculaire qui précède la décomposition.” Of the fact of the partial dissociation of carbonic acid and steam at the temperature of the detonation-wave, Mr. H. W. SMITH and I have found direct proof in the unburnt gases left on cooling.†

LIVEING and DEWAR‡ showed in 1884 that the flame of an explosion of gases in a glass vessel exhibited with marked brightness the lines of sodium and of calcium. When the explosion of electrolytic gas was made to pass along an iron tube about 3 feet long, closed at one end by a plate of quartz, a spectroscope placed in a line with the axis of the tube revealed the fact that the light was largely due to iron lines. When metallic salts in the form of powder were introduced into the explosion tube, the corresponding lines were visible in the spectroscope. These experiments showed how quickly the ignited gases could detach and volatilise solid matter from the side of the tube. But as regards the nature of the detonation-wave itself, the most interesting observation made by LIVEING and DEWAR was the reversal of the red lithium line when the explosion was made to travel *towards* the spectroscope. The natural interpretation of this reversal was that “there are gradations of temperature in the flame, and that the front of the advancing wave of explosion is somewhat cooler than the following part.” I would observe on this conclusion that in the many hundreds of photographs taken by me, the front of the detonation-wave is always shown as exceedingly sharp, and that probably the reversal observed was due to the wave reflected back from the quartz plate.

The light produced by the explosions of electrolytic gas is mainly due to particles knocked from the glass. In the faint continuous spectrum shown by the flame, the calcium lines stand out prominently. When the explosion travels first through a metal tube joined to a glass one in which the flame is photographed, the light is more intense near the junction. One can see the stream of luminous matter carried out of the metal tube, as in figs. 23 and 42 (lead) and 39 (copper).

The luminous particles, whatever their nature, follow very closely the movements of the gas in which they float.

The cyanogen explosions gave a continuous spectrum crossed by metallic lines and by the characteristic “cyanogen lines.”

* ‘Brit. Assoc. Report,’ 1900, p. 697.

† ‘Manchester Memoirs,’ 1889, p. 2.

‡ ‘Roy. Soc. Proc.,’ vol. 36, p. 471.

By the kindness of Dr. SCHUSTER I was enabled to use his apparatus to photograph the spectrum of the flame of cyanogen and oxygen as it exploded in a tube. With a single prism, a continuous spectrum was obtained showing a few bright lines. In order to get a detailed spectrum, the gases were fired in a gun-metal tube *end-on* to the spectroscope, the end of the tube next the spectroscope being open just before firing. By this means sufficient light was secured to employ considerable dispersion. A cadmium spark spectrum was taken on the same plate for comparison. No reversal of any line was shown on the photograph as in LIVEING and DEWAR's experiment where the explosion tube was closed by a quartz plate.

(2.) *Photographic Records of the Moving Flame.*

The discovery of the detonation-wave in gaseous mixtures by BERTHELOT and VIEILLE was followed shortly by the researches of MALLARD and LE CHATELIER on the initial phases of the explosion. The method they found most suitable for tracing the progress of the flame was a photographic one; the movement of the flame along a horizontal tube being recorded on a sensitised paper moving vertically. Failing to obtain any photographic image of the flame with mixtures such as carbonic oxide with oxygen, MALLARD and LE CHATELIER employed carbon disulphide with oxygen or nitric oxide, these mixtures yielding highly actinic flames.

When the gases were ignited by a flame at the open end of a long tube, the flame was propagated along the tube for some distance with a uniform slow velocity which MALLARD and LE CHATELIER regard as the true rate of propagation by conduction. In the case of mixtures of carbon disulphide with nitric oxide, this period of uniform movement is succeeded by oscillations of the flame, which sometimes become of larger and larger amplitude and then die down, and sometimes give rise to the detonation-wave. When carbon disulphide is mixed with oxygen the preliminary period of uniform movement is shorter and is immediately succeeded by the detonation-wave. These two mixtures appear to be typical of other gaseous mixtures, carbon disulphide with oxygen resembling oxygen mixtures generally, carbon disulphide with nitric oxide resembling air mixtures generally.

MALLARD and LE CHATELIER draw attention to the fact that in the explosions starting at the open end of the tube the development of the detonation-wave is not progressive, but always instantaneous. The detonation-wave is characterised not only by its great velocity of movement, but by its intense luminosity and the very high pressures instantaneously set up in it.

On the other hand, MALLARD and LE CHATELIER found that, when the mixture of carbon disulphide and nitric oxide was fired near the *closed* end of the tube the movement of the flame was uniformly accelerated, until the detonation-wave was set up. Their apparatus, in which the photographic paper could not be moved faster than 1 metre per second, failed to analyse the more rapid movements of the flame.

To make clear the movements studied by MALLARD and LE CHATELIER I reproduce on Plate 10 four of their photographs.

In fig. 1 the mixture of carbon disulphide with six times its volume of nitric oxide is ignited at the open end of a tube 3 metres long and 20 millims. in diameter. The tube is made up of three pieces, of 1 metre each, fastened together by rubber rings, which, eclipsing the flame as it passes, show as blank bands on the photographs. The flame advances with a uniform velocity of 1.25 metres per second to the point *b*, where the rate increases and vibrations begin. From this point the rest of the combustion takes place with strong vibrations of the flame.

In a tube of 10 millims. diameter the period of uniform velocity is shorter (fig. 2). The vibrations die down and recommence, but after the flame has traversed a little over 1 metre the flame is extinguished.

Fig. 4 is the photograph of the explosion of carbon disulphide and oxygen fired at the open end of the tube. A short period of uniform progression is followed abruptly by a flame which appears to be propagated instantaneously.

Fig. 3 shows the effect of igniting the mixture of carbon disulphide and nitric oxide near the closed end of a tube 2 metres in length. The velocity of the flame continually increases from the point of inflammation, *a*. This photograph* clearly shows a "rebound" wave, *c d*, from the end of the tube, which was not broken like the one described in the text ('Recherches,' p. 71).

Experiments of A. VON OETTINGEN and A. VON GERNET.

OETTINGEN and GERNET† set out to prove the truth of BUNSEN's principle of successive partial explosions. This discontinuous step-like combustion ("welches wir mit neuen Methoden experimentell zu prüfen unternahmen") should yield evidence of its real existence if the flame of the explosion were analysed by a rotating mirror. By an ingenious arrangement they contrived to pass a spark through a eudiometer tube at the moment when the image of the tube was thrown by the rotating mirror into a camera, so that the light of the flame might be drawn out and its movements recorded on a photographic plate. But although the flame of electrolytic gas appeared intensely bright, its spectrum only gave the sodium and calcium lines, and the most sensitive photographic plates showed "hardly a trace of the process." Failing to photograph the flame itself, they added finely divided salts to the tube, and found that the most brilliant pictures were given by cuprous chloride.

The pictures show the passage of waves sharply reflected backwards and forwards from the ends of the tube, and gradually diminishing in intensity and velocity.

* This photograph is not described in the text of MALLARD and LE CHATELIER's book. M. LE CHATELIER writes to me that my interpretation of the photograph is correct. I did not notice this "rebound" wave in the photograph until after my experiments of 1896.

† "Ueber Knallgasexplosion," 'Annalen der Phys. und Chemie,' 33, 586 (1888).

These visible waves, according to OETTINGEN and GERNET, are not a picture of the process of combustion itself, but are compression-waves moving through the products of combustion after the explosion is completed. The explosion itself, they say, is *quite invisible*.

Fig. 5, Plate 10, is a reproduction of their picture, showing the waves produced when the electrolytic gas is fired in the middle of a eudiometer 400 millims. long.

The image of the spark is lengthened into a vertical line by reflexion from the sides of the eudiometer. Nothing else is visible on the plate for some time except a thin wavy line, which, according to OETTINGEN and GERNET, is due to the particles of salt raised to incandescence by the spark. Then after the lapse of $\frac{1}{1000}$ second the material in the tube becomes luminous and a wave is seen starting from the upper end of the tube at *d* and traversing the gases to the lower end where it is reflected at *e*, and so on backwards and forwards, making nearly four complete vibrations before it becomes non-luminous. The photograph shows very beautifully the passage of these compression-waves through the heated gases, and the way in which the mass of gas (carrying with it the luminous particles from the salt) follows the compression-waves backwards and forwards. But I can see no evidence in this photograph for the conclusion drawn by OETTINGEN and GERNET, viz., that a true detonation-wave has proceeded from the spark, and its compression-wave has traversed the tube several times before it becomes visible by raising the salts to incandescence. The fig. 5A (reproduced from their Memoir) shows in outline the course of one of these hypothetical waves starting downwards from the spark and being reflected at *a*, *b*, *c*, and *d* before it becomes visible. According to this view, the wave has traversed the tube $3\frac{1}{2}$ times before it is photographed, and the true chemical combustion gives no light.

In the next photograph reproduced (fig. 6) the gases were fired at a point a quarter of its length from the bottom of the tube. In this case there is a displacement of the wavy line joining the spark to the luminous portion, but hardly sufficient to justify the conclusion that the visible wave proceeding downwards from *d* is the residue of a detonation-wave starting upwards from the spark and being reflected at *b*, and again at *c*. But this is the interpretation put upon the photograph by OETTINGEN and GERNET, as is shown in fig. 6A. By constructing these invisible waves backwards from the visible ones, it is possible, according to OETTINGEN and GERNET, to arrive at the velocity of the true detonation-wave at starting. In this way they calculate that the explosion travels from the spark at a velocity of about 2550 metres per second, and criticise BERTHELOT's statement that the flame increases regularly in velocity from the point of ignition. I will show in the sequel that the interpretation put upon these photographs by their authors is erroneous. The flame really starts slowly, but its rate of progress is remarkably affected when it is reflected from the end of a tube. It is quite true that the flame of electrolytic gas when first ignited has very slight luminosity, but this only holds during the period of slow motion. The

temperature and luminosity increase with the velocity; the flames are probably most luminous where they are hottest.

But besides these primary waves there are others to which OETTINGEN and GERNET call special attention; these are "secondary waves" running nearly parallel to the primary waves. The photograph given in fig. 5 is referred to as showing very clearly four of these waves near together running parallel to the chief wave, which starts downwards from *d*. Other photographs show somewhat similar appearances, *i.e.*, of weaker waves running nearly parallel with and sometimes coalescing with primary waves. OETTINGEN and GERNET say that they can find no other reason for these waves following one another at a short interval, but that successive explosions have taken place from the electrodes exactly as BUNSEN imagined. The evidence relied on for these successive explosions, even if we accept the general interpretation of the photographs given by the authors, appears to me exceedingly slender. My own photographs will show how complicated the reflexions are when gases are exploded in a short tube, and how readily "nearly parallel waves" are produced with a single ignition of the gases. In the many photographs of the initial period of the explosion taken by me and my fellow-workers there is no indication of any *second* flame starting from the region of the spark.

PART II.

PHOTOGRAPHIC ANALYSIS OF DETONATION-WAVES AND THEIR REFLEXIONS.

(*In conjunction with* E. H. STRANGE, *B.Sc.*, *and* E. GRAHAM, *B.Sc.*)

In 1895 we were engaged on an investigation* into the nature of the flame produced by the explosion of cyanogen with oxygen, when we made the observation which led to the present research. We found that the flame could be sharply photographed on EASTMAN'S films without the addition of any metallic salts, and that the films could be rotated very rapidly on a wheel without damage. In the experiments we then made, a mixture of cyanogen with oxygen was fired in a long vertical leaden pipe having a short piece of glass tube let into it to serve as a window. This window was focussed on to the vertically moving film. When the explosion passed through the glass tube an illuminated image of the window was thrown on the photographic film, and was drawn out to a length depending on the rate of rotation of the film and the time during which the gas in the glass tube was photographically luminous. We found that cyanogen with its own volume of oxygen (burning to carbonic oxide) gave a much brighter and shorter flame than cyanogen burning to carbonic acid, a result in accord with the hypothesis that carbon (in carbon

* "On the Explosion of Cyanogen," 'Journ. Chem. Soc.,' 1896.

compounds) burns first to carbonic oxide. In the earlier of these experiments we were puzzled by a re-duplication of the image which constantly appeared in the photographs, until at last this was found to be due to an unsuspected constriction in the lead pipe, which had sent back a reflected wave sufficiently intense to re-illuminate the window. This led us to examine more precisely the detonation-wave and its reflexions by means of the rotating film. The camera and film being arranged as before, so that the image was thrown on the film as the latter moved vertically downwards, and the explosion-tube being fixed horizontally, the photograph showed an inclined line of light compounded of the horizontal movement of the image of the flame and the vertical movement of the film. When the explosion-tube was placed at such a distance from the camera that the length of the image was $\frac{1}{30}$ that of the tube, a velocity of the flame of 3000 metres per second corresponded with a horizontal velocity of the image of 100 metres per second. When the wheel was rotated twenty-five times per second this caused the film to move vertically at a rate of 25 metres per second, the circumference of the wheel being 1 metre. The line described by the image on the film thus made an angle with the horizontal whose tangent was $\frac{1}{4}$ (nearly 14°); or when the wheel was rotated at twice this velocity (50 metres per second) the angle was nearly 27° . For most experiments a rate of rotation between these limits was used, although the wheel could be rotated 80 to 100 times per second.* On account of the slip of the catgut cord it was not found possible to determine the rate of the wheel from the gear and the rate of the motor. The true velocity of the wheel was determined by the trace of a tuning-fork, which showed that several revolutions were made at a sensibly uniform speed, and therefore that the speed of the film might be taken as rigorously uniform during its movement through the small arc affected by the photograph. The films were slowly developed in a long trough, which was set rocking by suitable gear. For the brighter photographs an hour was generally required for proper development, for the fainter images two or three hours were required. After fixing and washing, the films were soaked in glycerine and water, which prevented them from curling up on drying.

It was found that the detonation-wave of all the undiluted mixtures we tried could be photographed on the moving films, but the intensity of the images varied very considerably. Cyanogen, carbon disulphide, and acetylene fired with oxygen gave the brightest flames, hydrogen with chlorine and carbonic oxide with nitrous oxide gave the least bright flames. Thin strips of black paper were fastened round the explosion-tube to give reference marks on the photograph. These strips appear as black vertical bands on the prints, and are useful for measuring the angles made by the detonation or reflected waves.

Figs. 7, 8 and 9 (Plate 11) show the detonation-wave in a mixture of cyanogen with

* Our method of photographing on a rotating film has given most interesting results in Professor A. SCHUSTER's research on the constitution of the electric spark. See 'Phil. Trans.,' A, vol. 193, 1900, p. 189.

two volumes of oxygen travelling from right to left at the bottom of each photograph. In fig. 7 the wave strikes the end of the tube (a metal stopper being used), and a reflected wave is thrown back. The dark band was 35 centims. from the stopper. Fig. 8 shows the reflected wave between 35 and 70 centims. from the closed end, and fig. 9 shows the reflected wave travelling between marks placed at 70 and 105 centims. from the closed end. The film was moving slower when fig. 7 was photographed than in the other two cases.

The photographs in figs. 10, 11, 12, and 13 show the reflected wave (in different mixtures of gases) travelling backwards (from left to right) between 35 and 70 centims. from the closed end. When the end of the tube is open only a faint flame is projected from the tube, and no bright reflected wave is seen (fig. 14). A cork, loosely fitted, is sufficient to send back a bright reflected wave (fig. 15). Even when the tube is fractured by the explosion, the detonation-wave can be photographed and an indistinct reflected wave is visible (fig. 16).

The first points noticed in the photographs were (1) the sharpness with which the luminosity is set up, and (2) the uniformity of the detonation-wave. There is no evidence of any gradual heating up of the gases, but on the contrary the temperature appears to spring to its maximum with abrupt suddenness. This is, of course, in accordance with the views published by BERTHELOT and by myself as to the character of the detonation-wave, which we believe to be propagated by the shock of the molecules themselves moving forward with the velocity due to the whole heat of the chemical combination. The gas ignited by the detonation-wave (including dust and particles knocked off the tubes) remains luminous for some time after the wave has passed. As had been shown in the "window" experiments (previously referred to), cyanogen burning to carbonic acid left a longer trail of light than when burnt to carbonic oxide only, and the most prolonged images were obtained with the two mixtures $C_2N_2 + 2O_2$ and $CS_2 + 3O_2$.

Many of the photographs show very distinctly the movements of the gas *en masse* as it follows up the detonation-wave, comes to rest, and swings back again. Fig. 17 (taken in a long tube) shows the movements of the gas undisturbed by any reflected wave. These movements are analogous to the forward and backward movements in air produced by a vibrating body. According to the kinetic theory, the mean motions of the molecules of a gas may be resolved in any direction into equal and opposite movements. When a compressed tuning-fork is released, the forward movements of the molecules in contact with the prong have added to them the motion of the fork, and by exchange of velocities this added velocity is propagated from molecule to molecule, each molecule swinging forward with the increased velocity and returning with its normal velocity. In a sound-wave a number of these forward impulses is imparted to the molecules as the prong moves forward, and therefore the molecules move forward, causing a compression-wave, and afterwards, as the prong moves backwards, the reverse effect is produced, and a rarefaction follows the compression-

wave. The curious stratification of the light-giving particles behind the detonation-wave (observed in many of the photographs) allows this movement to be followed, and indicates the length of the excursion made by each layer of gas.* When the detonation-wave hits the closed end of the tube it is reflected back in a distinctly marked luminous-wave. What is most remarkable about this reflected wave is its great luminosity. As the reflected wave starts back from the end wall it has at first to meet the gas moving bodily forward in the wake of the detonation-wave. As it continues backwards the gas it meets has less and less forward motion, and at a certain point (usually some 400 to 500 millims. from the reflecting end) the gas it travels in is stationary. From this point the motions of the reflected wave and of the gas it travels in are in the same direction. It follows therefore that the velocity of the reflected wave is at first retarded and afterwards increased by the motion of the medium. This is probably the reason why the reflected wave sometimes appears to travel faster as it proceeds backwards along the tube, as in figs. 10, 11, and 15, and sometimes appears to travel at a nearly uniform velocity, as in figs. 7, 8, and 9, in spite of the cooling by radiation and conduction going on. The effect of these movements of the gas appears again in the photographs taken in repeating the experiments of OETTINGEN and GERNET (p. 348).

The reflexion-waves can be readily photographed in the brighter explosions as far back as 120 centims. (4 feet) from the reflecting end, and when the gases are fired in short tubes, waves that have been reflected from the ends eighteen to twenty times can still be photographed.

Three views may be held in regard to these reflexion-waves :—

(1) The combustion continues for a considerable time after the detonation-wave has gone by, and the returning compression-wave may owe part of its luminosity to increased chemical combination taking place in its path ;

(2) The combustion is practically completed in a short time, and the reflected wave travels in the burnt gases as an intense compression-wave of the same nature as a sound-wave of large amplitude ;

(3) The reflexion-wave travels as in (2) through the still combining gases without materially affecting the chemical changes proceeding.

According to the first view the continuation of the combustion may be due either to dissociation, which permits the gradual formation of compounds (*e.g.*, CO_2) only as the gases cool down, or to the time required for the natural completion of the reaction

* In some of the photographs this stratification appears almost too regular to be accidental ; can it be due to vibrations in the gas causing the dust to arrange itself in transverse streaks along a ventral segment, as the lycopodium arranges itself in KUNDT'S tubes when the length of the tube is not an exact multiple of the wave-length ?

owing to the last uncombined molecules being surrounded by the products of combustion. In either case it is not probable that the passage of a compression-wave would greatly influence the chemical combustion.

(a) If combination were limited by dissociation, *e.g.*, in the explosion of cyanogen with twice its volume of oxygen, there might be in the vessel some little time after the passage of the detonation-wave a mixture of CO_2 , CO , O_2 , and N_2 at a temperature (let us say) of 3000°C . and a pressure of 15 atmospheres. Let us take as a probable number the figure calculated by LE CHATELIER* for the dissociation of CO_2 under the partial pressure of 10 atmospheres and 3000°C . About one-fifth of the carbonic acid would be dissociated. Now if gases were compressed adiabatically (even if we take the molecular specific heat of CO_2 as 20 and the molecular specific heat of the diatomic gases as 7 at constant volume and 3000°C .), the rise of temperature due to compression would more than counteract the increase in pressure, and the dissociation would tend to increase.

(b) We have found that the addition of an inert gas to the mixture of equal volumes of cyanogen and oxygen prolongs the luminosity of the explosion.† As the inert gas must lower the temperature, and therefore reduce the possible dissociation, this prolonged luminosity must be due to a combustion of residual gases not caused by dissociation. But in undiluted mixtures this residue is small, as is shown by the very rapid fall in luminosity, and therefore the action of a pressure-wave on this residue can have little chemical effect.

We are thus brought to the conclusion that the reflected wave produced by the collision of a detonation-wave with the end of the tube is mainly a compression-wave.

The velocity of the reflexion-wave may be readily compared with that of the detonation-wave, their relative velocities being as the sines of the angles made by the two waves with the horizontal. These angles have been measured by fixing the films on a glass plate (illuminated from behind), and covering the films with tracing paper, on which the lines could then be traced and extended by the ruler. When the reflexion-wave was curved, a line was drawn touching the curve as nearly as possible at the point where the movement of the gas itself was nil. The velocity of the detonation-wave being known, the velocity of the reflexion-wave could readily be calculated. By this means the velocities given in the following table were measured, each velocity being the mean of those obtained from four to eight photographs:—

* 'Zeit. Phys. Chem.,' vol. 2, p. 782 (1888).

† 'Journ. Chem. Soc.,' vol. 79, p. 772.

TABLE II.—Velocity of Reflexion-Waves in Gaseous Explosions.

Mixture of gases.	Velocity of detonation-wave in metres per second.	Velocity of reflexion-wave in metres per second.	Ratio of velocities.
$2\text{H}_2 + \text{O}_2$	2820	1538	1.83
$\text{H}_2 + \text{N}_2\text{O}$	2305	1383	1.67
$2\text{CO} + \text{O}_2$	1676	1078	1.56
$\text{C}_2\text{N}_2 + \text{O}_2$	2728	1230	2.22
$\text{C}_2\text{N}_2 + 2\text{O}_2$	2321	1129	2.06
$2\text{C}_2\text{H}_2 + 5\text{O}_2$	2391	1133	2.11

Although the formula for the velocity of sound in gases is strictly valid for small displacements only, nevertheless it appeared of interest to calculate from the observed velocities of these reflexion-waves what temperature they indicated in the gas on the assumption that they were propagated as sound-waves. Of course, to calculate the temperature from the velocity of sound it is necessary to know the ratio of the specific heats γ , and since in the case of carbonic acid and steam this ratio is very doubtful, a corresponding uncertainty must exist in the temperature calculated. But in the case of cyanogen burning to carbonic oxide the products of combustion, carbonic oxide and nitrogen, are similar to air, and their specific heats either do not alter, or do not alter greatly with rise of temperature. The velocity of sound in such a gas would therefore give an approximation to the temperature.

Now the velocity of the reflexion-waves in cyanogen exploded with its own volume of oxygen is 1230 metres per second. Assuming γ to be unaltered by rise of temperature, and the velocity of sound in air at 0°C . to be 333 metres per second, the temperature of the gas where the reflexion-wave was measured is given by the formula :

$$T = \left\{ \left(\frac{v \sqrt{\frac{d_1}{d}}}{333} \right)^2 - 1 \right\} 273 = 3330^\circ \text{C},$$

where v is the velocity of sound, and d_1 and d the densities of the gas and air respectively under the same conditions. If, on the other hand, we assume (with LE CHATELIER) that the specific heat at constant volume of diatomic gases rises with the temperature and becomes 7 at the temperature of this experiment, then the ratio γ falls to 1.29, and the formula becomes :

$$T = \left\{ \left(\frac{v \sqrt{\frac{d_1}{d} \cdot \frac{\gamma}{\gamma_1}}}{333} \right)^2 - 1 \right\} 273 = 3672^\circ \text{C}.$$

In the case of cyanogen exploded with twice its volume of oxygen the first reaction probably consists in the burning of the cyanogen to carbonic oxide, which

combines more slowly to form carbonic acid. How far this second reaction is completed when the reflexion-wave is measured it is impossible to decide. On the assumption that the specific heat of nitrogen is constant and that of CO_2 is 7.2, the velocity of the wave in the completely burnt mixture would indicate a temperature of 4200°C .; on the assumption that the specific heats of CO_2 and N_2 are 20 and 7, the temperature indicated is 4780°C . On the other hand, if no carbonic acid had yet been formed, the temperature indicated for the mixture of diatomic gases (2CO , O_2 , N_2) is 2880° ($C_v = 4.8$).

In a similar manner, the temperatures corresponding to the velocity of the reflexion-waves have been calculated for the other mixtures, (1) assuming the ratio of the specific heats for a diatomic gas to be 1.41 and for a triatomic gas 1.28, and (2) assuming the ratio of the specific heats for a diatomic gas to be 1.29 and for a triatomic gas 1.11.

TABLE III.—Temperatures of Exploded Gases, Calculated from the Velocities of the Reflexion-Waves.

Mixture.	I. $\begin{cases} \gamma \text{ for diatomic gases} = 1.41. \\ \gamma \text{ for triatomic gases} = 1.28. \end{cases}$	II. $\begin{cases} \gamma \text{ for diatomic gases} = 1.29. \\ \gamma \text{ for triatomic gases} = 1.11. \end{cases}$
	$^\circ \text{C}.$	$^\circ \text{C}.$
$2\text{H}_2 + \text{O}_2$	3720	4830
$\text{H}_2 + \text{N}_2\text{O}$	3660	4130
$2\text{CO} + \text{O}_2$	4530	5250
$\text{C}_2\text{N}_2 + \text{O}_2$	3330	3670
$\text{C}_2\text{N}_2 + 2\text{O}_2$	4200	4780
$2\text{C}_2\text{H}_2 + 5\text{O}_2$	3980	4630

A glance at this table reveals the fact that, whether the specific heats vary or not, but on the assumption that combustion is complete in each case, the explosion of cyanogen to carbonic oxide (which according to all observers gives the brightest flash and the highest pressure) gives apparently the coolest combustion products a short time after the explosion-wave has gone by. Now I have found* that the velocity of the reflexion-wave is nearly equal to that of a true sound-wave of small displacement travelling through the flame produced by the explosion of cyanogen burning to carbonic oxide; it is not therefore open to us to reject the temperature calculated for this mixture as entirely wide of the truth. The natural inference to be drawn from the figures is, I think, that the combustion is not complete in the other mixtures (in all of which steam or carbonic acid, or both, are produced) at the moment the reflexion-wave is measured. If we suppose that the formation of carbonic acid and steam is incomplete, the densities of the products are less and γ is higher, and consequently the temperatures calculated from the velocities of the reflexion-wave would be lowered. For instance, if we assume in these several experiments that the hydrogen

* See Part III.

is two-thirds burnt, and the carbonic oxide is one-third burnt at the moment of measurement, then the velocities of the reflexion-waves correspond to the following temperatures:—

TABLE IV.

Mixture of gases.	Hypothetical state of reaction when wave is measured.	Temperature calculated from velocity of reflexion-wave. γ for diatomic gases = 1.41. γ for triatomic gases = 1.28.
$6\text{H}_2 + 3\text{O}_2$	$4\text{H}_2\text{O} + 2\text{H}_2 + \text{O}_2$	° C. 3010
$6\text{H}_2 + 6\text{N}_2\text{O}$	$4\text{H}_2\text{O} + 2\text{H}_2 + \text{O}_2 + 6\text{N}_2$	3290
$6\text{CO} + 3\text{O}_2$	$2\text{CO}_2 + 4\text{CO} + 2\text{O}_2$	3070
$\text{C}_2\text{N}_2 + \text{O}_2$	$2\text{CO} + \text{N}_2$	3330
$6\text{C}_2\text{N}_2 + 12\text{O}_2$	$4\text{CO}_2 + 8\text{CO} + 4\text{O}_2 + 6\text{N}_2$	3220
$6\text{C}_2\text{H}_2 + 15\text{O}_2$	$4\text{CO}_2 + 8\text{CO} + 4\text{H}_2\text{O} + 2\text{H}_2 + 5\text{O}_2$	2850

It will be observed that the temperatures calculated as above, in Tables III. and IV., are of the same order as those calculated by BUNSEN as the maximum temperatures of the explosion. But the photographs plainly show that they do not represent the temperatures reached in the detonation-wave itself. These calculated temperatures correspond to a period of the explosion subsequent to the passage of the wave—the period during which BERTHELOT and LE CHATELIER measured the “effective” pressures produced. The photographs therefore confirm LE CHATELIER’s statement that the “effective” pressures measured in the French experiments are not the maximum pressures produced in the explosion.*

PART III.

VELOCITY OF A SOUND-WAVE IN THE FLAME OF EXPLODED GASES.

(*In conjunction with R. H. JONES, B.Sc., and J. BOWER, B.Sc.*)

The interest attaching to the determination (even approximately) of the temperatures produced in the explosion of gases led us to attempt the measurement of the rate of a true sound-wave (of small displacement) in the gases produced by the detonation-wave.

In our first experiments the glass explosion tube A (fig. 18), was fitted to a steel piece, B, containing a tap of large bore and a small bye-tap, and connected by a pipe

* Some of our photographs showing the reflexion-waves were exhibited before the Chemical Section of the British Association at Liverpool in 1896. ‘B.A. Report,’ 1896, p. 746.

to a steel bomb, C, in which a small charge of fulminate could be fired. The bomb and connecting pipe were filled with air, while A was filled with a mixture of cyanogen with two volumes of oxygen. The fulminate in its copper case, *f*, was

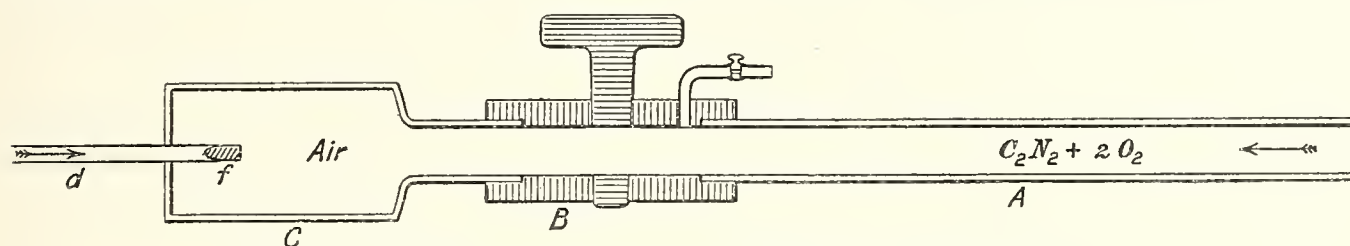


Fig. 18. Apparatus to make a sound-wave meet a detonation-wave.

inserted through a rubber stopper in the end of the steel tube, *d*, which was screwed into the bomb. A and *d* were connected with tubes fitted with firing wires, so that an explosion could be set up simultaneously in the cyanogen mixture in A and the electrolytic gas in *d*, the connecting tap being turned on immediately before the explosion. The lengths of the tube were so adjusted that the sound-wave started in the bomb by the detonation of the fulminate should be propagated through the air and cyanogen mixture so as to meet the detonation-wave coming in the contrary direction before the latter reached the end of the tube A. The detonation-wave was then photographed as it met the sound-wave.

In fig. 19 (Plate 12) the detonation-wave is seen advancing from right to left. The sound-wave strikes it near the centre of its path across the field of view, and the progress of the sound-wave is made visible as it traverses the heated gases from left to right. After meeting the sound-wave, the detonation proceeds some distance at full speed, and then its velocity slackens—as it meets the rush of air driven on to it from the detonation in the bomb. A second sound-wave, an echo of the first reflected from the ends of the bomb, next traverses the heated gas; and higher up the passage of a third sound-wave is registered on the photograph. The first sound-wave is no doubt retarded by the movement of the gas in the opposite direction, but the second wave appears to travel in almost stationary gas, while the third is only slightly retarded. The rates of these sound-waves have been measured, and the corresponding temperatures calculated. These values are given in Table V., on the assumption that the combustion was complete.

TABLE V.

Number.	Velocity of sound-waves in exploded mixture $C_2N_2 + 2O_2$.	Calculated temperatures.
		γ for diatomic gases = 1.41. γ for triatomic gases = 1.28.
1st	1116 metres per second	° C. 4100
2nd	1014 " " "	3330
3rd	893 " " "	2530

It will be seen that the temperature calculated for the first sound-wave (4100°) is in close accordance with that calculated from the reflexion-wave in the same mixture (4200°) given in Table III.

The experiment was next varied by the introduction of a thin iron membrane between the air and the explosive mixture. The shock transmitted through the air from the fulminate struck the flexible plate, and so propagated a wave of small displacement through the explosive mixture. This wave had very little effect on the movements of the gas in the wake of the detonation-wave, but its passage through the luminous gas was plainly marked. The gases were ignited as before, the lengths of the tubes being so adjusted that the first sound-wave met the detonation-wave about 1 metre from the membrane. Several of these sound-waves (produced by echo in the bomb) are seen in fig. 20 traversing the flame produced by the explosion of cyanogen with its own volume of oxygen; these sound-waves apparently increase in velocity owing to the movements of the gas through which they pass.

It is not so easy to measure the angles of these faintly-marked sound-waves, but three photographs have given as the mean of several independent measurements made on each photograph the velocity of the sound-wave *in the stationary gas* as 1250 metres per second. This velocity corresponds to a temperature of 3460° ($\gamma = 1.41$); a number in very fair agreement with that calculated from the reflexion-waves, viz., 3330° (Table III.). This agreement indicates that the reflexion-waves really travel with a velocity approximately equal to that of sound.

Similar sound-waves are shown in fig. 21, where the mixture was $\text{CS}_2 + 2\text{O}_2$.

Some other experiments were also made in which a compression-wave was made to follow closely behind a detonation-wave, both moving in the same direction.

A leaden tube had a by-pass inserted so that the explosion travelled both along the straight piece *a c* and along the curved piece *a b c*. When the flame travelling by the straight road reached the second opening of the by-pass *c*, it penetrated this opening and came into collision with the explosion travelling by the longer road. The reflected wave so produced followed the main detonation-wave into the glass tube *d*, where the two were photographed about 1 metre from the point *c*. The reflected wave is assisted by the forward movement of the gas, and consequently its apparent velocity is much greater than the sound-waves or reflexion-waves measured in stationary gas.

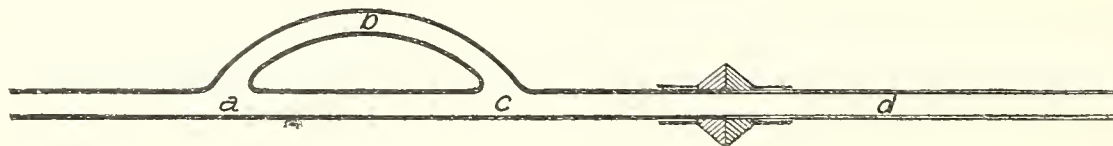


Fig. 22. Apparatus to make a sound-wave follow a detonation-wave.

In fig. 23, which shows the two waves in the explosion of cyanogen with two volumes of oxygen, the compression-wave is seen running alongside the detonation-

wave and inclined at a small angle to it. The velocity of the detonation-wave being 2320, the mean velocity of the following compression-wave is given by measurements of three photographs as 1635. The difference between this number and the velocity of the compression-wave measured in stationary gas (a difference of 500 metres per second) is due to the movements, and probably also to the higher temperature of the gas in the immediate wake of the detonation-wave.

PART IV.

ON THE COLLISION OF TWO DETONATION-WAVES AND THE EFFECT OF JUNCTIONS IN THE TUBES.

(In conjunction with R. H. JONES and J. BOWER.)

In 1897 we began the investigation (by means of the moving film) of the phenomena marking the collision of two explosion-waves meeting end-on in a glass tube. To make the two flames meet in the field of the camera, the glass firing piece A (fig. 24) was fastened to a lead pipe bifurcating at B into two arms of equal length, which were bent round at their extremities and opened out to hold the two ends of the glass tube. The junctions at C and D were made gas-tight by slipping pieces of rubber tubing over the ends of the glass tube and pushing them into the leaden caps. By shutting the tap B* and opening E, the whole apparatus could be filled through A with the explosive mixture; then the taps at A and E were shut and B was opened. On passing the spark the flame followed the two arms of the tube, and the two detonation-waves generated met in the centre of the glass tube C, D.

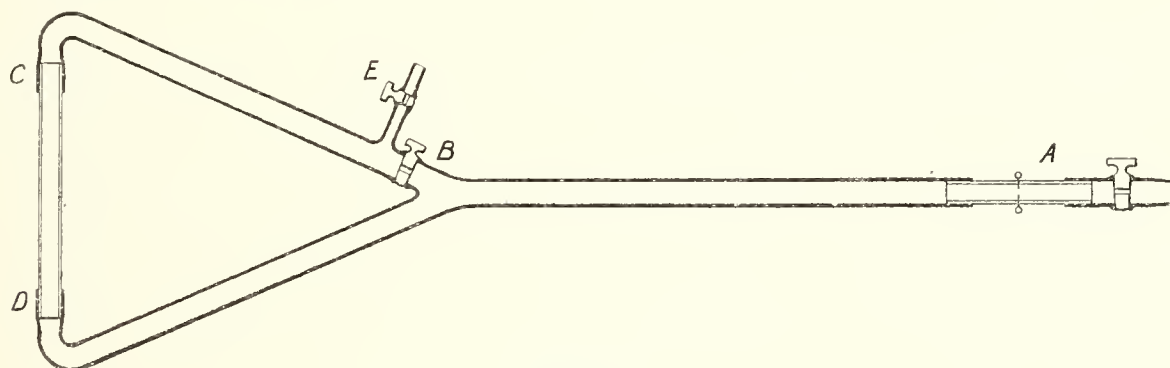


Fig. 24.

We were greatly puzzled by the photographs obtained with this apparatus. The first few experiments showed the rebound waves (after the collision) to be much brighter and to be travelling (backwards) much faster than the two detonation-waves themselves before the collision. Other photographs showed that the two flames were

* The bore of the tap B was the same size as the lead tube; it is shown in fig. 18.

not always symmetrical. Fig. 25 (Plate 12) shows the two flames meeting symmetrically, and giving rebound waves faster and brighter than the original waves before collision.

Fig. 26 shows the flame coming from the right hand to be much brighter and travelling much faster than that from the left hand (although the collision occurs only a little to the left of the centre of the tube). Fig. 27 shows these phenomena reversed. On repeating the experiments it was found that the faster flame had usually been affected by some impulses causing a sudden increase in its brightness and velocity, and also producing a backward wave (analogous to a reflected wave). Sometimes the impulses were exhibited by the left-hand flame, as in figs. 28 and 29 (Plate 13); sometimes by the right-hand flame, as in figs. 30 and 31. In some experiments a single impulse only is observed, as in figs. 29 and 31; in others several such impulses can be traced, as in fig. 30. Figs. 32 and 33 show both flames affected by these impulses; in fig. 32 symmetrically, in fig. 33 unsymmetrically.

The explanation of these appearances that first occurred to us was that the flame was preceded by invisible sound-waves, travelling more quickly than the flame in its initial phases; that these sound-waves became visible as soon as they met the flame moving towards them in the opposite direction (as in our previous experiments on sound-waves), and that, on the other hand, the visible flame meeting the sound-wave was affected by the sudden increase of pressure, and continued its journey with greater speed and luminosity. This explanation was at once destroyed when we found similar impulses in a flame which was sent through the apparatus in one direction only (the tap B being closed), as is shown in fig. 34.

It next occurred to us that these impulses might be due to the explosion *catching up its own sound-waves*. If sound-waves are propagated through the gas from the point of ignition, the flame might lag behind the sound-waves at first and catch them after a run more or less prolonged. The sound-waves when overtaken might cause reflected sound-waves (made visible in the luminous gases), and the explosion itself might become more intense owing to the collision.* Many experiments were undertaken to verify or disprove this hypothesis, but it will not be necessary to reproduce the large number of photographs we took in the course of this investigation. We found that no addition to the length of the lead pipe between the firing point and the bifurcation, and no addition to the length of the two arms affected the result. Finally, it was found that the explosion was affected as it passed through the junctions between the lead and the glass, and the impulses recorded in our photographs were due to the detonation-wave, damped down at the junction, being regenerated by fits and starts. One of the photographs showing this most plainly is reproduced in fig. 35. This was obtained in the following way:—Three straight glass tubes were fixed horizontally one above another and were joined together by curved lead tubes, into which the

* LE CHATELIER has published a similar hypothesis to explain the apparent break in the curve where the detonation is started; see p. 347.

glass was packed gas-tight by indiarubber, as in fig. 36. The gases were ignited by the spark at *a*, and the flame travelled towards *b*. Fig. 35 shows the result of firing equal volumes of cyanogen and oxygen in this apparatus. The photograph does not show the spark or first part of the flame.

The flame, starting from the right-hand lower corner, is seen to become more luminous, and to move faster as it approaches the middle of the tube. The detonation-wave is then set up, its initiation being marked by a luminous-wave thrown backwards from the point where the

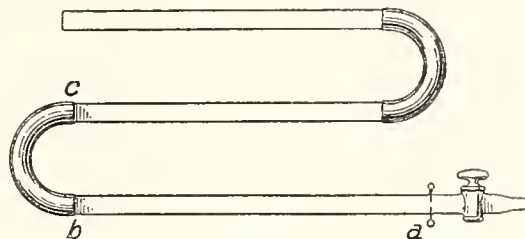


Fig. 36.

detonation begins. To the end of the first glass tube the detonation-wave then travels with uniform speed and luminosity. On emerging from the lead bend into the second glass tube the flame is seen to be less luminous, but in a short period detonation is again set up, with the accompaniment of the backward thrust, and with the formation of a dark patch of gas, which is also characteristic. On passing the second lead bend the luminosity and speed are again reduced, and the detonation-wave is not again determined until the flame has traversed half the third glass tube. It was now evident that what had happened in our "collision" photographs was this damping down of the detonation-wave as it passed through the lead bend and entered the glass tube. It was, however, difficult to imagine that the curvature of the tube had affected the rate, for BERTHELOT had determined the rate of explosion of electrolytic gas in a leaden pipe with many sharp bends, and I had found the same velocity in a straight pipe and in one coiled round a small drum. An experiment made with a glass tube bent to the same curvature showed no retardation (fig. 37). BERTHELOT had also found the velocity of explosion unaltered in a rubber tube with many bends; and when we proceeded to substitute a stout rubber tube for a lead bend (fig. 38, Plate 14), we found the wave traversed it without retardation. Therefore it was also equally hard to imagine that the use of a rubber tube to make a gas-tight joint between the glass and the lead could affect the explosion, and yet this turned out to be the origin of the disturbance.

We had connecting-pieces made of different metals and of varying curves and angles, but after many trials we found that the only thing which mattered was the rigidity with which the glass and metal were connected together. Any packing (such as rubber) which gave to a shock caused a retardation; when the glass was firmly cemented to the metal no retardation occurred. Fig. 39 shows the explosion passing through two glass tubes rigidly connected to a copper junction, which was equally without retarding action whether curved or bent at a sharp angle. This retardation was not confined to curved junctions. When a straight piece of lead pipe was fastened by rubber to two glass tubes, the wave of detonation was damped down on traversing the junction; when the metal was cemented to the glass no retardation occurred.

The knowledge gained from these experiments with junctions made it now possible to investigate the phenomena of collisions, but the experiments had also shown that it would be of interest to explore photographically the region of the explosion prior to the initiation of the detonation-wave.

When two detonation-waves come into collision, the tube remains brightly luminous at the point of contact for some time,* and two reflected waves are sent backwards with velocities which increase at first, owing to the movement of the gas through which they are propagated.

The photographs of two detonation-waves meeting in collision are shown in figs. 40 and 41, the explosive mixture being cyanogen with two volumes of oxygen, and in fig. 42 (Plate 14), the mixture being two volumes of hydrogen with one volume of oxygen.

A comparison of all the photographs shows that the gases are more luminous after a collision than when the explosion-wave strikes a flat surface of metal fastened at the end of the tube. The reflected waves in the two cases are similar in character, but the reflexion generated by collision with another detonation-wave seems always to travel slightly faster. If we were dealing only with waves produced mechanically, the reflected waves would be exact copies of the incident-waves with velocities reversed in both cases. But in the detonation-wave we have chemical as well as mechanical action, while the reflected wave is mainly mechanical. We should expect therefore the reflected waves to travel more slowly than the incident-waves, but we should also expect the reflected waves to travel with the same velocity whether they were produced by collision with a rigid diaphragm or with a similar and equal wave travelling in the opposite direction, unless there was some chemical difference involved in the two kinds of collisions. Now in the hypothesis† I have advanced for the mode of propagation of the detonation-wave I have assumed that the explosion is propagated not only by the forward movements of the molecules produced by the chemical change, but also partly by the movement of the yet unburnt molecules. For instance, in the explosion of hydrogen and chlorine, molecules of hydrogen chloride just formed in the wave-front may move forward until they come into collision with molecules of unburnt hydrogen or molecules of unburnt chlorine. These molecules (by exchange) now move forward with increased velocity, and in turn meet molecules of the opposite kind, with which they combine. The combination therefore does not proceed between cold molecules entirely, nor between heated molecules entirely, but mainly between molecules half of which are at the ordinary temperature and half are heated by collision with the products of combustion. If this roughly represents the state of the wave-front, there would be a chemical difference between the collision with a diaphragm and with another explosion-wave. For in the latter case

* General HESS, of the Austrian artillery, has photographed a luminous band at the point of collision of the two compression-waves produced by exploding two cartridges suspended in the air a short distance apart. See 'Bulletin Soc. de l'Industrie Minérale,' Saint Etienne, 1900, vol. 14, 3, p. 116.

† 'Phil. Trans.,' A, vol. 184, p. 131 (1893).

the unburnt molecules would meet, at the moment of impact, the unburnt molecules travelling with an equal velocity in the opposite direction, and chemical combination would ensue between molecules both of which were highly heated before they met.

The difficulty of obtaining an exact measure of the velocity of the waves reflected after collision is very considerable. With all the care we could exert, we found it impossible to obtain concordant readings for the angles made by the collision-waves with the horizontal, but we believe that the difference between the rate of collision-waves and waves reflected from the end of the tube is sufficiently large to be fairly evident from a number of observations. For instance, six experiments with cyanogen and two volumes of oxygen gave as the ratio of the velocity of the detonation-wave (D) to the collision-wave (C):—

$$\frac{\text{sine of angle of C}}{\text{sine of angle of D}} = \left\{ \begin{array}{l} 1.58 \\ 1.40 \\ 1.38 \\ 1.19 \\ 1.44 \\ 1.47 \end{array} \right\} \text{Mean . . } 1.41,$$

whereas measurements of the reflexion-wave with the same mixture gave:—

$$\frac{\text{sine of angle of R}}{\text{sine of angle of D}} = \left\{ \begin{array}{l} 2.11 \\ 2.03 \\ 2.32 \\ 2.11 \\ 1.97 \end{array} \right\} \text{Mean . . } 2.11.$$

In a similar manner, measurements made on the relative rates of the collision-wave and the reflexion-wave in the mixture of cyanogen with one volume of oxygen, gave:—

$\frac{\text{sine of angle of C}}{\text{sine of angle of D}}$	$\frac{\text{sine of angle of R}}{\text{sine of angle of D}}$
1.91	2.22
1.79	2.15
2.13	2.33
1.98	2.08
—	2.22
Mean . . 1.95	Mean . . 2.20
—	—

In this case the numbers are too near for us to draw any certain conclusion, but in the first case it is difficult to suppose that the difference of 50 per cent. can be entirely experimental error. This inequality is an argument in favour of the view that there is a difference between a collision with a wall and with an approaching wave, but I would not lay stress on it to support my hypothesis of detonation.

Our photographs have shown that the wave of detonation has certain characteristics by which it may be readily recognised:—

- (1) It starts suddenly, throwing back a strongly luminous wave through the burning gases and leaving a dark space where it started ;
- (2) It travels with constant velocity, unless it traverses a junction not rigidly attached ; after being damped down by such an obstacle, it recoups itself and again starts with abruptness ;
- (3) On collision with a similar detonation-wave moving in the opposite direction, or with a rigid diaphragm, it sends back a reflected wave not so rapid as itself, and as a rule not so luminous.

In the case of the more luminous explosions, *e.g.*, those of cyanogen, acetylene, and carbon disulphide mixtures, the reflected waves were less luminous than the detonation-wave ; but in the case of the less luminous explosions, *e.g.*, those of hydrogen and carbonic oxide, which depend largely for their light on the particles detached from the tubes, the waves reflected from a collision were sometimes more luminous than the detonation-waves themselves, for instance in fig. 42.

With regard to the pressures produced in the detonation-wave, our experiments have repeatedly shown that glass tubes more readily fracture at the point of collision of two detonation-waves than when the detonation-wave traverses the tube in one direction only.* A tube has stood half-a-dozen passages of the detonation-wave in one direction, and has been shattered by the first collision. A closed tube breaks most frequently at the end furthest from the firing point. Again, the point at which the detonation-wave is set-up (and at which the corresponding "retonation-wave" is driven backwards) is frequently found to be the place where the tube is broken. Of course with a weak tube this would mean nothing but that the tube would be fractured by the detonation, as it would naturally break at the point where the fracturing force was first applied. But we have found that strong tubes have withstood the passage of the detonation-wave which was already determined before entering the tube, and have been fractured when the detonation-wave was started in the tube itself. Again, the tube has broken at the point where the detonation began, and has for the rest of its length withstood the passage of the detonation (as shown photographically).†

* In 1894 I was unable to show this experimentally ('*Manch. Memoirs*,' IV., 8, p. 180). In the experiments then made the flame was not photographed, and it is possible that the explosions were damped down at the junctions.

† Messrs. JONES and BOWER have made many attempts to obtain quantitative measurements of the pressures produced in the detonation of gaseous mixtures by the method of fracture of glass tubes, but the results cannot be regarded as certain. They found for the detonation of the mixture $C_2N_2 + O_2$ a pressure between 58 and 75 ats., while Dr. CAIN and I found for the same detonation a pressure between 60 and 140 ats. For the mixture $C_2N_2 + O_2 + 2N_2$, Messrs. JONES and BOWER found the pressure between 74 and 93 ats., while Dr. CAIN and I found it between 63 and 84 ats. Cf. '*Manch. Memoirs*,' 1894 IV., 8, p. 174, and 1898, No. 7.

PART V.

ON THE INITIATION OF THE DETONATION-WAVE AND ON THE WAVE OF
"RETONATION."

(*In conjunction with R. H. JONES and J. BOWER.*)

Several of the photographs previously described have illustrated the abrupt spring with which the detonation-wave is started. Sometimes, apparently, one such abrupt change alone occurs, marking the place where the gradual acceleration of the explosion changes; on other photographs one or more abrupt changes occur in the acceleration before the final spring which marks the detonation-wave. Figs. 29, 30, and 33 show one or more such sudden changes in the curve always accompanied by a luminous wave thrown back through the ignited gases.

The strongly luminous wave thrown back from the point where the detonation is started I propose (*nominis egestate*) to call the *retonation-wave*. This wave has not the same constant characteristics that mark the detonation, but when generated under certain conditions it resembles detonation most closely. Fig. 43 (Plate 14) shows a wave of retonation travelling parallel with and therefore at the same velocity as the collision-wave. Fig. 44 shows a retonation-wave travelling more quickly than the wave reflected from the end of the tube. A study of a number of photographs leads to the conclusion that the retonation is faster and more luminous when no other bright waves have been thrown back by the advancing flame before the point of detonation is reached.

In considering the cause of the intensity of the retonation-wave we must remember the facts illustrated in many of our photographs on collisions. The collision of two flames, in which detonation had not yet been determined, gave rise to reflected waves more rapid and more luminous than the incident waves. Fig. 25 shows this for both. Now these reflected waves could not owe their increased velocity to the mechanical impact, which could only result in the reflected waves being copies of the incident waves. It is evident then that chemical action must occur to assist these reflected waves, and therefore the combustion is obviously not complete when these waves return. From this it would appear probable that the period before the detonation is distinguished not only by a slower propagation of the flame, *i.e.*, of ignition, but also by a slower process of combustion. In the initial period the molecules of the gas might meet many times before chemical reaction occurred; in the detonation-wave the molecules might be so intensely agitated that most collisions between chemically opposite molecules would result in chemical change.

At the point of detonation the rise of pressure must be exceedingly rapid owing to

the increase of chemical action, and this pressure would produce not only the forward wave of detonation, but also a sudden backward wave of compression into the gases still slowly burning behind it. This compression-wave must raise the temperature of the ignited gases and so quicken the residual burning; its propagation would then be analogous to that of the detonation-wave, but modified by the extent to which the slow combustion has proceeded. I believe this view accounts for the facts described and also for the phenomena observed in the initial period of the explosion (see Part VI.).

The retonation-wave attains its greatest rapidity and brightness when it is developed at the closed end of a tube, *i.e.*, when the gas is fired at such a distance from the closed end that the explosion, gradually increasing in intensity, just reaches the detonation point as it arrives at the stopper. Under such conditions the reflected wave is superposed on the wave of retonation, and the result is a wave which cannot be distinguished from a true detonation. Fig. 45 shows the retonation-wave developed at the closed end of the tube in the mixture $C_2H_4 + 2O_2$.

Fig. 46 shows the formation, propagation, and collision of two retonation-waves produced by firing the gases in the middle of a long tube. The picture shows the two flames travelling from the spark to the right and the left, and the two points of detonation at either edge of the film. The two retonation-waves return symmetrically to the centre and there come into collision, producing reflected waves. If we compare this photograph with fig. 47 or with fig. 48, we see the effect of the retonation-waves starting from the two ends of the tube. If the point of detonation occurs just before the end of the tube is reached, the reflected wave runs back nearly parallel with the retonation-wave, as is shown in fig. 49.

In order to make a direct comparison of the velocity of the retonation-wave starting from the end of the tube with the detonation-wave, two tubes were fixed parallel one above another, and were filled with the same gases. One tube was connected with a long leaden pipe, in which the explosion was started, so that the detonation-wave alone traversed the glass tube; the second glass tube was fitted with firing wires at one extremity, so that the initial phase of explosion up to the detonation point should just occupy its length. The two flames were then photographed simultaneously on the moving film.

Fig. 50 shows the detonation-wave in the mixture $C_2H_2 + O_3$ moving from right to left in the lower tube, and just above the point where it meets the closed end of its tube the beginning of the explosion in the second tube can be seen. The second flame travels slowly from left to right, striking the end of the tube at the detonation point. Its intense retonation-wave, returning from right to left, travels in a line parallel with the detonation in the first tube, *i.e.*, their velocities are apparently equal. The same thing is shown in fig. 51, where the mixture used was $CS_2 + 3O_2$. In the lower portion of the picture the flame is seen travelling slowly from left to right in the first tube; it crosses the image of the detonation-wave, which is travelling

from right to left in the second tube, and sends back a wave of retonation when it reaches the end of the tube. The retonation and detonation-waves are apparently parallel.

As regards the dark space formed at the point where the detonation and retonation-waves originate, it is no doubt a space of cooler gas. It persists for some time, and its damping effect on the passage of the collision-wave can be observed in several of the photographs. In fig. 31 this effect is evident just before the collision-wave cuts the black reference line on the right of the picture.

In 1898, a few of our photographs showing the point of detonation and the collision and reflexion of waves in explosion were published in the 'Manchester Memoirs.' Attention was drawn to the wave of retonation and the dark space. It was pointed out that the detonation "was represented by the straight and intensely-luminous line as distinct from the curved and less luminous line of the recouping period."

"Further, a new and very luminous wave is observed starting from the point of re-determination of the explosive-wave proper, and travelling in the opposite direction with a speed almost, if not quite, equal to that of the explosive-wave itself."

In June, 1900, LE CHATELIER published* an account of his photographs of the development and propagation of the detonation-wave (taken on a moving plate). Without knowing of our work, he observed and recorded the "wave of retonation" and the "dark space" formed in the explosion of acetylene with oxygen, and the waves reflected from the end of the tube and at the point of collision of two waves:—

"1. Au moment du développement spontané de l'onde explosive une onde condensée *rétrograde* est toujours lancée en arrière dans les gaz déjà brûlés ;

"2. L'arrêt complet ou partiel de l'onde explosive contre l'extrémité fermée (ou dans une région étranglée) d'un tube lance en arrière une onde condensée *réfléchie* ;

"3. Au point de rencontre d'ondes explosives allumées simultanément en différentes parties d'une masse gazeuse, leur extinction simultanée donne naissance à des ondes condensées *prolongées* qui progressent dans la même direction que les ondes explosives auxquelles elles succèdent."

For the velocities of these waves in the mixture $C_2H_2 + O_2$ LE CHATELIER gives the following values:—

Wave.	Rates, metres per second.
L'onde explosive (detonation-wave)	2990†
L'onde rétrograde (retonation-wave)	2300
L'onde réfléchie (reflexion-wave)	2250
L'onde prolongée (collision-wave)	2050

* 'Comptes Rendus,' vol. 130, p. 1755.

† I found the rate of detonation in the mixture $C_2H_2 + O_2$ to be 2961 metres per second, by measurements in a long tube ('Phil. Trans.,' 1893, A, p. 161), a rate in close agreement with LE CHATELIER's number.

While our observations and those of LE CHATELIER are mainly in accord, we differ from him in his conclusion that the wave of retonation (*l'onde rétrograde*) is propagated in the *burnt* gas; we believe it is propagated in the *still burning* gas. With regard to the effect of collisions between two detonation-waves, LE CHATELIER appears to consider the *act of crossing* of one wave by another to be sufficient to damp down their velocity; we, on the other hand, regarding the "prolonged wave" of the one as the reflected wave of the other (which is a mere verbal distinction), and attribute the retardation of the reflected wave to its altered character and to the movement of the gas which meets it.

PART VI.

ON THE INITIAL PHASES OF THE EXPLOSION.

(*In conjunction with R. H. JONES and J. BOWER.*)

The interest attaching to the development of the explosion, as well as a desire to investigate the anomalies shown by some of our pictures, led us to attempt to photograph the flame from the beginning.

In fig. 28 (Plate 13), the initial flame, starting from the left-hand lower corner of the photograph, is overtaken by a faster-moving flame. Fig. 39 (Plate 14) shows a similar phenomenon in the right-hand corner. On photographing in a straight glass tube the region of explosion prior to the detonation, we found (fig. 52, Plate 16) the initial flame was overtaken by a bright and well-marked faster flame, and, at their point of meeting, a reflected wave was driven back, and the advancing flame of explosion became faster and more luminous. What is the origin of this faster wave which overtakes the advancing flame?

A mixture of cyanogen with twice its volume of oxygen was found to give a sufficiently luminous flame from its start to be photographed on the moving film, but mixtures of carbon disulphide with oxygen gave still better images. Fig. 53 (Plate 16) with the cyanogen mixture, and fig. 54 with the carbon disulphide mixture, showed that the bright wave which overtook the initial flame *came from the end of the tube near the firing wires*. The wires in the tube used were sealed through the glass 3 inches from one end, and the glass had been so much distorted at the point of sealing that a dark band was shown on the photograph where the wires were inserted.

Another tube was prepared with firing wires inserted 4 inches from one end; the slight distortion at the wires acted like a convex lens and increased the light at this point. Fig. 55 shows the development of the explosion with $\text{CS}_2 + 5\text{O}_2$; fig. 56

with $\text{C}_2\text{N}_2 + \text{O}_2$, and fig. 57 with $\text{C}_2\text{N}_2 + 2\text{O}_2$. In all cases the flame begins to travel right and left from the wires with equal velocity in both directions. In fig. 56 the flame develops the retonation-wave at the near end of the tube (on the left) as is shown by the intense-wave running nearly parallel to the detonation, which is started at about the same distance on the other side of the firing point. In the less rapid explosions it is seen that the flame does not travel direct to the near end of the tube, but while still a short distance from it recedes and again approaches with an oscillatory motion which is repeated before the flame finally reaches the end of the tube. From the point where the flame is first checked, a luminous wave is seen (in fig. 55) running back and overtaking the main flame, which at this point acquires greater brightness and velocity. How does this new "return-wave" arise?

Now, when an explosive mixture in a tube is fired by a spark, the suddenly ignited gases must expand and transmit a compression-wave in both directions. This travels with the velocity of sound in the unburnt gas, and will be reflected from the end of the tube. The propagation of the flame from the firing point is in most gaseous mixtures less rapid than the velocity of sound in the unburnt gas, but the rate of propagation of the flame augments much more rapidly in some mixtures than in others. If the tube is a long one the flame will overtake the sound-wave after a more or less prolonged chase, according to the nature of the mixture. But if the tube is short, the sound-wave may reach the end of the tube and return as a reflected-wave to meet the flame which is still advancing. This seems to be the origin of the "return-wave."

Fig. 57A gives in outline the movements of the flame in fig. 57. The flame, starting at A, moves to the left, tracing the curve A to C. At C the detonation is set up, and the retonation-wave C D is thrown back. Now the velocity of sound in the unburnt mixture $\text{C}_2\text{N}_2 + 2\text{O}_2$ is about 312 metres per second. If a sound-wave of this velocity had started towards the left-hand from A at the same time as the flame, it would have been overtaken by the flame at the point B; but the sound-wave starting to the right-hand would reach the end of the tube at E and been reflected to G before meeting the flame. On constructing the figure from the approximately-known velocities of the film and the flame, the point G is seen to be the spot where the movement of the flame is retarded and the "return"-wave is visible—within the limits of experimental error.

The photograph 58 permits this point to be determined with considerable accuracy. The gases $\text{C}_2\text{N}_2 + \text{O}_2$ were fired *close** to one end of a tube 8 inches long (205 millims.). In a second tube, parallel with the first, the detonation-wave in the same mixture was set up and the two explosions photographed together. The detonation is shown at the top of the picture. Fig. 58A gives an outline of the explosion of the first tube. Starting at A, the flame moves to the left, but is checked

* The spark was passed between wires which just penetrated the stopper.

at C, where a "return"-wave is propagated backwards. The return-wave is reflected at E and meets the flame (which has slightly receded) at D. The flame then advances rapidly to the end of the tube and sends back a retonation-wave from G. Now in this mixture a sound-wave travels about 300 metres per second. If a sound-wave started from A at the same time as the flame, it would reach the end of the tube at B and be reflected before meeting the advancing flame. On constructing the figure this reflected sound-wave was found to hit the point C.

In another experiment the mixture $C_2N_2 + O_2$ was fired *at* one end of a tube $8\frac{1}{2}$ inches long (215 millims.). No slackening of the flame was found, and no return-waves are visible. A very intense retonation-wave was thrown back from the end of the tube. On setting out the sound-wave from the firing point we find that the sound is overtaken before it reaches the end of the tube (fig. 59.)

In fig. 60 we see the effect of firing the mixture $C_2N_2 + O_2$ 4 inches from one end of a tube 13 inches long. Fig. 60A shows the outline of the visible waves and the path of the two sound-waves. The sound starting from A reaches the near end of the tube just before the flame; its reflexion coalesces with the intense retonation-wave. The sound-wave marching to the left is overtaken about $5\frac{1}{2}$ inches from the spark. In fig. 61 the same mixture is fired in the centre of a tube 8 inches long. It confirms the result obtained in fig. 60; two intense retonation-waves are started simultaneously at either end. From these experiments we should infer that, if this mixture were fired at a point less than 4 inches from the end, the flame would be checked and return-waves would become visible. Fig. 62 shows this mixture fired in the centre of a tube 6 inches long. The flame is checked symmetrically and the sound-waves produced cross and recross with great intricacy.

With the less rapid mixture $C_2N_2 + 2O_2$ some experiments were made by firing the gas at the extreme end of the tube. A sound-wave starting at the firing point and travelling at the rate of 312 metres per second would be overtaken by the flame just before reaching the end of the tube, which was 12 inches long (fig. 63). On the other hand, when the tube was shorter the flame is checked by the returning sound-wave, which becomes visible as it traverses the incandescent gases (fig. 64). When the same mixture was fired in the centre of a tube 12 inches long, the sound-waves reach the two ends first, and produce symmetrical reflexions (fig. 65).

These measurements afford, I think, conclusive evidence that compression-waves advance in front of the flame at the beginning of the explosion. When the mixture is fired the gases, as they ignite, expand and send a series of pulses through the unburnt gas, driving the molecules at every pulse bodily forward, and so increasing the pressure in the column of gas ahead. When the firing point is near the end of the tube, the head of this compression-wave returns and meets the advancing flame, and, of course, is propagated with increased velocity through the ignited gases. The return of the compression-wave, of course, checks and may reverse the bodily forward movement of the molecules in the flame, and thus oscillations are set up.

Many of the photographs show what appears to be a definite inferior limit to the flame in its initial phase, *e.g.*, figs. 46, 59, 63, and 64, as if the flame died out promptly. We attempted to explore the dark region left inside the flame by passing sparks through it. But no effect whatever was produced in the photographs—doubtless because the gases were still burning. Indeed, most of the photographs show that the region is only comparatively dark.

I would draw attention to the fact that when no sound-waves interfere with the flame, the retonation-wave is very brilliant; when the flame is crossed several times by reflected waves, the light is more evenly distributed. Nos. 63 and 64, taken under the same conditions, illustrate this fact. The phenomenon is explained on the assumption that in the region traversed by the flame the gases are still burning, and the sound-waves, when they enter this region, bring about an increase in the rate of combination. They are, therefore, of the same nature as, but differ in degree from, the wave of retonation. The more intense the compression-wave, the more rapid the combustion in its path, and the more rapid the cooling in its wake.

The “kick-off” which the explosion gets when the gases are fired near one end of a tube considerably modifies the initial progress of the flame, especially when this is comparatively slow: compare fig. 66 where the spark is passed *at* the end of the tube, with fig. 67, where the spark is passed 4 inches from the end of a similar tube. For this reason the experiments made on the time required in different gases to develop the detonation showed anomalies which only disappeared on firing the gases by means of wires which just penetrated through the stopper at the end of the tube. The mean distance in which the detonation is developed in different gases is shown in the following table, according to the position of the spark:—

TABLE VI.—Distance from the spark at which Detonation is set up.

Mixture.	Spark <i>at</i> end.	Spark 3 inches from end.
$2\text{H}_2 + \text{O}_2$	4 feet	12 inches
$2\text{H}_2 + \text{O}_3$	—	4 feet
$6\text{H}_2 + \text{O}_2$	—	16 „
$\text{C}_2\text{N}_2 + \text{O}_2$	$8\frac{1}{2}$ inches	4 inches
$\text{C}_2\text{N}_2 + 2\text{O}_2$	12 „	10 „
$2\text{C}_2\text{H}_2 + 3\text{O}_2$	$4\frac{1}{2}$ „	$2\frac{1}{2}$ „
$\text{C}_2\text{H}_4 + 2\text{O}_2$	9 „	5 „

The mixtures most affected by the position of the spark among those experimented with are those of hydrogen and oxygen. The considerable “run” required before the detonation is set up when electrolytic gas is diluted, confirms my previous experiments on the rate of explosion of these mixtures.

The facts established above concerning the initial phases account for the great difference observed in the appearance of an explosion produced by a spark in the

centre of a tube. If the flame overtakes the sound-wave before it reaches the ends of the tube, as in fig. 61, the explosion exhibits intense retonation-waves followed by rapid cooling; but when the sound-waves are first reflected from the ends of the tube, as in fig. 65, the explosion shows less intense waves and a longer illumination.* These differences in the mode of burning must affect the maximum temperatures and pressures produced in an explosion.

Gases Fired at Open End of Tube.

We have made a few experiments on the appearances presented when gaseous mixtures are fired at or near the open end of the tube.

As is well known, from LE CHATELIER'S researches on the vibratory period, a mixture of carbon disulphide and nitric oxide gives oscillations of large amplitude. We show one photograph of the explosion of the mixture $\text{CS}_2 + 8\text{NO}$ fired at the open end of a tube 5 feet long and 1 inch bore. Only the last foot of the tube is shown in fig. 68.

The effect of firing the mixture $\text{C}_2\text{N}_2 + \text{O}_2$ 3 inches from an *open* end is the same as that of firing it *at* the closed end. Fig. 69 shows the detonation set up in about 8 inches, and of course no rebound from the open end. The mixture $2\text{C}_2\text{H}_2 + 3\text{O}_2$ lighted at the open end produced the detonation in $4\frac{1}{2}$ inches; lighted by a spark 3 inches from the open end, it set up the detonation in $2\frac{1}{2}$ inches from the spark.

PART VII.

FURTHER EXPERIMENTS ON THE INITIAL PHASES.

(*In conjunction with B. DAWSON, B.Sc., and L. BRADSHAW, B.Sc.*)

1. LE CHATELIER'S *Hypothesis of Discontinuity in the Explosion.*

Many of the photographs previously referred to show a peculiarity at the point where a less luminous line is succeeded abruptly by a more luminous one. The lines photographed do not appear *continuous*, but the more luminous line appears to start from a point not yet reached by the less luminous one. Figs. 28 to 31 (Plate 13) illustrate this discontinuity. The point of collision also of two waves appears to project in front of the waves which are meeting. Figs. 29 to 31 illustrate such a collision. It appeared to me at first as if these appearances might be due to invisible waves advancing in front of the visible ones, but as I found that they only showed

* The very interesting photographs, obtained by Mr. PETAVEL, of the movements of a piston connected with a chamber in which gases under high pressure are exploded, show vibrations which may be due to reflexion-waves.

where the luminosity of the lines was in marked contrast, and disappeared entirely when the films were less sensitive, or the contrast of luminosity was diminished for other reasons, as in figs. 40 and 41 and many other photographs (which are not reproduced), I came to the conclusion that the effect was due to halation on the photograph, the brighter lines being enlarged.

But in 1900 LE CHATELIER, relying on the same kind of evidence, put forward the view that the wave of detonation starts *in front of* the variable wave (which is increasing in velocity), and originates in an invisible wave, which is proceeding in front of the visible wave and with a velocity equal to it. He says:—

“ Il n’y a, dans aucun cas, continuité entre la période variable et l’onde explosive. Celle-ci prend naissance à une certaine distance en avant de la flamme à vitesse variable, à 0·05 mètre dans le mélange $C_2H_2 + O_2$. Ce fait est accusé par un ressaut de la courbe photographique enregistrée; . . . Dans cette période variable, la flamme est précédée d’une onde comprimée qui marche devant elle avec une vitesse égale, comme le font à la surface de l’écran les ondulations qui précèdent la proue d’un navire. Une fois l’onde explosive développée, les deux phénomènes se superposent, c’est-à-dire que le front de l’onde comprimée coïncide avec la tranche gazeuse en combustion, au lieu de la précéder.”

This definite judgment of the brilliant French experimenter compelled us to re-examine the question. We attempted at first to decide the matter by photographing an explosion as it passed from a less luminous mixture into a more luminous one; but we could not succeed in making the transition sufficiently sudden. However quick we were in pushing a short column of $C_2N_2 + O_2$ into one end of a tube filled with $2H_2 + O_2$, and firing the latter, the gases had diffused enough to prevent any abrupt change in the brightness of the explosion. We did, however, succeed in obtaining sudden changes of brightness by introducing a layer of “Welsbach” salts (a mixture of thoria and ceria), and having the rest of the tube quite clean. Although to raise the salt from the glass and to render it incandescent must take some time, nevertheless the photograph shows a small but distinct break in the line of detonation similar to that in question. Figs. 70 and 71 show the passage of the wave of detonation in electrolytic gas through such a “salted” tube. If the salts could be raised instantaneously, it is possible that the break would be as well-marked as any observed in the development of the detonation.

It is of course easy to show the enlargement due to brightness. If a tube is filled with a mixture giving a luminous explosion, and the explosion is photographed while half the tube is covered over, and if the tube is then filled with a mixture giving a less luminous explosion, which is photographed on the same film while the first half of the tube is covered, a photograph is obtained (fig. 72) which shows a greater discontinuity than any of those in question.

Another way of showing the same thing is to photograph a thin platinum wire

stretched by weights and rendered luminous by an electric current. If a second wire is brought to touch the first so as to divide the current, the portion of the wire which carries the whole current is more luminous than the other portion, and the photographs make it appear of far greater diameter. Fig. 73 is a photograph taken in this way, with the same camera and films as were employed for the explosions. If the effect in the explosions is due to halation, we ought never to see the bright line displaced so much as not to overlap the duller line *on both sides*. None of our photographs show such a displacement, the effect might therefore be caused by halation alone.

We may thus summarise the evidence against the existence of LE CHATELIER'S "invisible wave":—

1. Its supposed effect is only seen when the contrasts are strong, and not on photographs of the same phenomena in which the contrasts are not brought out.
2. It can be imitated in various ways by means of contrasts.
3. The same effect is seen in the collision of two detonation-waves, but LE CHATELIER does not suppose that the "invisible wave" can precede the detonation.

2. *Repetition of v. OETTINGEN and v. GERNET'S Experiments.*

The very short time required for the explosion in electrolytic gas to raise the Welsbach oxides to incandescence (as shown in figs. 70 and 71) was strong evidence against the view held by v. OETTINGEN and v. GERNET, viz.: that the detonation of electrolytic gas is invisible, and that the salts present in their experiments only became luminous after the combustion had been for some time complete. Our previous experiments had also shown conclusively that the detonation is not set up *at once*, but only after the flame has run some distance, which varies with the nature of the mixture and the position of the spark. But to place the matter beyond all doubt we have repeated their experiments, using a tube of the same size and construction as theirs, filled with electrolytic gas, but without the addition of any salts. By careful development, the course of the flame can be seen on the negatives from the firing wire. In all cases the explosion begins slowly, and has slight luminosity until the detonation-waves are started by reflexion from the ends of the tube.

In fig. 74 the explosion was started in the centre of the tube (400 millims. long). Just as in v. OETTINGEN'S "fig. 8" (our fig. 5, Plate 10), a line of light joins the spark to the more luminous portion of the explosion; but instead of a detonation-wave travelling three and a half times the length of the tube before the first visible compression-wave descends from the top of the tube (as in their explanation), the flame is seen to travel slowly right and left until it meets with the return sound-wave from the end of the tube. The flame is checked while these two sound-waves cross the ignited gases,

and then moves on rapidly from the point where the reflected sound-wave reaches the front of the flame. The movement on each side is now seen to be unsymmetrical; the flame to the left does not reach its end of the tube before it is again checked, that to the right reaches the end of the tube and sends back a powerful retonation-wave. The consequence of this dissymmetry is the greater intensity and rapidity of the wave started from the right over that from the left. Although the reflexions of these waves at first run nearly parallel, yet after four journeys to and fro the stronger wave catches the weaker and coalesces with it, and the rest of the picture shows only repeated reflexions of a single wave. A similar coalescence of two waves is shown in fig. 7 of v. OETTINGEN and v. GERNET, and this figure closely resembles a portion of our photograph. It is thus evident how "secondary" waves running parallel with the "primary" are produced; there is no need to invoke any "successive partial explosions" to account for them.

Fig. 74 shows in outline the path of the flame and the sound-waves, which may be compared with the "schema" of fig. 8, according to v. OETTINGEN and v. GERNET.

In fig. 75 the gas was also fired in the middle, but the explosion is more symmetrical, and "nearly parallel" waves can be seen to be produced by reflexion from both ends of the tube.

Our next photograph (76) shows the effect of firing the gas near one end of the tube. The flame proceeds with increasing velocity to the further end, where a strong wave is sent back; this single wave shows less complications than the double wave started in 74.

In 77 the gases are fired at one-fourth the length from one end. Very complicated reflexions are produced, which only gradually become absorbed. In this photograph the mode of formation of "nearly parallel waves" can easily be traced.

In fig. 78 is seen the effect of lighting the explosive mixture at one end, and at a point one-quarter from the further extremity simultaneously. The effect of the sound-wave proceeding from each point of ignition is plainly seen when it reaches the *other* flame. The *reflected*-waves, visible in the burning gases, cross the unignited mixture, and again become visible in the flames beyond. When the two flames coalesce the picture resembles the other photographs. In fig. 79 the gases were ignited simultaneously at each end of the tube.

[The last three photographs, reproduced in figs. 80, 81, and 82, show the detonation-wave set up in a long tube reaching a portion of gas which (having been independently ignited) is still in the initial stage of combustion. In fig. 80 the gas was lighted in the centre of the tube just before the detonation-wave arrived. The detonation is slightly damped down on meeting the already ignited gas. In fig. 81 the detonation-wave strikes a flame started *at* the end of the tube. In fig. 82 the initial flame had spread considerably before the detonation struck it near the left-hand edge of the photograph. In each case the wave is propagated through the already burning gas like a retonation-wave.—H. B. D. and L. B., *October*, 1902.]

DESCRIPTION OF THE PLATES.

PLATE 10.

MALLARD and LE CHÂTELIER'S photographs of explosions in tubes fixed *vertically*.

- Fig. 1. Explosion of the mixture $\text{CS}_2 + 6\text{NO}$ ignited at the open end of a tube 3 metres long and 20 millims. in diameter.
 Fig. 2. Same mixture fired at open end of a tube 10 millims. in diameter.
 Fig. 3. Mixture of $\text{CS}_2 + 6\text{NO}$ fired near closed end of tube 2 metres long. A "rebound-wave" from the further end of the tube is shown.
 Fig. 4. Mixture of CS_2 and oxygen fired at open end of tube. A detonation-wave is set up.

VON OETTINGEN and VON GERNET'S photographs of explosions in a *vertical* tube containing metallic salts.

- Fig. 5. Electrolytic gas fired in centre of tube 400 millims. long.
 Fig. 5A. Supposed path of waves in fig. 5.
 Fig. 6. Electrolytic gas fired 50 millims. from one end of tube.
 Fig. 6A. Supposed path of waves in fig. 6.

PLATE 11.

PHOTOGRAPHS of explosions in *horizontal* glass tubes.

- Fig. 7. Reflexion of wave from closed end of tube ($\text{C}_2\text{N}_2 + 2\text{O}_2$).
 Fig. 8. Reflected wave proceeding backwards ,,
 Fig. 9. ,, ,, ,, ,, ,,
 Fig. 10. ,, ,, in mixture $2\text{H}_2 + \text{O}_2$.
 Fig. 11. ,, ,, ,, $2\text{CO} + \text{O}_2$.
 Fig. 12. ,, ,, ,, $\text{C}_2\text{N}_2 + \text{O}_2$.
 Fig. 13. ,, ,, ,, $2\text{C}_2\text{H}_2 + 5\text{O}_2$.
 Fig. 14. Explosion reaching open end of tube.
 Fig. 15. ,, ,, end of tube loosely corked.
 Fig. 16. Tube broke as explosion reached end of tube.
 Fig. 17. Explosion passing through long tube; movements of the gas are not affected by reflexions.

PLATES 12 AND 13.

- Fig. 19. Sound-waves *meeting* detonation ($\text{C}_2\text{N}_2 + 2\text{O}_2$).
 Fig. 20. ,, wave meeting ,, ($\text{C}_2\text{N}_2 + \text{O}_2$).
 Fig. 21. ,, ,, ,, ($\text{CS}_2 + 2\text{O}_2$).
 Fig. 23. ,, ,, *following* ,,
 Fig. 25. Two explosions meeting end-on.
 Fig. 26. ,, ,, ,, ,,
 Fig. 27. ,, ,, ,, ,,
 Fig. 28. ,, ,, ,, ,,
 Fig. 29. ,, ,, ,, ,,
 Fig. 30. ,, ,, ,, ,,
 Fig. 31. ,, ,, ,, ,,
 Fig. 32. ,, ,, ,, ,,
 Fig. 33. ,, ,, ,, ,,
- } Showing irregularities and sudden increments of brightness and velocity.

- Fig. 34. Sudden increment shown by flame travelling one way only.
 Fig. 35. Detonation passing through three parallel glass tubes joined together by lead junctions. In passing each junction the explosion is damped down and recoups itself suddenly.

PLATE 14.

- Fig. 37. Detonation passing through glass bend *without* retardation.
 Fig. 38. „ „ „ flexible rubber junction *without* retardation.
 Fig. 39. „ „ „ rigid copper junction „ „
 Fig. 40. Collision of true detonation-waves ($C_2N_2 + 2O_2$).
 Fig. 41. „ „ „ „ „ „
 Fig. 42. „ „ „ „ „ ($2H_2 + O_2$).
 Fig. 43. "Retonation"-wave running parallel with collision-wave.

PLATE 15.

- Fig. 44. Retonation-wave and reflexion-wave.
 Fig. 45. „ „ „ started at end of tube.
 Fig. 46. Explosion started in centre of long tube; retonation-waves return from points where the detonation-waves are set up.
 Fig. 47. Explosion started in centre of short tube; retonation-waves are set up at either end of tube ($C_2N_2 + O_2$).
 Fig. 48. Explosion started in centre of short tube; retonation-waves are set up at either end of tube ($2C_2H_2 + 3O_2$).
 Fig. 49. Velocity of retonation and reflexion-wave compared.
 Fig. 50. „ „ „ detonation and retonation-waves (in separate tubes) compared.
 Fig. 51. „ „ „ „ „ „ „ „

PLATE 16.

- Fig. 52. Explosion showing initial wave overtaken by faster wave.
 Fig. 53. „ „ „ that the overtaking flame comes from the near end of the tube.
 Fig. 54. „ „ „ „ „ „ „ „
 Fig. 55. „ „ „ „ „ „ „ „
 Fig. 56. „ „ of $C_2N_2 + O_2$, showing retonation-wave started from near end of tube and running nearly parallel with detonation-wave.
 Fig. 57. Explosion of $C_2N_2 + 2O_2$, showing reflexion of *sound* from near end of tube checking the flame and producing a visible wave in the burning gases.
 Fig. 57A. Plan of 57, showing path of waves.

PLATES 17 AND 18.

- Fig. 58. Explosion of $C_2N_2 + O_2$ in lower tube, showing reflexion of sound-wave from further end of tube. Upper photograph shows detonation of same mixture in another tube.
 Fig. 58A. Plan of 58, showing path of sound-wave.
 Fig. 59. $C_2N_2 + O_2$ fired *at* one end of tube $8\frac{1}{2}$ inches long.
 Fig. 60. Mixture $C_2N_2 + O_2$ fired 4 inches from one end of tube 13 inches long.
 Fig. 60A. Plan of 60, showing paths of sound-waves.
 Fig. 61. $C_2N_2 + O_2$ fired in centre of tube 8 inches long; intense retonation-waves are sent back from each end.

- Fig. 62. $\text{C}_2\text{N}_2 + \text{O}_2$ fired in centre of tube 6 inches long ; the flames are checked symmetrically by sound-waves reflected from the ends of the tube.
- Fig. 63. Explosion of $\text{C}_2\text{N}_2 + 2\text{O}_2$ where sound-wave is overtaken by explosion before reaching end of tube.
- Fig. 64. Explosion of $\text{C}_2\text{N}_2 + 2\text{O}_2$ where sound-wave reaches end of tube first.
- Fig. 65. „ „ „ sound-waves return from both ends of tube and cross flame symmetrically.
- Fig. 66. Explosion of $\text{C}_2\text{N}_2 + 2\text{O}_2$ fired *at* closed end of tube.
- Fig. 67. „ „ „ 4 inches from closed end.
- Fig. 68. Oscillations of flame in explosion of $\text{CS}_2 + \text{SNO}$ after passing 4 feet down tube.
- Fig. 69. $\text{C}_2\text{N}_2 + \text{O}_2$ fired 3 inches from open end of tube.

PLATE 19.

- Fig. 70. Effect of "salting" part of tube traversed by detonation of $2\text{H}_2 + \text{O}_2$, showing "break" in line of detonation.
- Fig. 71. Effect of "salting" part of tube traversed by detonation of $2\text{H}_2 + \text{O}_2$, showing "break" in line of detonation.
- Fig. 72. Glass tube magnified by halation.
- Fig. 73. Platinum wire magnified by halation.
- Fig. 74. Explosion of electrolytic gas in centre of short tube, showing effect of reflected sound-waves.
- Fig. 74A. Plan of 74, showing path of sound-waves.
- Fig. 75. Explosion of electrolytic gas in centre of short tube. The reflexions are nearly symmetrical.
- Fig. 76. „ „ „ fired near one end of short tube. The flame is less complicated.

PLATE 20.

EXPLOSIONS of the mixture $\text{CS}_2 + 3\text{O}_2$.

- Fig. 77. Mixture fired at one-fourth the length from one end: nearly parallel waves are produced by reflexions.
- Fig. 78. Mixture fired at same time at one end and at one-fourth the distance from other end. The sound-waves are visible as they cross the burning gases.
- Fig. 79. Mixture fired at both ends of the tube at same time.
- Fig. 80. Explosion started in centre of tube, and the initial flame struck by detonation coming from left hand.
- Fig. 81. Explosion started at one end of tube, and initial flame struck by detonation coming from left hand.
- Fig. 82. The explosion has spread considerably before it is struck by detonation as before.
-



Fig. 1.

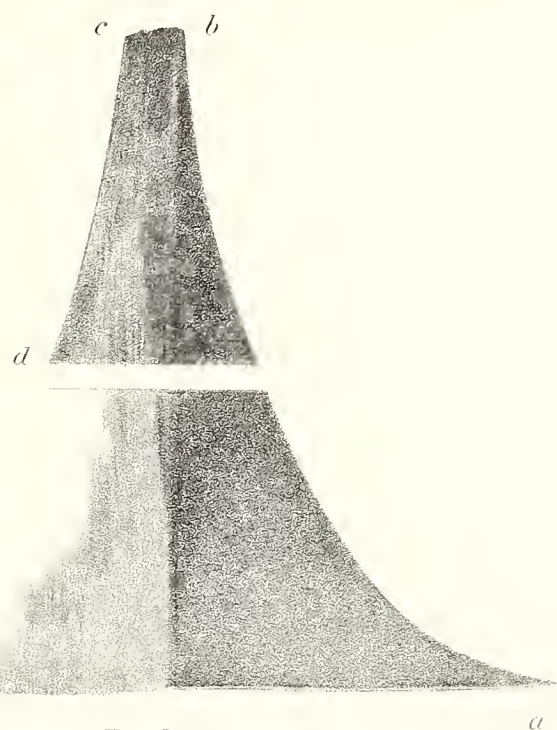


Fig. 3.

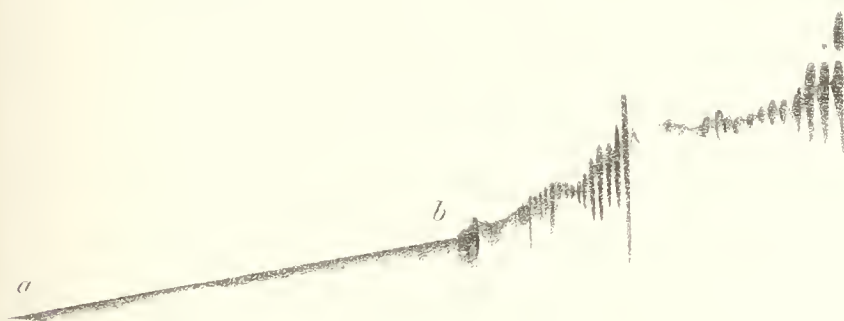


Fig. 2.



Fig. 4.

Mallard and Le Chatelier's Photographs.

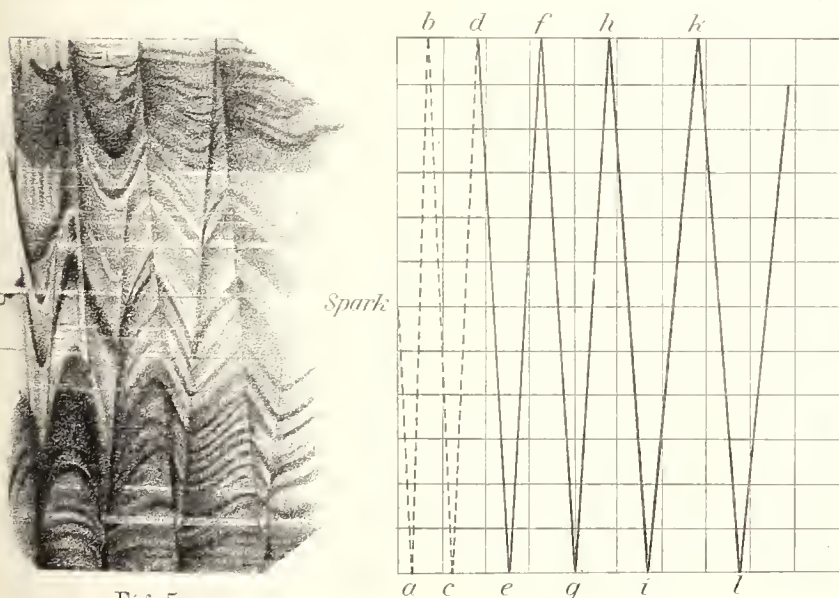


Fig. 5.

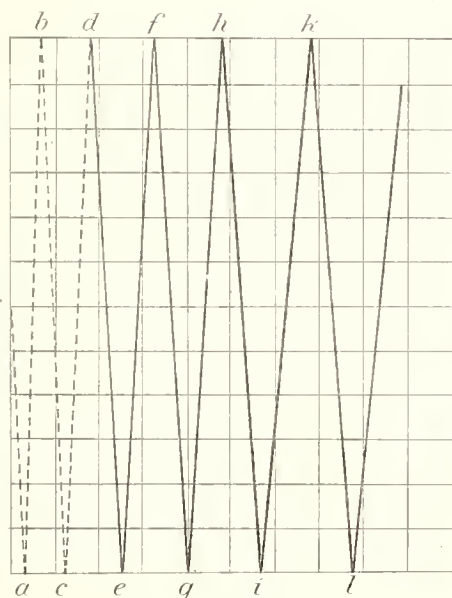


Fig. 5a.

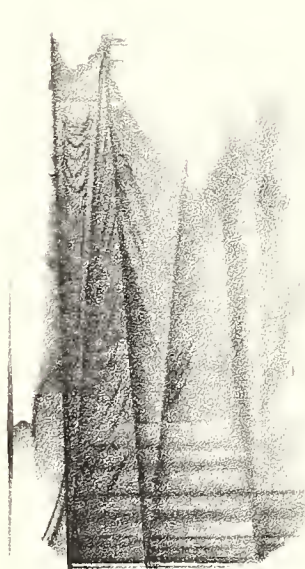


Fig. 6.

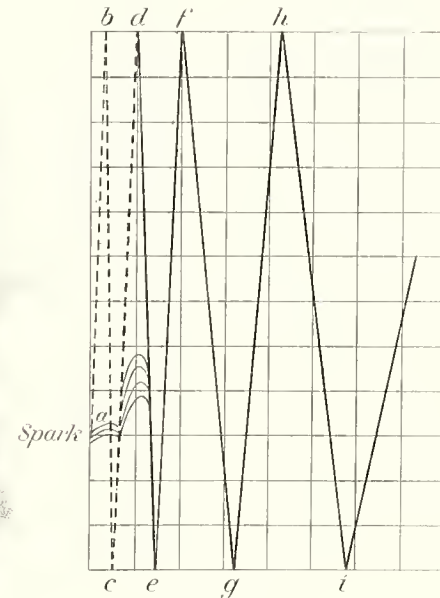


Fig. 6a.

von Oettingen and von Gernet's Photographs.



Fig. 7.



Fig. 8.



Fig. 9.

Explosions of mixtures of $C_2N_2 + 2O_2$, showing waves reflected from closed end of tube.

Fig. 10. $2H_2 + O_2$.Fig. 11. $2CO + O_2$.Fig. 12. $C_2N_2 + O_2$.Fig. 13. $2C_2H_2 + 5O_2$.

Fig. 14.—With end open.

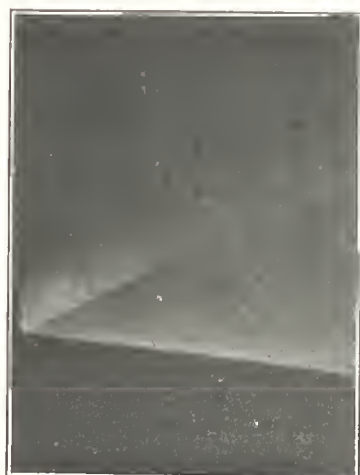


Fig. 15.—End loosely corked.



Fig. 16.—Tube broke.



Fig. 17.—Movements of gas following detonation-wave.

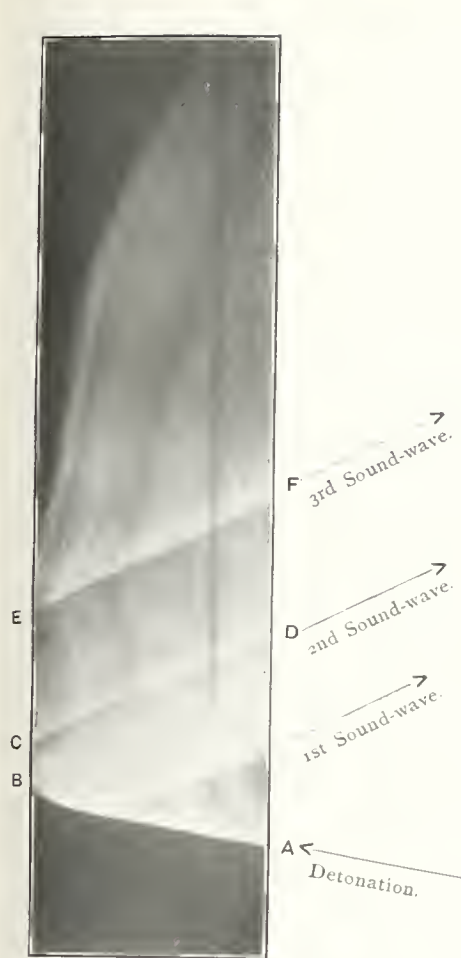


Fig. 19.—Sound-waves meeting detonation of $C_2N_2 + 2O_2$.

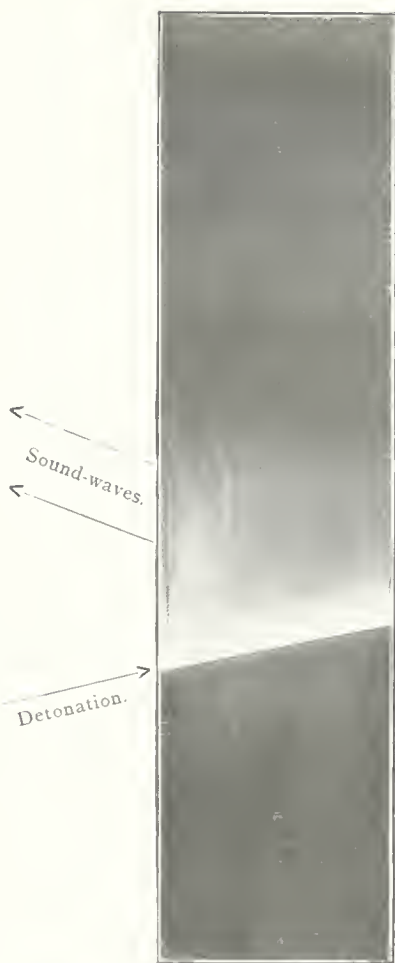


Fig. 20.—Sound-waves meeting detonation of $C_2N_2 + O_2$.

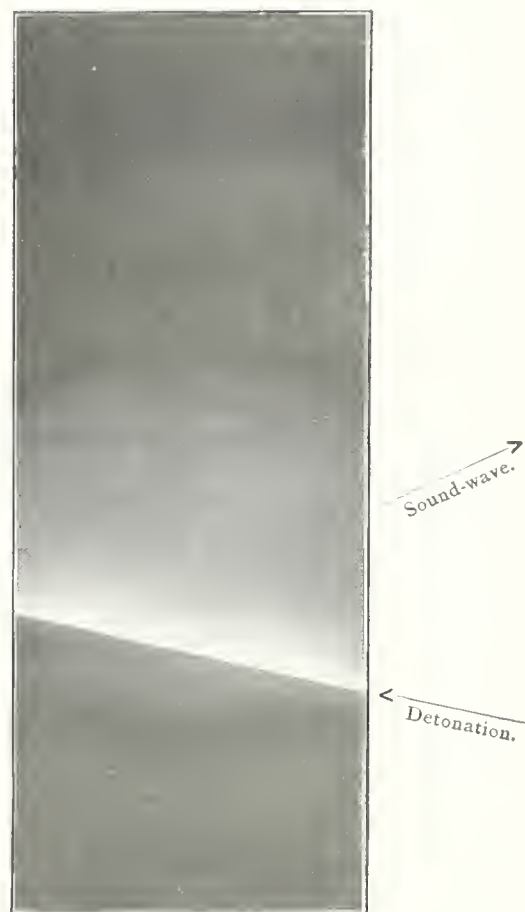


Fig. 21.—Sound-wave meeting detonation of $CS_2 + 2O_2$.

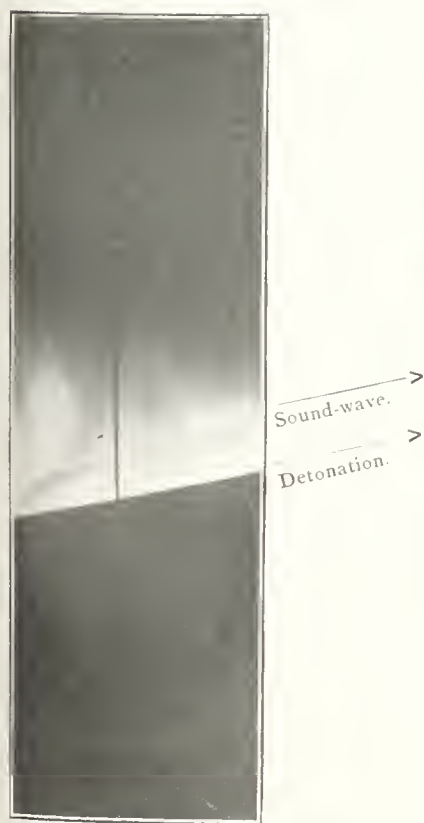


Fig. 23.—Sound-wave following detonation of $C_2N_2 + 2O_2$.



Fig. 25.—Two Explosions meeting.



Fig. 26.—Two Explosions meeting.



Fig. 27.—Two Explosions meeting.



Fig. 28.—Two Explosions meeting.



Fig. 29.—Two Explosions meeting.



Fig. 30.—Two Explosions meeting.



Fig. 31.—Two Explosions meeting.



Fig. 32.—Two Explosions meeting.



Fig. 33.—Two Explosions meeting.



Fig. 34.—Explosion going one way only.

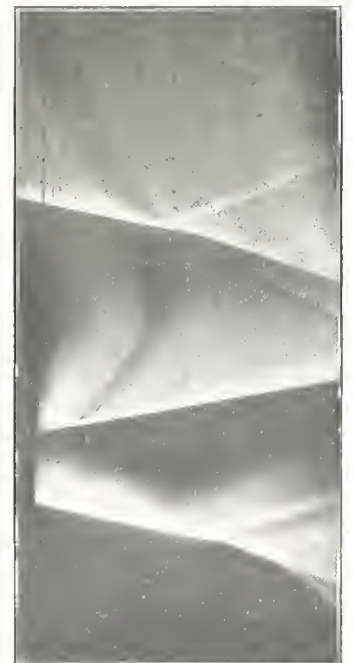


Fig. 35.—Detonation passing through three tubes with lead junctions.



Fig. 37.—Detonation passing through glass bend.

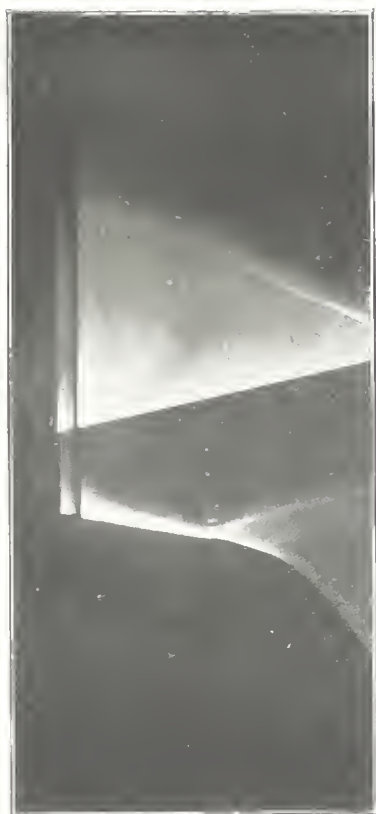


Fig. 38.—Detonation passing through flexible rubber junction without retardation.



Fig. 39.—Detonation passing through rigid copper junction without retardation.



Fig. 40.—Collision of two detonation-waves.
 $C_2N_2 + 2O_2$.



Fig. 41.—Collision of two detonation-waves.
 $C_2N_2 + 2O_2$.



Fig. 42.—Collision of two detonation-waves.
 $2H_2 + O_2$.



Fig. 43.—Retonation-wave running parallel with collision-wave.



Fig. 44.—Retonation-wave and Reflexion-wave.



Fig. 45.—Retonation-wave started at end of tube.



Fig. 46.—Retonation-waves formed by explosion in centre of long tube.



Fig. 47.—Retonation-waves formed at ends of short tube.
 $C_2N_2 + O_2$.



Fig. 48.—Do. $C_2H_2 + O_3$.

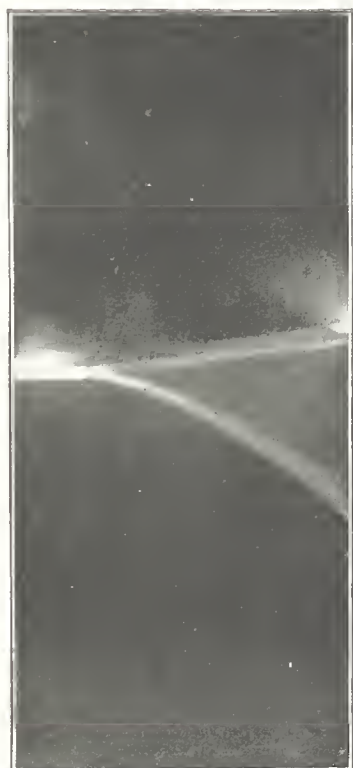


Fig. 49.—Retonation-wave and Reflexion-wave.



Fig. 50.—Detonation in lower tube. Retonation in upper tube.



Fig. 51.—Detonation in one tube; Retonation in other tube.

← Retonation.
← Detonation.



Fig. 52.—Initial wave overtaken by faster wave.



Fig. 53. $C_2N_2 + 2O_2$.



Fig. 54. $CS_2 + 3O_2$.



Fig. 55. $CS_2 + 5O_2$.

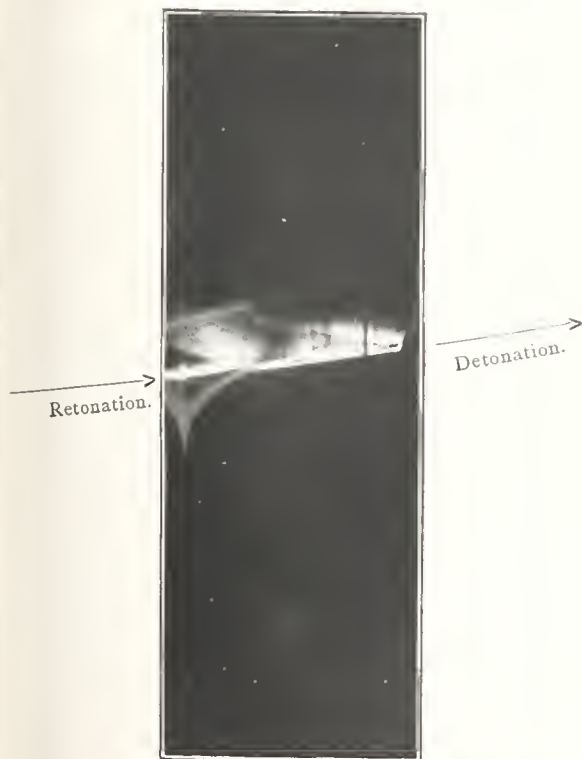


Fig. 56. $C_2N_2 + O_2$.



Fig. 57. $C_2N_2 + 2O_2$.

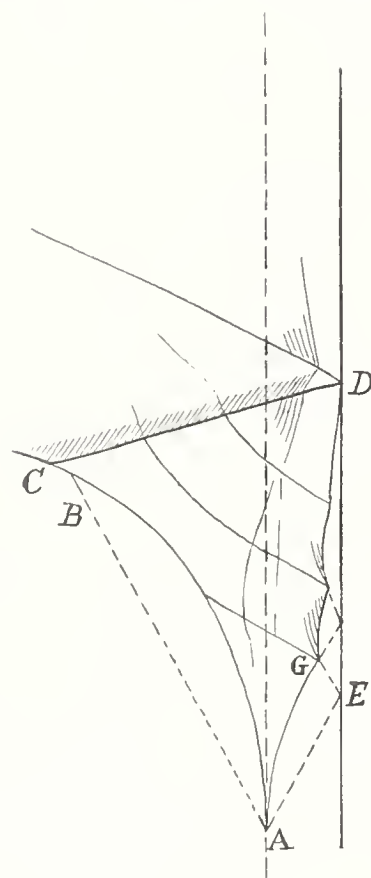


Fig. 57A.



Fig. 58. $C_2N_2 + O_2$.
Detonation in upper tube.
Initial flame in lower tube.

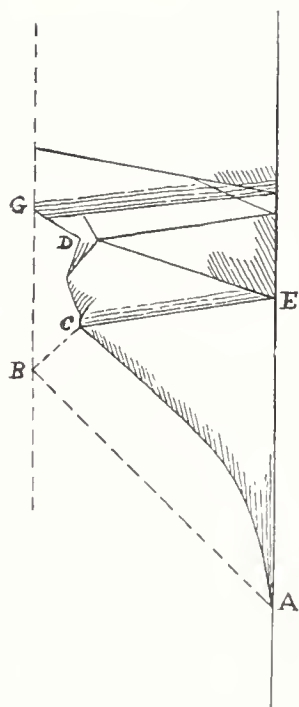


Fig. 58A.

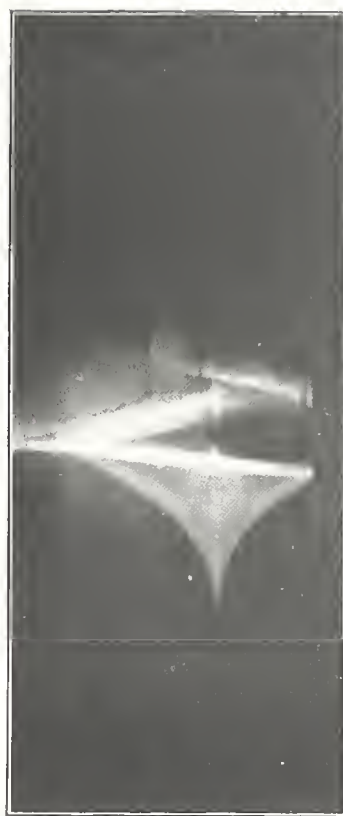


Fig. 60. $C_2N_2 + O_2$.

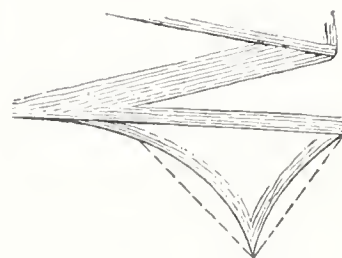


Fig. 60A.

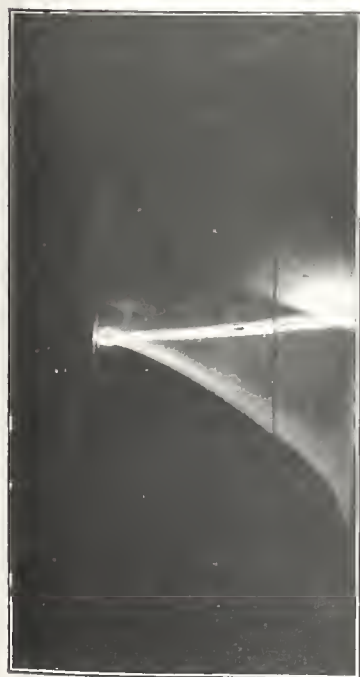


Fig. 59. $C_2N_2 + O_2$.



Fig. 61. $C_2N_2 + O_2$.



Fig. 62. $C_2N_2 + O_2$.

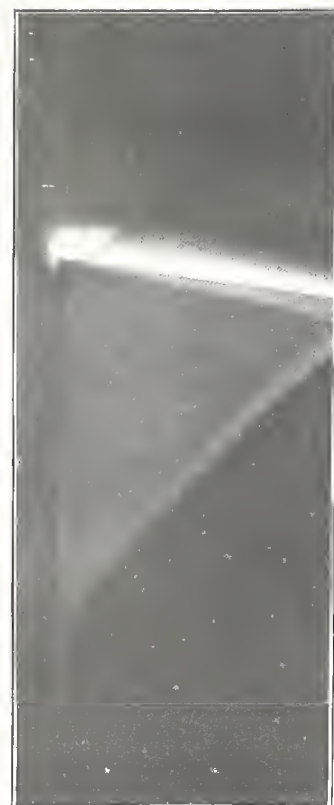


Fig. 63. $C_2N_2 + 2O_2$.



Fig. 64. $C_2N_2 + 2O_2$.
Showing sound-wave return-
ing from right-hand end.



Fig. 65. $C_2N_2 + 2O_2$.
Showing sound-waves return-
ing from both ends.



Fig. 66. $C_2N_2 + 2O_2$.
Fired at closed end.



Fig. 67. $C_2N_2 + 2O_2$.
Fired 4 inches from closed end.

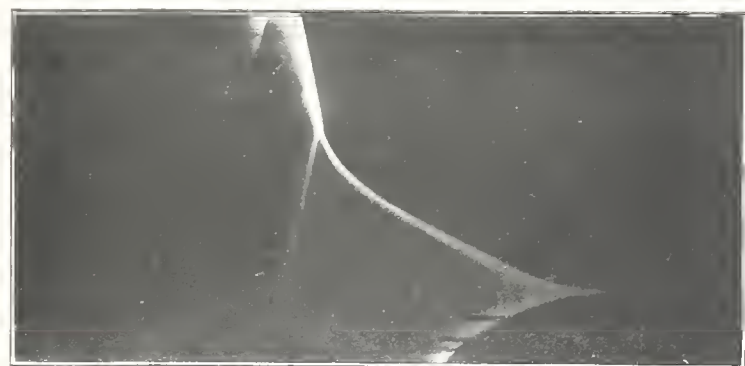


Fig. 69. $C_2N_2 + O_2$.
Fired 3 inches from open end.



Fig. 68. $CS_2 + 8NO$.
Oscillations of flame between 4 and 5 feet from firing-point.



Fig. 70.—Part of tube salted.
 $2\text{H}_2 + \text{O}_2$.



Fig. 71.—Part of tube salted.
 $2\text{H}_2 + \text{O}_2$.

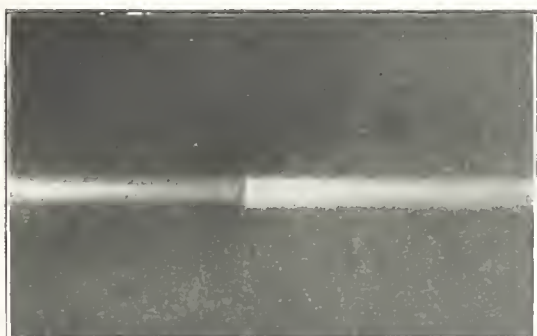


Fig. 72.—Glass tube magnified by halation.

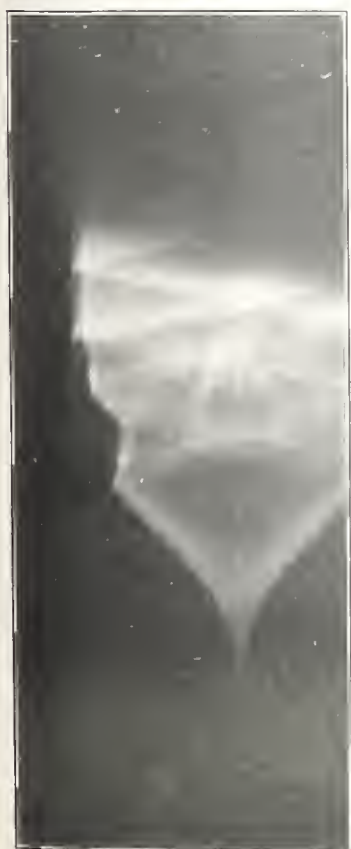


Fig. 75.



Fig. 76.

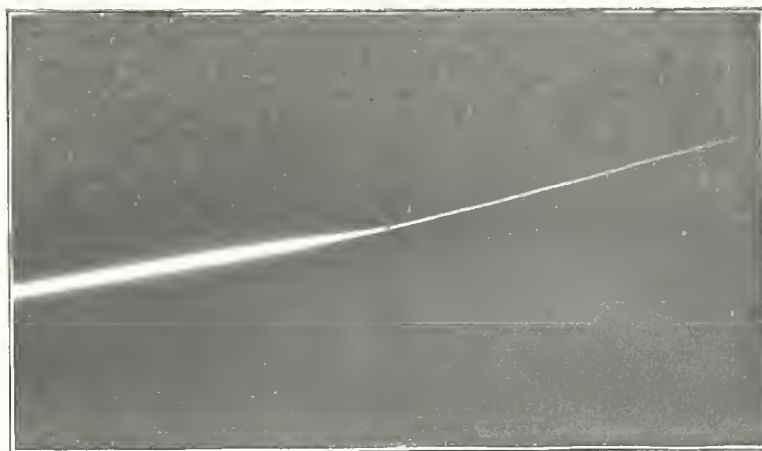


Fig. 73.—Platinum wire magnified by halation.



Fig. 74. $2\text{H}_2 + \text{O}_2$.

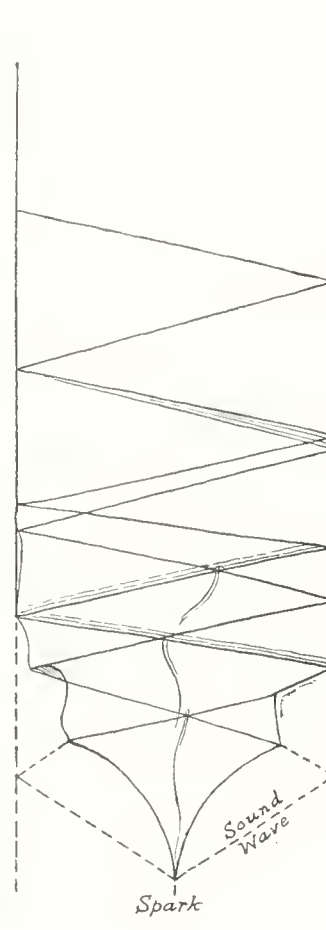


Fig. 74A.



Fig. 79. $\text{CS}_2 + 3\text{O}_2$.



Fig. 78. $\text{CS}_2 + 3\text{O}_2$.

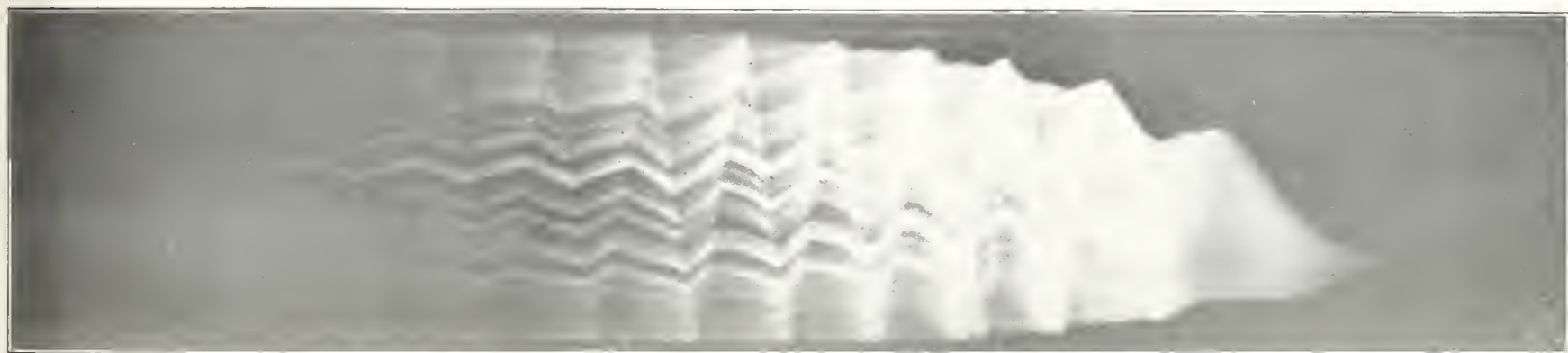


Fig. 77. $\text{CS}_2 + 3\text{O}_2$.



Fig. 80.



Fig. 81.



Fig. 82.

Detonation-wave passing through flames of gas ignited independently.

INDEX SLIP.

ANDERSON, Tempest, and FLETT, John S.—Report on the Eruptions of the Soufrière, in St. Vincent, in 1902, and on a Visit to Montagne Pelée, in Martinique.—Part I.

Phil. Trans., A, vol. 200, 1903, pp. 353-553.

FLETT, John S. and ANDERSON, Tempest.—Report on the Eruptions of the Soufrière, in St. Vincent, in 1902, and on a Visit to Montagne Pelée, in Martinique.—Part I.

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IX. *Report on the Eruptions of the Soufrière, in St. Vincent, in 1902, and on a Visit to Montagne Pelée, in Martinique.—Part I.*

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INTRODUCTION.

THE islands of the Caribbean chain have been occupied by European colonists for several hundred years, yet they cannot even at the present day be said to be thoroughly known or sufficiently explored. Though small, they are for the most part mountainous, and present usually a ridge or backbone of high land forming the main axis of each island, with sharp spurs on each side running down to the sea. Cultivation is practically confined to the lower grounds, where alone there are good roads, and the interior is covered with dense tropical forest, the aspect of which varies greatly with the altitude, and through which there are only rough bush paths. The valleys are usually very deep and narrow, and the steep slopes are covered with plantations of arrowroot, limes, cocoa, coffee, banana or plantain, while most of the level alluvial ground in the valley bottoms is given up to the growth of sugar cane.

In all the British islands, at any rate, the principal peaks and ridges have been ascended, and the main features of the country are delineated on the Admiralty charts, which are the best, and in fact the only available maps. As regards the

coast-lines and the lower grounds generally, they are very accurate; but in the interior only the more important points, the principal mountain summits and the like, have had their position sufficiently determined. The rest of the country has apparently been sketched in more or less carefully—but many of the details as, for example, the courses of the smaller streams, and the number of their branches, cannot be relied on. The want of a good map on a fairly large scale is a great drawback in geological work, and prevents the structure of the country being laid down with any approach to minuteness.

This difficulty is increased by the rich mantle of tropical forest which covers all the surface except those parts which are highly cultivated. In the forests one may walk for hours along narrow winding paths, obstructed by fallen trunks and branches, by the roots of the larger trees, and the tough stems of the lianas, without seeing any exposure except of the deep soft soil, which in a thick layer covers the rock beneath. The branches of the trees meet overhead, shutting out the sun and affording a delightful shade; but to the right and left it is impossible to see for more than a few yards, so dense is the vegetation. Even where in such forest paths the surface of the rocks is exposed, the moist climate and the abundance of decaying vegetation are so powerful in effecting decomposition, that it may be a very difficult matter to ascertain what is their nature, and impossible to obtain specimens sufficiently fresh for microscopic examination. The great rapidity with which the volcanic rocks weather under these conditions, and the great depth to which they are covered with soft earthy decomposed material, is a source of never-failing astonishment to the geologist, and powerfully impresses on his mind the rapidity with which denudation may take place in tropical countries. The rivers and streams in the lower parts of their courses flow, as a rule, through alluvial deposits of gravel and boulders, but in the higher valleys good rock exposures are more frequent, and much may be learned from an examination of their banks. This is, however, no easy task; so dense is the growth, so frequent are cascades and waterfalls, and so numerous the large boulders obstructing the streams, that such exploration is a most laborious undertaking. If we add to this the difficulty of continuous exertion in a temperature approaching 80° F., and an atmosphere saturated with moisture, it will be seen that minute geological mapping is not to be accomplished under the circumstances.

On the other hand, the broad general features of the geology of the country usually can be easily made out with fair accuracy. As already stated, the conditions are all in favour of rapid erosion. The islands are so narrow and their central mountain axes so high that the ground slopes steeply on each side into the sea. The rainfall is heavy, averaging perhaps 100 inches on the low grounds, and much greater at higher elevations, and it is so distributed as to produce the maximum of geological effect. It is greatest in the rainy season, which in most of the islands lasts from the beginning of July to the end of September. In the dry season there is sometimes no rain for days. It usually takes the form of short heavy showers which are soon

over and leave the streams in raging flood, and not unfrequently an inch of rain falls in an hour, and a rainfall of several inches in a day often takes place. This is sufficient to make streams which are usually quite dry, rise and overflow their banks in floods which sweep everything before them to the sea. Even the smaller brooks, which are often dry in fine weather, have their channels filled with very large boulders, 8 feet or more in diameter, which show by their rounded and worn surfaces that they have been transported for some distance. When we remarked this to one of the planters in St. Vincent, he gave us numerous instances in which streams apparently of trifling size had moved large rock masses, and had done great damage to bridges, retaining-walls and other structures. Engineers have encountered great difficulties in making roads through the British colonies to withstand the devastation occasioned by the heavy rains. All the less important thoroughfares and the bridle-paths are frequently rendered impassable by the water cutting across the road and washing out deep channels. These are then filled up by a band of labourers with large hoes, who dig up the soft rotten rock from any convenient spot by the road side and cast it into the furrows, some of them 6 or 10 feet deep.

These deluges of rain not only greatly increase the cutting power of the streams, but they also saturate the soil and render it particularly liable to slip when it is lying on steep ground. Much of the ground under arrowroot cultivation on the leeward side of St. Vincent lies on the side of ridges and slopes at angles approaching 40° . When the plants have been dug up and the surface is left nearly bare, a few showers of heavy rain may cause extensive landslips, and the planter may have all the valuable soil stripped from a considerable part of a field in a few minutes. So deep and narrow are the valleys, that on each side the material is resting practically at the angle of repose, and anything which facilitates interstitial movement or lessens the cohesion of the particles, is sufficient to cause a landslide. Thus it will be readily understood that in a typical Caribbean valley the experienced eye can perceive how every feature of the surface is determined by the underlying geological structure. The coulées of lava stand out as vertical cliffs, which give foothold to only a few scattered plants which have managed to take root in the crevices and cracks. The columnar jointing, though often rude, is usually sufficiently pronounced to yield characteristic surfaces, and to indicate the nature of the rock from a distance. This, however, is not always a safe criterion, for it is possible to find beds of tuff which weather in much the same way. As a whole, the tuffs are softer and crumble more readily than the lavas, yielding abundance of loose material which gathers in sloping taluses on the flanks of the valleys. These weathered ash beds form rich soils, and, where possible, are cultivated; but even in the higher elevations, where there is little cultivation, they are clothed with luxuriant vegetation, and contrast strongly with the bare cliffs of lava.

In most of the islands the coast sections are very fine, and as it is possible to row

in smooth water along the leeward shore, much information can be obtained in this way. The sharp spurs running down from the main axis end as a rule in prominent headlands with high rocky cliffs which give good natural exposures of the geological structure. Between these promontories the streams debouch in little bays, which show a sandy or gravelly beach in their centre. A sail along the leeward shores in a small boat, which keeps close to the land, is a very pleasant and instructive experience for the geologist. Each headland furnishes a clean cut geological section, while in crossing the bays magnificent views are obtained of deep valleys, steep sided, gloriously picturesque, cultivated below, but higher up clothed in tropical forest, and in the background lofty peaks with their summits veiled in mist.

In many respects the investigation of the geological history and structure of the Antilles is one of the most fascinating branches of geological research. The evidence of repeated and prolonged eruptive activity, of great changes of level in comparatively recent times, of profound alterations in the distribution of land and sea, of enormous erosion, and the accumulation of great detrital deposits and masses of organic limestones, have combined to excite the interest and awaken the enthusiasm of geologists. But so far as concerns the recent eruptions and the islands involved in them, there is no need to enter into a discussion of the many interesting and difficult problems of Caribbean geological history.

St. Vincent, Martinique, Dominica, St. Lucia, and Grenada are almost entirely volcanic, and there is no reason to believe that their rocks are older than the Pleistocene. They are built up principally of tuffs and agglomerates, with a smaller proportion of lava flows—the preponderance of fragmental ejecta being typical of the group. The epoch of maximum volcanic activity has probably long since passed away, though there are records of eruptions in historic times in Martinique, St. Vincent, St. Lucia, and Dominica; and in Grenada there is a well-preserved crater with a lake—the Grand Étang. All the islands show abundant solfataric action; most of them contain one or more “Soufrières” emitting steam and sulphuretted hydrogen.

In Guadeloupe and in Antigua there are fossiliferous deposits possibly of Miocene age interbedded in the volcanic tuffs, and it may be that eruptive activity in the Leeward Islands dates back to the Eocene. But in the Windward Islands the oldest fossiliferous beds are Pleistocene or recent. Rocks of this age occur at elevations of several hundred feet above the sea in Martinique, St. Lucia, and Grenada, and to whatever period the outbreak of volcanic activity is to be assigned, there can be no doubt that it has continued through the Pleistocene up to the present day.*

* R. T. HILL, “The Geology of Jamaica,” ‘Bulletin of the Museum of Comparative Zoology at Harvard College, 1899,’ vol. 34 (Geological Series, vol. 4), p. 223.

THE PHYSICAL FEATURES AND GENERAL GEOLOGY OF ST. VINCENT.

The Lavas and Truffs of the Old Volcanoes of the South End of the Island.

The outline of St. Vincent is the elongated oval which is characteristic of the Windward Islands, the main axis lying north and south, and corresponding with the direction of the group as a whole. The length of the island is 18 miles, the greatest breadth 11, and the total area is estimated as 150 square miles. In the coast line there are no important indentations, the largest being Kingstown Bay, near the south end, on which the principal town stands. The only villages of any size are Georgetown, on the windward side of the island, near its north end, and Barrualli and Chateaubelair on the leeward, the latter almost opposite Georgetown. The population of Kingstown is estimated at 7000, that of the whole island at 40,000, and is for the most part black or coloured, there being only 1500 whites. Arrowroot is the staple product—St. Vincent arrowroot has long had a high reputation. Sugar has almost gone out of cultivation, but cocoa, coffee, cocoanuts, nutmegs, bananas, and a great variety of fruits grow well, and are to some extent exported.

The island may be described as almost mountainous, there being little flat land except in the valley bottoms and around the shores of the larger bays. On the windward or eastern side there are old terraces or benches of marine erosion which form a narrow rim of comparatively level country skirting the coast. On the leeward side the land slopes steeply to the sea. The average density of the population is high—nearly 300 per square mile—or almost equal to that of the Isle of Wight and some of the less populous English counties. Yet it is only the valleys and the low ground near the coast which are really inhabited, for the higher ridges are covered with forest and bush, through which there are few paths. Some of the mountain peaks have rarely been ascended, as may be understood when it is explained that a journey in these regions cannot be made without several sturdy labourers armed with cutlasses with which to hew a way through the dense undergrowth which often obstructs and may obliterate the paths. Owing to the fertility of the soils, the warm climate and the abundant rains, a very small plot of ground is sufficient to support a black man and his family in comfort. They require little clothing, and there is almost no demand for luxuries. Many of the blacks are employed as labourers and as servants on the estates, but it has of late years been the policy of the Government to plant them out on the Crown lands as peasant proprietors, each on his own small allotment. The best ground, however, is always occupied by the arrowroot and sugar estates, many of which have been in cultivation for nearly 200 years.

A ridge of hills runs along the centre of the island in a north and south direction, the principal peaks being the Grand Bonhomme (3193 feet), the Morne Garu and

Richmond Peak (3528 feet), and the Soufrière (4048 feet). If we except the last, which forms the north end of the chain, we may say that the form of all these mountains indicates that they have suffered prolonged and intense erosion. Though of volcanic origin, none of them shows a crater or a well-preserved cone. A series of radiating valleys, very deep and narrow, has been cut into the old volcanic pile, and between these valleys there are high steep-sided spurs, the summits of which are knife edges often only broad enough to serve as a footpath. In the recent geological history of the island erosion has been of vastly more importance than accumulation. This is due to the causes already enumerated; and nowhere better than in St. Vincent can the rapidity with which denudation takes place in a moist tropical climate be studied and exemplified. Excepting on the steepest slopes, where loose accumulations will not rest, the rocks are deeply covered with weathered material, which has either been formed *in situ* or has been carried down by landslips or as rain-wash.

The rainfall is not equally distributed over the year, as June, July, and August are usually the most rainy months, while January and February, March and April, constitute the "dry season." It averages about 110 inches in the year, according to the records kept for many years in the Botanic Gardens at Kingstown.* Among the mountains, however, especially on the windward side, it is much greater than this. Most of it falls in short heavy showers, which fill the rivers up to their banks, and make them difficult or dangerous to cross when they are of any size. Even after gentle showers the water is muddy from the quantity of sediment it carries. The smaller brooks have usually their channels encumbered with large boulders, which have obviously been brought down in floods and left stranded when the water subsided. The steep and winding road which passes along the leeward shore is in this way often obstructed by falls of earth or rock. A very brief residence in the Windward Islands is sufficient to convince a geologist that erosion is there proceeding with great rapidity, and that although the volcanic masses have been deeply sculptured, they are not necessarily of great geological age.

The axial mountain ridge is as a whole fairly continuous, at any rate from Mount St. Andrew, which overlooks Kingstown to Richmond Peak, which is a few miles south of the Soufrière. Some of the streams, however, in cutting back the heads of their valleys from opposite sides of the island, have nearly succeeded in meeting across the main ridge and uniting to form a low open channel from side to side of the island. The valley of the Buccament has been cited by Professor SPENCER as "crossing the

* Mr. HENRY POWELL, Curator of the Botanic Gardens in Kingstown, St. Vincent, has kindly furnished us with tables of the rainfall at that station during the last seven years (including this year up to the end of November). The average monthly rainfalls for that period are as follows:—January, 5·78 inches; February, 4·29 inches; March, 3·79 inches; April, 4·22 inches; May, 9·85 inches; June, 12·87 inches; July, 11·34 inches; August, 11·97 inches; September, 13·69 inches; October, 10·84 inches; November, 12·05 inches; December, 9·78 inches. It will be seen that the first four months of the year constitute the "dry season." The average yearly rainfall is about 110 inches.

island to the sea, so that a comparatively small submergence would convert it into a strait.”* But a better instance of this is the broad open valley on the south side of the Soufrière, between that mountain and the Morne Garu. It is a significant fact that there is no good road from the leeward to the windward side over the mountains, and that the only passable footpath was an old Carib track which started from Chateaubelair and went to Georgetown. It led right up to the edge of the crater of the Soufrière before descending on the other side, an ascent of 3000 feet. It stuck to the summits of the spurs between the valleys, where the vegetation was least dense, and there was least probability of a wash-out due to the sudden rains and floods. At first glance it would have appeared easier to follow the course of the valley across the island, but this would have passed through dense forests, and the lateral streamlets descending the side of the valley to join the rivers would have been difficult to cross in time of flood, and would have caused much damage to any road carried over them. In such a situation a road would also have been exposed to landslides and falls of rock, which would have often rendered it impassable.

In consequence of the small size of the island, none of the rivers of St. Vincent are of any magnitude. Their courses are remarkably short and straight, being mostly nearly at right angles to the coast-line. The drainage is of a very simple “consequent” type, and intimately connected with the geological structure of the island. The most striking feature of the stream-valleys is their great depth, especially in their upper parts, where they present many of the characteristics of cañons. This is best seen in the devastated country, where the surfaces are now quite bare and have a desert aspect.† The casual streamlets which descend the valley walls after sudden rains form niches which furrow the surface, and the alternation of vertical cliffs of lava with sloping taluses of weathered ash produces features which strongly recall those of the cañons of western North America. As the streams are still cutting rapidly along most of their length, it is rare to find any extensive plains of alluvial deposit except near their mouths. The sea is too deep for the formation of deltas, but in many of the bays there is a small stretch of flat ground with usually a curved storm beach facing the sea. Higher up the valleys narrow flood plains may be found, but they are few; and sugar cultivation, which is best carried on on level ground, is almost entirely confined to the coast.

The whole of St. Vincent is of volcanic origin. There are no marine sedimentaries, and no organic limestones. All the available evidence goes to show that the island is the product of prolonged volcanic activity, and that the materials accumulated on a land surface, without any intervals of depression during which marine sediments were formed. By far the most common type of rock is a coarse volcanic tuff or agglome-

* J. W. SPENCER, “The Windward Islands of the West Indies,” ‘Trans. Canadian Inst.,’ vol. 7, p. 364.

† TEMPEST ANDERSON and JOHN S. FLETT, “Preliminary Report on the recent Eruption of the Soufrière in St. Vincent,” ‘Proc. Roy. Soc.,’ vol. 70, Plate 2, 1902.

rate, sometimes containing blocks 20 or 30 feet in diameter, and nearly always full of bombs and ejected fragments a foot across or more. These beds are red or brown in colour, and are principally andesitic or basaltic in composition. As a rule, their bedding is very rough, many thick masses showing only faint and inconstant bedding planes several feet apart. Splendid sections of these agglomerates are to be seen along the leeward coast, where cliffs several hundred feet high are built up entirely of such materials (see Plate 22, fig. 1). It gives the impression that in the Antillean volcanoes violent explosive action has been far more frequent than the outpouring of streams of lava, and that the recent eruptions are in this respect typical of the region to which they belong. But when we compare the beds of ash which have been laid down during the present year with those of older date which may be traced in the cliff sections, we are led to the conclusion that in comparison with former convulsions this last is a pigmy affair.

The possibility that in these thick masses of coarse volcanic ejecta we may have the remains of former craters—necks, dissected and exposed in the cliffs by the erosive action of the sea—is one which at once suggests itself to the mind of the observer. But nowhere did we see a section which could be regarded as that of a typical neck, whether plugged by agglomerate or by crystalline rock. Wherever the structural relations of these masses of coarse ash to the surrounding rocks were well seen, they proved to be those of thick irregular beds, lenticular in character, and passing laterally into thinner and more uniform sheets. The rapid variation in coarseness of material when traced along the strike, the irregular thickness of the beds, and the imperfect development of the bedding planes, together with the absence of fossiliferous intercalations, are best explained on the supposition that we are dealing with a series of purely sub-aerial volcanic deposits. The rarity of necks is partly a consequence of the simple character of the Antillean volcanoes, as exemplified by the Soufrière and several others recently extinct but still well preserved—where parasitic or lateral craters are exceptional.

Further evidence of land conditions is furnished by the numerous sections of old stream-valleys in the agglomerates (see Plate 22, fig. 1). Some of these are well shown in the cliffs near Cumberland, a little north of Chateaubelair. Deep gullies have been cut out of the beds of ash, and these have been filled up with material, which at the sides slopes steeply towards the centre of the trough, while in the middle the bedding planes are flat or slightly concave. Some of these valleys must have been veritable gorges, deep and narrow. A peculiar volcanic conglomerate, composed of blocks weathered out of the old ash beds or lava streams, but water-worn and well rounded, is the deposit which usually occupies them. It is easily recognised after a little practice, as it is very different in appearance from the purely volcanic tuffs. The sorting action of running water is often exemplified in these conglomerates. Some beds consist almost entirely of large, well-rounded boulders, while others are much finer grained, and composed of sandy sediment deposited by gently

flowing streams. Such alternations give a pronounced stratification to these accumulations, yet the individual beds are never persistent but always very local, and the irregularities of the bedding and of the dip are explained by the rapid variation of tropical streams in volume and velocity of flow.

Although lava flows are not the predominant feature of the coast sections, and are far less important than the ash beds—yet it is not to be imagined that they are rare or absent. There are many fine examples of them on both the leeward and the windward shores of St. Vincent. Their mass is often considerable, as they are frequently 40 feet and sometimes 80 or 100 feet thick, and some are nearly a mile in length. They are mostly andesites or andesitic basalts, porphyritic, with large crystals of pyroxene, plagioclase felspar, and commonly olivine. On the fresh fracture their dark colour and vitreous lustre often indicates the presence of considerable amounts of a glassy base.

The thickness of these lava flows and the large area they cover are two of the most striking features of the geology of the island. Among the more important examples may be mentioned the lava which outcrops on the south side of Kingstown Bay, that which occurs below Petit Bordel, at Chateaubelair, and that which forms the Black Point, south of Georgetown. On the leeward coast, near Cumberland, and also near Barrualli, thick streams of lava may be seen in the sides of the valleys a little distance inland. By the vertical cliffs they form they can be easily traced as they run up the valleys, and it is clear that they are of considerable age; for, as they appear on both sides of the stream at the same level, they were at first continuous, and have been cut up into separate blocks as the streams deepened their course and worked through them into the softer ashes below. It is this alternation of ash beds with columnar-jointed lavas which yields under the influence of sub-aerial erosion those picturesque and varied effects which make the island of St. Vincent famous for the beauty of its scenery.

It is possible that a certain number of those crystalline rock masses may really be not lavas which flowed out on the surface, but massive intrusive sheets injected between the bedding planes of the ashes; and in that tropical climate, amid the dense forests, and on such difficult ground, it would be a task of no small labour to establish in every case what is their true nature. Wherever it was possible to investigate thoroughly their structural relationships, they proved, with very few exceptions, to be true lava flows, though they are not frequently scoriaceous, even on their upper surfaces. The bedding of the ashes in which they lie is, as already stated, rough and irregular, so that it is not easy to show that they are strictly interbedded or conformable. Small discrepancies between the apparent bedding of the tuffs and the lava flows are not to be regarded as important. On the whole the greater lavas form flat sheets with a gentle uniform dip which agrees with that of the tuffs where these exhibit a good bedding. Owing to their greater regularity in this respect, the lavas afford better evidence to the geologist as to the true dip and

structure of the country than the irregular, inconstant, and often tumultuous deposits of tuff.

The lavas, like the ash beds, have certainly accumulated on a land surface. Occasionally we may see where a lava flow has entered an old valley and has partly or completely filled it up, and frequently the base of the lavas is so irregular as to show that the surface over which they flowed had been sculptured by sub-aerial erosion into numerous ridges and furrows. The curved outcrops not unfrequently suggested that folding had taken place and thrown the beds into little anticlines and synclines, but examination always revealed the eroded character of the underlying material, while no proof of folding on such a small scale was ever obtained.

A layer of bright red earth very commonly lies directly below the lavas, and where the rocks are utterly decomposed, this may be one of the best indications of the nature of the overlying mass. It is undoubtedly an old terrestrial soil in which the hydrous iron oxides have been changed to hematite by the action of the hot lava flow. This layer is usually about a foot in thickness and consists of a decomposed ash bed (only very rarely do two lavas come together in the sections) in which the fragmental structure is becoming obliterated, baked and hardened into a splintery red clay or porcellanite, especially where it is fine grained and in immediate contact with the lava. We did not find any remains of burnt wood or other fossils in these old soils or in any part of the tuffs.

Intrusive sheets also are not lacking in St. Vincent, though according to our experience they are few and small. There are some sections which show junctions so decidedly transgressive that they can only be due to the injection of molten masses into fissures in the tuffs. Good examples are to be seen at Dark Head, on the leeward coast, and Duvernette Island, near Calliaqua. They are so rare, however, that it would not be difficult to enumerate every instance which came under our notice. In no case are these masses of great size, but some of the more important lava flows attain to such great thickness, and are so lenticular in character, that they greatly resemble laccolites. One of these in the Cumberland Valley, about a mile above its mouth, forms a beautiful columnar cliff nearly 300 feet high, overlooking the stream. There was not, however, sufficient available evidence to prove that it might not be a lava flow occupying an old valley channel, and very thick in consequence. There is another great cliff of crystalline and columnar-jointed rock in the Mariaqua Valley, and from the regularity of this mass in dip and thickness it is most probably a lava flow.

In the splendid sections on the leeward side of the island, dykes are as rare as intrusive sheets. The exposures are so good that this is sufficient to indicate their scarcity, but in the interior of the island, owing to decomposition of the rocks and the thick growth of forest, they would be practically certain to escape detection. Three or four dykes were seen cutting the tuffs near Layu.

The rarity of dykes, intrusive sheets and necks, are characteristic features of

the Antillean volcanoes, which find their explanation in the peculiarities of their methods of eruption. They have been mostly explosive volcanoes, emitting great quantities of ashes, bombs, and ejected blocks, with occasional flows of lava. In this way great simple cones have been built up, mostly with one or more craters near their summits. Parasitic craters and lateral outlets have been rare, owing to the great strength of the volcanic structures, and as a rule the old orifices have been repeatedly made use of in the eruptive history with few and unimportant modifications. This has been the story of the recent outbursts, and is in accordance with the structure of the best preserved dormant or extinct volcanoes of the Caribbean chain. They are simple cones, sometimes breached, or surrounded by an old "Somma" wall, and consist for the most part of ash with more or less abundant coulées of lava.

So far as it was possible for us to ascertain, the geological structure of St. Vincent is very simple. All around the coasts the dip of the rocks as indicated by the great lava flows and the more persistent beds of ash, is outwards from the centre. On the leeward side the lavas dip to the west at gentle angles (averaging about 10°). At the south end and near Kingstown the dip is mostly south, while on the east or windward side the dip is also towards the sea. The ash beds are more irregular in this respect, and within short distances may show considerable variations. But, taken as a whole, they always agree fairly well with the dip of the lava flows, and they practically never show an inclination towards the centre of the island. We are, in fact, dealing with a highly eroded volcanic pile, a chain of old volcanoes, and the position of the craters and outlets must have nearly corresponded with the central ridge of hills. The radial outward dip is a reflex of the slopes down which the lavas flowed, and on which the ash beds gathered. Over these surfaces the streams ran in more or less direct courses to the sea.

Along this central ridge, apparently, no craters, except that of the Soufrière, can now be found, and none are indicated on the map. Erosion, most rapid on the higher grounds, has already obliterated them. But it is there they are to be searched for, and though the task is a well-nigh hopeless one, the remains of them may yet be found. In all probability most of them are filled with agglomerate, but some may be represented by great bosses of crystalline rock like the Pitons of St. Lucia.

We searched carefully in the streams which descend from the main ridges, and along the shores of the whole island for specimens of true plutonic rocks of dioritic character, but none were found, and this helps to demonstrate the comparatively recent nature of the whole island, and corroborates the evidence drawn from other sources as regards its geological history.

It is, of course, possible to regard the island as an anticline or dome rising on the summit of a great earth fold, such as is indicated by the soundings on the charts as separating the Caribbean Sea from the Atlantic. But there are certain facts to be enumerated shortly which this would not explain, and in the whole island there is a

remarkable absence of folding, contortion, crushing and faulting. Small faults were occasionally seen, but very rarely. None of the big masses of lava could be proved to be cut by any considerable faults, and the irregularities in their outcrops were to be explained by the variation in thickness due to outflow over a rough eroded country rather than as a consequence of folding or deformation. In some of the other islands evidence of faulting is far more abundant than in St. Vincent, though it may be admitted that, except they cut the more persistent lava streams, it might be very difficult to establish the existence of faults, as the ash beds are so lenticular in character and so irregular in their dip.

The Morne Garu and Richmond Peak, which stand just south of the Soufrière, are the best examples of highly eroded extinct volcanoes in the whole central mountain chain. We lived for several weeks at either Chateaubelair or Georgetown, on opposite sides of this mass, and had many opportunities of observing the disposition of its rocks, especially as on its north side the bush had been burnt or overthrown during the recent eruption. On the west, great lava streams descend by Richmond, Richmond Vale, and Chateaubelair to the sea. On the east there are important lavas behind Georgetown and at Black Point, with an easterly inclination towards the coast. On the north side the dip is northwards, towards the Soufrière (see fig. 2, Plate 29), and on the south side of Morne Garu at least many of the lavas have a dip towards the south. Hence it appears that the rocks are so disposed as to dip outwards on all sides from the centres or summits of these mountains, and this agrees with the supposition that they are the remnants of a highly eroded volcano (or less probably two adjacent volcanoes), which in its prime was probably of considerably greater magnitude than the Soufrière. Few men have ascended to their summits. We could find only one who had been there—a black guide, and from what he told us, it seemed certain that there are no craters, but that the extreme top is formed by sharp knife-edged *arêtes* between deep valleys as is indicated on the map (see Plate 39). Even in that case it is likely that of all the extinct volcanoes of St. Vincent, this was the last to die out. In the deep radial valleys which have been carved out of the mass, we find the consequences of its former conical surface. It seems probable that volcanic activity has persisted in the north end of the island long after it ceased at the south.

The Upraised Sea Beaches.

One of the most marked features of the geography of St. Vincent is the presence on the windward side of considerable stretches of comparatively level country, which contrast strongly in general character with the highly eroded and deeply sculptured uplands. There is no such belt along the leeward shore, and consequently the largest and richest estates are to be found on the east side of the island, where also the population is most dense. The road from Kingstown to Georgetown, the only carriage road in the island, after crossing the ridge behind Kingstown, passes entirely

through this flat country, with rich cultivated land on each side, except where it skirts the sea shore. The Carib Country at the base of the Soufrière, north of Georgetown, is part of this tract of low ground, and on it stood some of the largest and best estates in the island, now devastated and covered by volcanic ashes (see Plate 21, fig. 1).

The difference in aspect between the flat-terraced coastal plains on the windward, and the sharp spurs and deep valleys which on the leeward side run right down to the sea, is so striking as to call for explanation. The windward plains are eroded and cut into by valleys, but these have none of the rugged steepness which makes the leeward coast so picturesque. The ridges between the valleys are broad and flat-topped, with fine fields of sugar-cane and arrowroot. They differ entirely from the knife-edges which separate the ravines on the higher grounds and the narrow ridges between the deep valleys on the western shore. In some places the low, round-backed hills have rather the appearance of a chalk country in the south of England.

The line of demarcation between these two terranes is in many places fairly well defined. Its altitude is about 700 feet above the sea. Not unfrequently, however, the one type of country blends with the other; there is no sharp boundary, but the hollow, concave, deeply incised features of the uplands soften and gradually give place to the rounded, convex, somewhat flowing outlines of the plain country beneath.

In some places this coastal plain is beautifully terraced. Flat benches, separated from one another by steep declivities, mark successive levels of the sea during some previous epoch of submergence. One very persistent terrace is almost 200 feet above the sea level. It is very conspicuous about Mount Pleasant and Brighton on the windward road, 6 or 8 miles from Kingstown. Below it two lower flats can also be seen, and above it there are others. The highest well-preserved beach which we saw was on the south side of the Mariaqua Valley, at about 690 feet above sea level, but, as a rule, the higher terraces have suffered from erosion far more than the lower, and are much less perfect and less easily traced. In all there are at least six or seven of these old sea levels, though, in the absence of an accurately contoured map, much time would have been required to ascertain their exact number. In some places near Brighton it is possible to make out four, one above another.

They are never capped with uplifted coral reefs as are some of the raised beaches of Dominica and Grenada. Careful inquiry showed that there was no limestone in St. Vincent, and that all the lime used was obtained from blocks of coral taken from the living reefs along various parts of the shore. Nor, so far as we saw, are they covered with deposits of beach gravel, as in the more recent raised beaches of Dominica. This may be so in some places, but it cannot be by any means common. Good sections of these terraces are quite frequent on the windward road, which often

runs upon their surfaces for short distances, and these show them to be rock platforms carved by marine erosion out of the tuffs and lavas of the island.

The absence of coral and gravel deposits, the depth and number of the valleys cut across the beaches, the imperfect preservation of the lower ones, and the fragmentary condition of the higher, all prove that they are not of recent date, but have been for a very long time exposed to sub-aerial denudation, and are in process of decay and disappearance. The cliffs behind the ledges have had their outlines softened and rounded, the streams have cut deep channels in the rock platforms, reaching new base levels at greater depths, and as the valleys widened they have encroached, to a large extent, on the flat-topped ridges between. Any superficial accumulations which formed on them have long since disappeared. They may, however, have been quite unimportant. Coral is not very abundant around St. Vincent, possibly because the streams flowing through the soft and weathered tuffs discharge great quantities of mud into the sea, discolouring it, after heavy showers, for several hundred yards from the shore. The surf, which constantly beats on the weather side of the island, has carved a broad submarine plateau opposite the headlands. In the bays and across the stream-mouths sand and gravel gather, but on the points between the bays there is usually no beach, but only a flat platform, all awash and covered by the sea. It is on these headlands that the old raised beaches are most conspicuous, and in such situations there is no mistaking their meaning. As seen in profile they have mostly a slight but perceptible slope towards the water. In section parallel to the coast they are sensibly horizontal. But in the sides of the valleys and up the stream-courses it is far less common to find these beaches well preserved. For this there are several reasons. The cutting power of the sea is least in these places, and accumulation of gravel is more common than erosion and removal of the solid rock. Rock ledges may never have been cut, or, if cut, must have been quite small. Any gravel beaches formed in such bays would be rapidly carried away by the streams which took possession of them when the sea retreated. Moreover, it seems probable that the lower courses of most of the less important streams are subsequent to the terraces and have been carved out of them, so that in the formation of the stream-valleys the beaches must have been removed by erosion.

Further evidence of the great age of these features is afforded by the depth to which the rocks composing them have weathered under the attack of atmospheric agencies. On the leeward side of St. Vincent, and everywhere also in the higher elevations, although decomposition is rapid, there are abundant exposures of fresh rock, for the steep slopes facilitate removal of loose material, and landslips and the washing action of the rain keep the surfaces clear. On the lower grounds often a great thickness of rotted rock may be found in certain situation. The taluses on the valley sides, where not exposed to the main streams, may be very deep. The soft ash weathers readily into a fine, friable dark soil, and after a heavy rainfall the lateral rivulets may cut into this to depths of 30 feet without exposing solid rock

beneath, but this is local. On the windward side, however, it is the exception to see fresh rock exposed on the surface of the terraces. So advanced is the decomposition that it is often very difficult to distinguish a weathered lava from an ash bed. A brown earth covers everything, fine as a whole, but often full of rounded stones, which are usually the bombs and ejected blocks of the tuffs, more resistant to decay than the finer matrix between. Occasionally a lava assumes this appearance, and then it is only by proving that all the rounded masses have the same petrographical character, or by finding that the matrix shows the same porphyritic crystals as the harder kernels it incloses, that we can establish what the rock was originally. The presence also of the red layer of baked soil beneath the lavas often confirms the diagnosis. The more basic lavas weather spheroidally, and the brown, rotten earth shows then innumerable spheroids, large and small, composed of concentric shells. The weathered material is known in St. Vincent as "Pozzuolana," and is much used for dressing the surface of the roads.

On the coast, a little south of Georgetown, a peculiar brown earth covers the volcanic rocks to a considerable depth. The road just north of Colonarie Point passes beneath cliffs of this material, over 50 feet in height, which continue up to near Black Point. It is dark-yellow or brown in colour, soft, fine-grained, with numerous blocks and rounded fragments scattered through it. Occasionally it shows a fairly good bedding, the division planes being 2 or 3 feet apart. Some of the larger stones are vesicular volcanic bombs, others are pieces of various lavas, all of types frequent in the island. This deposit contains no fossils, and apparently has not accumulated beneath water, for there has been no sorting of the materials, and the rough imperfect bedding is quite unlike that of an aqueous formation.

Its most significant feature is the dip of the rude bedding planes. When the material lies on slanting ground the dip is down the slope and parallel to the surface. On the rounded backs of the ridges between the valleys the beds lie flat. In other words, the bedding shows an intimate relationship to the surface configuration of the country. Not infrequently "unconformabilities" are to be observed (see Plate 22, fig. 2), slanting layers overlapping others nearly horizontal. The nature of the material above such an "unconformability" and below it does not differ.

The origin of this formation is somewhat of a problem. In many features it resembles a pulverulent ash which has been showered down from above, and has covered every irregularity of the surface with a layer of fine ejecta. Each bed might be supposed to represent an eruption, and the larger fragments might be bombs and ejected blocks. Such loosely aggregated ash might be expected to be especially liable to rapid decomposition. But a coating of this nature would certainly be readily eroded by the tropical rains, and washed off the slopes into the valleys. It should occur mostly near the Soufrière, as that is the only volcano in the island which shows any evidence of having been in activity for a long time past. This deposit, however, is not well seen on the Soufrière, but may often be found in patches near the south

end of the island, and is fairly common around Kingstown. It is only found on the low grounds which are covered with vegetation. There decomposition goes on rapidly, but erosion and deportation are least effective.

A more satisfactory hypothesis is that this is a weathered rock rubbish, which has accumulated on the flatter grounds where the rain has little power of washing away fine fragments, especially as the roots of the vegetation serve to hold the soil together and prevent its removal. When saturated with rain such incoherent materials will tend to slip under the action of gravity, and the slow creeping of the soil down the principal slopes may sometimes give it a roughly-bedded appearance. The bombs and crystalline blocks are the larger masses enclosed in the old tuffs from which the brown earth has mostly formed. In many respects it bears a strong resemblance to the "head" which covers the rocks in Cornwall, only the appearance of bedding is more frequent.

A fact of some importance in connection with the geographical development of

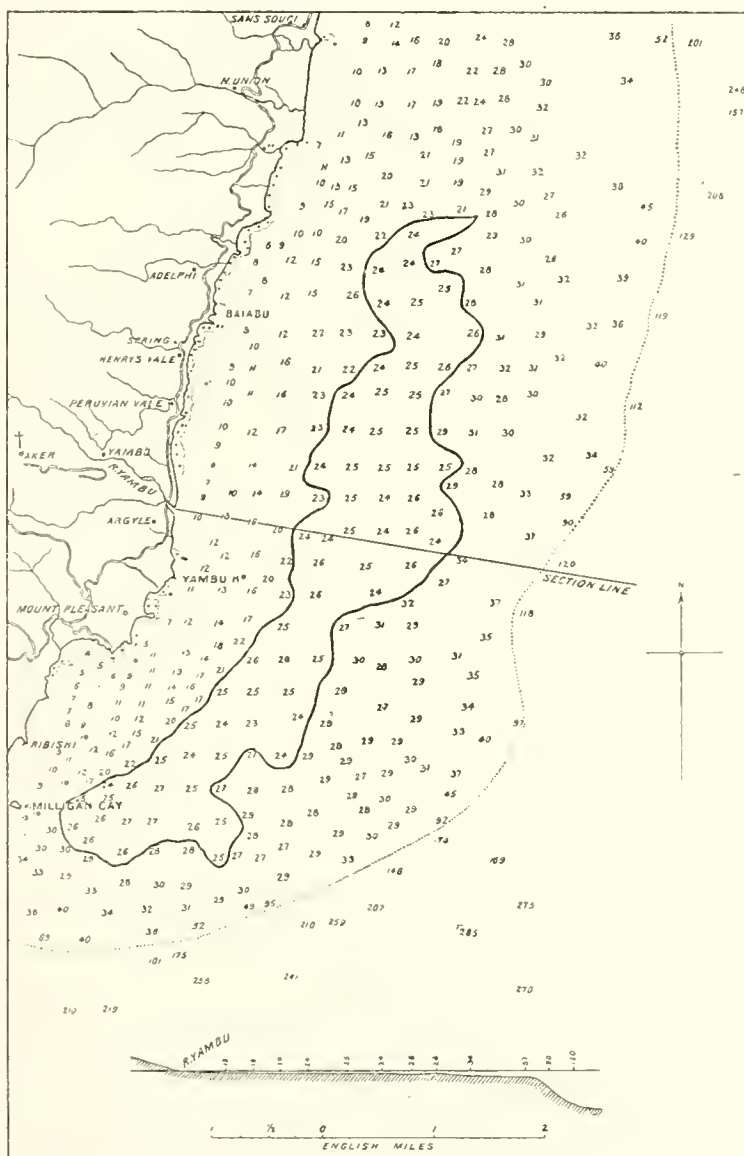


FIG. 1. Map of the south-east corner of St. Vincent. A line is drawn to enclose depths of 24 to 26 fathoms, indicating the submerged beach platform. The section at the base shows the coastal shelf.

St. Vincent is the occurrence of a well-marked submerged terrace off the south-east coast of the island. The soundings of the Admiralty chart show that opposite the mouth of the River Yambu the sea bottom slopes continuously down to a depth of 150 feet in a distance of a mile, then for nearly a mile the depth does not perceptibly increase, but the soundings are uniformly 24, 25, and 26 fathoms. To seaward of this flat ledge there is a gradual descent, and the 50-fathom line is often within a mile of its outer edge. Thereafter the slopes are almost precipitous, and there may be only one-third of a mile between the depths of 200 fathoms and of 50 fathoms—a slope of nearly 1 in 2 (see fig. 1).

Evidently the 50-fathom line separates a gentle slope from one which plunges into the depths which lie to eastward of the lesser Antilles. Above that line we have a surface which has at some time or other been part of the land; below that we have the face of the great earth fold on the crest of which the Windward Islands stand. This is the real boundary of St. Vincent in a strict geographical sense. The flat ledge with an average depth of 150 feet can only be regarded as an old sea-cut terrace. Like the beaches on the windward shore it has no relationship to the geological structure of the country, and the slopes cannot be old escarpments, as the dips are all seawards. It proves that the island formerly stood at least 150 feet higher, and was slightly more extensive in area than at present.

Were it the case that on the windward side of St. Vincent there is a well marked series of raised beaches, while none can be traced on the leeward shores, this would be sufficiently remarkable to call for explanation. Undoubtedly there is no stretch of level ground on the western coast, nor are there well-developed terraces like those to be seen near Brighton, and elsewhere to windward, and it was not till after we had seen these, that it was possible for us to recognise and identify the scanty remains of similar features near Chateaubelair and Barrualli (see Plate 21, fig. 2). But even here there are occasionally level stretches on the headlands at heights of 200 feet and more above the sea. They are narrow, and are most easily recognised from some short distance out at sea, where they appear as obvious breaks in the sloping sky-line presented by the ridges of the spurs. In one or two places (near Cumberland) two or three can be made out, one above the other, with steeper intervening slopes representing the old bluffs behind the shores. Their insignificant development is a natural consequence of the sheltered condition of this coast line. Erosion is much more rapid on the windward side, where the billows, driven before the trade wind, pound steadily against the rocks. The land-slopes on the western side also are steeper than on the eastern, so that sub-aerial erosion is more effective and the terraces (originally smaller) have been more rapidly cut down and effaced. They occur, as might be expected, mostly on the rocky capes, and are never to be seen in the interior of the bays and valleys. They extend probably to as great a height here as on the other side of the island, and belong to the same period of submergence and marine erosion.

Coastal terraces of another type, only indirectly connected with the eroded rocky platforms already described, deserve also to be mentioned here. Although they cover no very extensive area, they are important, as on them stand many of the most important villages and towns, as, for example, Kingstown, Georgetown, and Barrualli. They consist of alluvial material which has been transported down the steep slopes behind the shore, partly as taluses and landslides, but chiefly by the rivulets and streamlets, which wash away the fine earth after heavy falls of rain. Their upper surfaces are by no means plane, but have always a definite slope towards the shore, and are often somewhat hummocky, uneven, and rounded, as they consist of coalescing fans of alluvium, each of which diverges from the streamlet which has deposited it. The rain rills and small rivulets cannot carry the same load when they reach the gently sloping country at the foot of the hills as they did on the hill sides, and the material sinks and is laid down as sheets of gravel and fine earth. This deposit is often full of vegetable matter and the calcareous fragments of land shells. It does not seem to have gathered under the surface of a sheet of water, as it is never stratified, and contains no marine fossils.

In that stretch of level ground which lies at the eastern base of the Soufrière, and stretches from Georgetown to Overland Village, we have the two types of terrace blended inextricably together (see Plate 21, fig. 1). Here are the remains of the old sea benches, much eroded, rounded off, and nearly effaced; yet that it is a rock platform can be proved in many places by the tuffs and lava-flows which are exposed in the stream sections and the coastal cliffs. The surface of these terraces is deeply weathered, and often covered over by alluvial fans laid down by water descending from the higher grounds. Shortly before we reached Georgetown heavy rains had fallen, and the fields of arrowroot behind the village had been covered by flows of mud which had descended the slopes above Grand Sable House. These mud streams had done great damage to the crops. They had entered the village and knocked down a hut, in which were two black people, who were buried in the falling débris, and were drowned.

This country is partly a terrace of marine erosion, partly a terrace of accumulation. It forms the most extensive stretch of flat ground in the island, and is now covered with the ashes of the eruption and the sand and mud deposited on it by the rains. Formerly it was the pride of the island, as on it stood many fine estates, and it was inhabited by a large and contented population. Georgetown, which contains two fine churches and many good houses, was the centre of trade for the surrounding region. It is covered 2 feet deep with ashes; the gardens of the town and the fields around are desolate as a cinder heap, though here and there the vegetation is reasserting itself. There is hardly an unbroken pane of glass in the whole town, as nearly all were shattered by falling stones, and when we were there a large part of the population was receiving relief from the funds at the disposal of the Governor.

The geological history of the island of St. Vincent, as recorded in the rocks exposed to view within its boundaries, may be summarised as follows:—

The whole island consists of volcanic rocks and the products of their disintegration. The only active volcano is the Soufrière, and all the accumulations may be considered as belonging to recent geological times.

The epoch of maximum volcanic activity was a period of elevation during which the island was certainly more extensive than at present. The eruptions were sub-aerial. This was followed by depression to, at any rate, 700 feet below the present level, and during this subsidence, or the subsequent elevation, a series of rock terraces was cut by the action of the waves.

The elevation which followed was carried to a stage at which the land stood at least 150 feet higher than at present.

Since then there has been a second depression of 150 feet. In all probability this has been the last chapter in the island's history, as the absence of uplifted coral makes it unlikely that there has been any recent elevation.

The Soufrière; its Configuration and Structure.

In external configuration the Soufrière is a very typical example of a volcanic cone. Nearly 8 miles across at its base, it rises to a height of 4000 feet, so that the average slope of its sides is about 15° . Its summit is occupied by a large crater, nearly a mile across, and this gives it a rounded or flat-topped appearance, as seen from Chateaubelair, which is on its south-west side. To the north of the crater there rises a semi-circular ridge, separated from the crater wall by a deep valley, which it overlooks in a series of precipitous cliffs of bare rock, nearly 1000 feet high, fringed with taluses of débris. This ridge forms the extreme summit of the mountain, being 4050 feet in height, while the lip of the crater is only 3000 to nearly 3500 feet high. It surrounds the crater on the north and north-east sides, forming an amphitheatre, the inner side of which is nearly vertical, while externally it slopes down to the sea with the conical form of an eroded volcano. This ridge bears exactly the same relationship to the present crater as Somma does to Vesuvius,* and has undoubtedly originated in the same way by some great explosion, which has blown away the summit of the hill, and left a gigantic depression nearly 2 miles in diameter.

From Georgetown and the base of the mountain on the windward side the conical shape is less obvious than from Chateaubelair, though, so far as our experience went, it was not often that a clear view of the summit could be obtained from this quarter, as the upper regions were nearly always veiled in cloud. From the sea to the north of the island the appearance of the mountain is that of a round-backed deeply furrowed pile, in which there is nothing to suggest strongly the volcanic origin of the

* H. REUSCH, 'Nature,' vol. 66, p. 132, June 5, 1902.

structure. But the encircling Somma wall overlooking the actual crater is so well seen from the south-west that one can divine at a glance the history of the mountain. (see Plate 25, fig. 1). It is in the north-west quadrant of the hill that the conical slopes, determined by the internal structure, are least modified and most apparent. So active are the processes of erosion in the Windward Islands, that there is no part of the hill that does not bear conspicuous evidence of their operation. Deep ravines radiate out on all sides from the summit, as is indicated (perhaps a little diagrammatically) on the Admiralty chart (see Plate 39).

The profound gorges which score the sides of the volcano give it a picturesqueness which otherwise would have been lacking. When we saw them they were bare and naked, all the green vegetation had disappeared during the eruption; only blackened trunks were left to bear witness to the tropical forest which had once clothed the surface. But though the mountain had lost in beauty and variety of colour, it had gained in interest and impressiveness to the geologist. Every detail of cliff and scar was visible from a distance. The Soufrière had been one of the beauties of the West Indies, and travellers had come from far to gaze on its richly-wooded slopes, and to see the marvellous lake which nestled in the crater on its summit. To-day it looks more like a skeleton; the ribs of bare rock stand out everywhere plainly to be seen.

In more than one respect the gorges on the mountain deserve the name of cañons. Some of them must be nearly 1000 feet deep, and they are often so narrow and steep-sided as to render the country almost untraversable. We were told that along the leeward shore there is an old Carib track, but communication is kept up entirely by means of row boats. The mountain was very frequently ascended, in fact this was one of the commonest pleasure trips for the inhabitants and tourist visitors. But, as already mentioned, there was only one track, which served also as the main road from Chateaubelair to Georgetown. After crossing the low ground it struck the slopes of the hill and ascended along the knife-edges between the ravines. The road was fairly good for a bush path in the West Indies, and horses could be ridden along most of the way. As the valleys are radial, the ridges between them swept right up to the edge of the crater, so that the track led up to the lip of the depression, then along it for a short space, and down another spur to Lot 14 on the windward side. The slopes on each side were often 40° , and sometimes still higher, and the back of the ridges so narrow that in many places there was only width sufficient for the path. After the eruption, the bush was all destroyed, and as the roots had served to hold the loose materials together, landslides were frequent, so that the path was dangerous in one or two spots, especially as one never knew what weight the new ashes which had gathered on it were able to bear without slipping. Except by this path, it would have been a matter of the greatest difficulty to reach the summit, and so dense was the undergrowth, and so numerous and profound the ravines, that no one attempted to explore the surface of the hill, and few, except a handful of wandering Caribs, were

acquainted with any other part of the higher slopes than those adjoining the footpath.*

Another element of danger is the suddenness and severity of the rain-showers which send torrents down the valleys, which sweep everything before them. The Somma wall behind the crater can only very seldom have been climbed. Certainly the white inhabitants of St. Vincent are not lacking in energy or enterprise, but we did not meet anyone who had ever stood upon the actual top of the Soufrière. During the three weeks we spent at the base of the mountain, we did not once get a clear view of the summit of this ridge, and to climb it in a dense mist would have been not less arduous than unprofitable.

Magnificent sections are afforded by the sea cliffs on the leeward side of the Soufrière, between Morne Ronde and Quashie Point, and in the deep ravines of the Rozeau Dry River, Larikai, and other streams which descend the mountain slopes. On the windward side also there are very fine inland cliffs above Overland and Sandy Bay. We had also frequent opportunities of examining the sections in the gorges on the south side of the mountain in the course of our ascents to the crater. As a result of our study of all the available exposures, it is clear that the Soufrière is a massive volcanic cone, consisting of alternating beds of lava and of ashes which on all sides dip outwards from the summit craters at angles of 12° to 15° .

Some of the lavas are thick (50 to 100 feet), but on the north-west side especially they are thin and numerous. Occasionally seven lava flows with sheets of ashes between can be traced above one another in a single cliff or the side of a ravine. They contrast in this respect with the thick lavas of the southern end of the island and the very massive accumulations of tuff in which they occur.

The centre from which the lavas diverge and have proceeded is the summit of the hill and the craters which occur there. We saw no evidence of parasitic cones or minor eruptive foci. There are also few or no dykes, and intrusive sheets are rare or absent. There is abundant proof that the eruptions took place on a land surface. In these respects the volcano is of the usual Antillean type, and differs only from its extinct neighbours in that its activity has shown less of explosive violence, and more frequent gentle outpouring of lava has been habitual in their case.

The structural relationships of the gigantic crater of the Somma to the present craters within it are not entirely understood, but as the northern wall of the larger of the present craters is composed of lavas and ash beds with a northerly dip, it seems that a new cone has been built up within that depression by repeated flows of lava and showers of ash. On the south side of this crater are tuffs and lavas dipping south, which must have been emitted from the present vent, though no line of demarcation could be drawn between them and the older rocks which belong to

* See JAMES ANDERSON, "An Ascent of Morne Garu," 'Phil. Trans. Roy. Soc.,' vol. 75, p. 16, 1785. F. A. OBER, 'Camps in the Caribbees,' 1880. GILBERT HAWTREY, "Lost on La Soufrière," 'The Wide World Magazine,' vol. 9, p. 523, September, 1902.

that earlier cone in which the great Somma crater had been formed. It is possible that the great explosion which removed the summit of the old hill left a crater one lip of which was more than 1000 feet lower than the other, and on this lower lip the present craters have built up their later cone. Or it may be that, after that paroxysm, the hill presented the appearance of a breached cone with a wide, open trough facing the south, and that the actual craters stand on the line of this fissure.

The great valley which has already been described as lying on the south side of the Soufrière, between it and the Morne Garu, is deserving of special description, as it is here that the ejecta of the recent eruption have chiefly accumulated (see Plate 25, fig. 2). It runs across the island, and on the west is occupied by the Wallibu and Wallibu Dry Rivers, on the east by the Rabaka Dry River. The dividing ridge is about 2000 feet high, and is the lowest part of the axial chain of the island, so that it is quite possible to regard the Soufrière as a detached mountain not directly connected with the main ridge to the south. The rivers mentioned run in narrow ravines which have been incised in the floor of an open valley, so that it seems improbable that as they are at present they are the main factors in its formation. It seems rather that during some past epoch, when the physical conditions were different from what they are now, this wide hollow was scooped out, and that subsequently the present rivers have cut depressions in its floor.

It should be remembered that this is a structural depression separating the extinct and much eroded volcano of the Morne Garu from the still active and better preserved cone of the Soufrière. As such it has been a valley from the beginning, and is not entirely to be ascribed to erosion. This is shown by the dip of the rocks, which on both sides is towards the centre of the valley (see Plate 32, fig. 1). It must have been from the earliest times one of the main drainage systems of the surface of St. Vincent.

The streams flowing in this depression appear to have at some time or other reached a base-level, so that their cutting power was in abeyance and their valleys broadened and opened out laterally. This may have been during the subsidence, the climax of which is marked by the 700 foot terrace. Subsequent elevation has enabled the streams to deepen their courses and to incise deep channels in the old valley bottom.

Another factor which must have been of importance in determining the formation of this trough is the absence of any hard lava streams in this part of the Soufrière—one of the most perplexing features in the structure of the mountain (see Plate 34, fig. 2). Lava flows are abundant on the north, the west, and the east sides of the volcano, but they are not found at the base of the hill on the south side, or in the part out of which these valleys have been eroded, though at higher elevations (above 1200 feet) they are fairly numerous and often seen in the stream sections. As these lavas always offer much greater resistance to the deepening of the river channels than do the tuffs and agglomerates, and generally stand out as cascades and waterfalls, it is easy to see

why base levels should have been reached at an earlier date by these streams than by the others in the middle and south of the island, which had thick lavas to cut through.

These masses of tuff are fairly well stratified, and dip towards the south, parallel to the slopes of the volcano. They have obviously proceeded from the Soufrière, and appear to be the latest rocks of the hill. They show that the most recent outbursts have been of purely explosive type, and that lava has not flowed from the craters for a prolonged period. The volcano has undergone a change in the manner of its action, and has passed from what may be called a normal to a virulent and spasmodic condition. This may indicate approaching extinction. It is the opinion of most geologists who have studied the volcanoes of the Antilles, that the heyday of their activity is over, and that a general decadence has taken place. If so, the Soufrière, the last of the St. Vincent volcanoes, may be no exception to the general rule, and its decreasing vigour may have resulted in a change in the *modus operandi* of its outbursts.

It is not easy to explain why these latest tuffs should have gathered to thicknesses of several hundred feet in this particular valley, and not elsewhere round the cone. It may be that since the eruption which resulted in the formation of the Somma wall, the ejecta have been discharged mostly over the lower or southern lip of the crater, while the lofty northern wall has protected the country behind it. Possibly, also, the "sand avalanche" type of eruption may have been characteristic of the later activity, and great masses of fine ash and débris, as in the recent eruption, may have overflowed the southern rim of the crater and lodged in the depression at this side of the cone. As we shall see later, there is historical evidence in support of this view.

Of the two craters of the Soufrière, the larger is known as the "Old Crater," while the smaller, which originated during the eruption of 1812, has for that reason been called the "New Crater." These names are no longer very appropriate, as the old crater is that which took the principal part in the eruption of this year, while it is doubtful whether any material was emitted from the smaller or "New Crater." In this report the larger or "Old Crater" is meant whenever we refer to the crater of the Soufrière.

This crater (see Plate 37, fig. 2) is very nearly circular in shape, and, according to the map (see Plate 39), appears to have been a little less than a mile across and somewhat smaller in the north and south direction than from east to west. Its northern lip was 3623 feet above the sea, and about 600 feet higher than the southern. But the crater rim did not slope regularly down from its northern to its southern side, but was slightly serrate with alternating projections and depressions. These are clearly indicated on the Admiralty chart.* The slopes on both sides of this rim (which was

* On the western rim of the crater there was a moderately deep notch leading into the valley of the Larikai. This is mentioned by Mr. E. O. HOVEY ("Martinique and St. Vincent: a Preliminary Report," 'Bulletin American Museum of Natural History,' vol. 16, p. 336, 1902). We never had a good view of it, owing to the mist which hung around the crater.

really a knife-edge like that of the spurs between the valleys) were steep, averaging 30° to 40° , and were furrowed by rivulet channels which were shallow, especially near the summit of the ridge.* The water descending to the interior had gathered to form a large lake, rather more than half a mile across, with an outline which was approximately circular. The funnel shape of the crater walls was due to erosion by running water and to the action of landslips, and not to the accumulation of cinders and loose ejecta at the angle of repose for such material. If beds of ash dipping inwards had ever cloaked these slopes, they had been entirely removed, for, from the descriptions of those familiar with the crater lake and from photographs, it is certain that it consisted of lavas and tuffs dipping outwards on all sides, and presenting escarpments and talus-covered edges to the interior cavity. It is more likely, however, that the crater walls had been in large part vertical cliffs which had reached a condition of stability by repeated rock falls and continued erosion.

These rain-furrowed slopes were covered with bush and short timber, but it was a matter of no very great difficulty to descend them to the shores of the lake, though in one or two places the lava sheets formed low vertical escarpments.

The water of the lake was greenish and opalescent, probably from finely-divided particles of precipitated sulphur.† It smelt strongly of sulphuretted hydrogen, especially on a hot day (hence the name Soufrière), but was not warm or even tepid, and the adventurous occasionally bathed in the lake. The level of the surface is given on the chart as 1930 feet, so that the southern wall was 1100 feet high, and the northern about 1700 feet. Probably the amount of water varied slightly with the season of the year. There was no outlet, but the excess of rainfall over evaporation leaked away through the bedding planes and joints of the surrounding rocks.

The depth of this lake is not very accurately known. Mr. P. FOSTER HUGGINS,‡ in 1896, built a small raft of logs and made a series of soundings of the crater lake in that and succeeding years. He found the depth at the centre to be $87\frac{1}{2}$ fathoms, and believes that to the north of this it would have been found to be still greater, but he was never able to complete his soundings in that quarter. The underwater slopes were very steep, as might be expected from the origin of the lake, and on the north side, below the great vertical precipice, a sounding of 43 fathoms was obtained at a distance of only 60 feet from the shore. Apparently the floor of the depression was bowl-shaped, but not uniformly concave, as the deepest part lay to the north of the centre. As measured from the lower lip of the crater the total depth must have been over 1600 feet.

* F. A. OBER, 'Camps in the Caribbees,' 1880, p. 203.

† HANS REUSCH, 'Nature,' vol. 66, p. 132, June 5th, 1902.

‡ P. FOSTER HUGGINS, 'An Account of the Eruptions of the Saint Vincent Soufrière.' Kingstown, St. Vincent, 1902.

The smaller or "new crater" was really parasitic on the north-east lip of the larger. They were separated by a narrow saddle, lowest in the middle, a steep-sloped, knife-edged ridge which it was considered dangerous to pass along. Comparatively few visitors undertook the scramble along the sharp rim of the main crater to the smaller one, which was not so deep as the principal crater, and its bottom was dry. The diameter of the "new crater" was one-third of a mile and the outline nearly circular. On the north side of the two craters was the deep valley, already mentioned as lying at the foot of the precipitous wall of the Somma.

The crater lake of the Soufrière is described by all who had the good fortune to see it as having been a thing of beauty. The mists which roll across the mountain top before the steady trade wind too often obscured the view and cheated the traveller of the reward of his arduous climb. But when the clouds lifted, and the sun breaking through the veil shone on the pearly-green sheet of water, reflecting from its placid surface the swelling mists above, and set in the sloping verdant crater walls, like an opal surrounded by emeralds, the sight was one the memory of which was cherished for a lifetime.

THE ERUPTION OF MAY 7TH, 1902.

Premonitory Symptoms.

For 90 years the Soufrière had slumbered, and few were living who had seen it in eruption. But in the early part of the year 1901, an uneasy feeling began to arise in the minds of those who dwelt on the flanks of the mountain. Earthquakes are by no means infrequent in St. Vincent, though usually of slight severity, but in February and March, 1901, they became so numerous around the north side of the volcano as to awaken suspicion that danger was threatened. There are two settlements of Caribs—the remains of the original inhabitants of the island, most of whom were exported, after the close of the Carib War of 1796, to the island of (Rattan in the Bay of Honduras)—at Morne Ronde on the west, and at Sandy Bay on the north-east of the mountain. It may be that traditions linger among them regarding former disastrous eruptions, or superstitions connected with the crater, for though the Caribs are certainly not deficient in courage, they became so apprehensive that they petitioned to have their settlements removed to some other part of the island, and officials had to be sent to quiet their fears. There is evidence in letters written to the Kingstown newspapers in March, 1901, that general interest had been awakened, and it was widely known that anxiety was being felt regarding the condition of the volcano.

These premonitory earthquakes were accompanied by subterranean rumblings, which appeared to come from the interior of the hill. Apparently they did no

damage to buildings or other structures, but so much alarmed were some of the black labourers that the managers and owners of estates had, in some cases, to take steps to allay their apprehension and encourage them to continue working in the district. They appear to have been very local and to have differed greatly in intensity at different parts of the hill. In the Carib Settlement at Morne Ronde more were experienced than anywhere else. Earthquakes and noises continued through the whole of the ensuing twelve months. They were neither violent nor loud, but were more numerous than was usual for that part of the island. The white inhabitants regarded them with indifference or curiosity, for as earthquakes are by no means rare in St. Vincent, the repeated small shocks felt during 1901 were not regarded as necessarily the precursors of a cataclysm. There were some, however, who, aware of the suddenness with which the eruption of 1812 had broken out, could not help suspecting that they foreboded another outburst.

In the latter half of April, 1902, they increased considerably in number and intensity. At Owia, on the 13th of April, at 12.20 P.M., there was a sharp shock, and about that time as many as eight tremors were sometimes experienced there in 24 hours, all slight and of short duration. Along the whole north-west quadrant of the hill the violence of the earthquakes in the last week of April, 1902, was exceptional. They dislodged stones from the cliffs of lava and sent them tumbling down the slopes, and occasioned small landslips in the loose material of the taluses, damaging the crops which were planted there. On Monday, April 29th, three well-marked shocks were experienced at Windsor Forest and Campobello, while at Morne Ronde about this time eighteen earthquakes were counted in one day. It will be remembered that the first signs of activity on Montagne Pelée were observed about the 23rd of April, and that a great earthquake shook Guatemala on the 18th April.* As yet the Soufrière had shown no symptoms of actual eruption. The Caribs at Morne Ronde, however, fully anticipating an outburst, were preparing to evacuate their houses and flee to Chateaubelair and other places of safety.

Preliminary Stages of the Eruption, Tuesday, May 6th.

By 1 o'clock on the afternoon of Tuesday, the 6th May, there could no longer be any doubt that the Soufrière was in eruption. It is said that during the previous night the Caribs had seen a cloud of steam emitted from the mountain and a bright glow on the summit. This is doubtful, but certainly on that morning they were flocking into Chateaubelair, and a general anxiety was awakened in the neighbourhood.

At Wallibu† a rumbling sound like distant thunder was heard from 6 to 8 A.M. on the 6th May, and between 8.30 and 12 that forenoon, seven distinct shocks of earthquake were felt. These tremors were not experienced in Chateaubelair, which is about

* 'Nature,' vol. 66, p. 150. 1902.

† For the position of Wallibu relatively to the Soufrière, see Plate 25, fig. 2.

a mile further south. The day was very clear, and the whole lip of the crater could be perfectly made out by those on the south-west side of the hill. From the north side the crater is concealed by the intervening Somma wall, while the inhabitants of the windward quarter could see none of the upper parts of the mountain owing to the thick trade-wind cloud which covered them.

Between 11 A.M. and 12 noon the labourers on Richmond Estate noticed puffs of steam arising from the crater, and reported the fact to the manager. This seems to be the first trustworthy observation of activity on the part of the volcano. News travels fast among the black people, and word of this was not long in spreading to the adjoining estates. At 1.30 P.M. steam was seen by an observer at Wallibu ascending in a great pillar which spread out like a palm tree. It rose from the centre of the old crater. From Chateaubelair nothing suspicious had yet been noticed, but at 2.40 P.M. there was a loud report like that of cannon, and a great cloud of white steam shot up into the air to an immense height. The people gathered in the streets to see the strange spectacle. There is an excellent system of telephonic communication around the coast, and word was at once sent to Kingstown by the officer in charge of the police station at Chateaubelair, and the news that the Soufrière was in eruption was soon flashing over the island.

In a very short time Chateaubelair was thronged with Caribs from Morne Ronde on their way back to Rosebank, a little further south, where there was another Carib settlement. Some had passed through earlier in the forenoon, and their fears had been mocked at by those whom they met. By 5 o'clock the village of Morne Ronde was deserted.

As the afternoon advanced matters got distinctly more threatening in appearance. The trade-wind cloud hovered as usual over the mountain, but the ridge on the south side of the crater was visible from Chateaubelair. About 4.30 P.M. a big column of steam was seen to ascend from it, and several observers remarked that at the base of this there was a red glow like the reflection of fire. About half-past 5 o'clock there was another outburst, and the red reflection from the cloud was again visible. At 6 o'clock a thick cloud of steam was emitted, and from Wallibu fire was seen forming a ring around the lip of the crater, and a crackling sound was heard, which sounded like "brush on fire." A general stampede ensued in the villages and plantations of Wallibu and Richmond, only a few remaining to look after their property and domesticated animals. The sun was now setting. In Chateaubelair all was animation and excitement. The little village was crowded with refugees from the western base of the hill, and the whole population was standing in the streets and on the beach watching the progress of the eruption. Mr. T. M. McDONALD, who had received warning by telephone from his manager at Richmond Vale, that the Soufrière was emitting steam clouds, landed on the beach a little north of Chateaubelair pier at 6.30 P.M. He says—"The summit of the Soufrière was enveloped in the usual white cloud, and at first nothing unusual was visible. Within

a minute or two of landing, however, one of the party exclaimed, 'Soufrière is bursting now,' and on looking toward the mountain I saw enormous vertical columns of white vapour being ejected, virtually noiselessly, and was now quite convinced that an eruption had been, and was now taking place."

At 7.30 P.M. a great explosion followed, accompanied by a loud noise and by slight earthquakes. The discharge of vapour was greater than any of those which preceded it. Mr. McDONALD notes that he observed "flame along the whole line of the top of the crater, forming a thin red sparkling line between the base of the column of vapour and rim of the crater."*

Thereafter at intervals of about two hours similar discharges took place, and some of them were reported to be also accompanied by flame.

By means of the telephone, constant communication was kept up between the police station at Chateaubelair and the head office in Kingstown, and seeing that as the afternoon and evening passed on, Chateaubelair was becoming more and more crowded with refugees, and the excitement hourly became more intense, Captain CALDER, the Chief Constable, decided to proceed thither with an additional force of policemen. Dr. DUNBAR HUGHES, the district Medical Officer, and Dr. CHRISTIAN BRANCH, Medical Officer at Kingstown, accompanied him, and the written notes supplied to us by these three gentlemen form a very valuable body of evidence as to the course of events from this point onward.

Their boat was entering Chateaubelair Bay just about midnight. The night was calm and clear, and on the way north they had passed several boats laden with fugitives fleeing southward. Just as they reached the wharf, a powerful explosion took place. A column of steam ascended "rather higher than the mountain itself, that is to say, about 4000 feet." Dr. BRANCH continues: "The whole width of the mountain top was outlined with leaping flames, red, and having the same appearance as a cane field on fire . . . Immediately, as the flames died down, a rattling roar like that of burning canes reached us." Captain CALDER describes it as follows: "The whole top of the mountain burst into flame, the long flashes of deep red fire travelling from the top downward in a circular track, just like fire bursting from a head of smithy coal when fanned by a strong draught from the bellows. This was immediately followed by an explosion as of much heavy ordnance, dying away in a long-drawn angry grumble." Dr. DUNBAR HUGHES compares the noise to the "sound of a fast-going cab on cobble stones, but much exaggerated." They found the black population madly excited, and standing gazing on the mountain. During the night at least two outbursts took place with a loud noise.

The results of the observations made by persons at Morne Ronde, Wallibu, Richmond, Richmond Vale, and Chateaubelair (all on the sea coast along the south-west side of the mountain) during the day and night of Tuesday, May 6th, may be summed up as follows:—The first emissions of steam were seen somewhat before

* See Appendix II.

mid-day, and earthquakes were experienced about the same time by those at the base of the volcano, but were local. A violent explosion alarmed the whole neighbourhood about 2.40 P.M., and no doubt remained regarding the reality of the eruption. Others followed at intervals of two or three hours, accompanied by loud noises, the most severe being at 7 o'clock, and at midnight. Flames, or the reflection of fire on the steam cloud, were noticed first at 4.30 P.M., and thereafter were seen repeatedly by various observers.

We shall now consider the evidence of a party of fish sellers who left Chateaubelair and crossed to Georgetown on the forenoon of Tuesday. They followed the track already mentioned as leading up the crests of the spurs to the lip of the crater. The party consisted of several black or coloured women and one boy. Fish are obtained only on the leeward side of the island, as the heavy seas on the windward side render small boats unsafe, and these women were accustomed regularly to cross the island, and were in consequence very well acquainted with the appearance of the path and of the crater under ordinary conditions. We cross-questioned them at the refugee camp at Barrualli, in presence of Mr. T. M. McDONALD, whose assistance was invaluable, as it is only after long acquaintance with these people that one can enter into their ways of thought and understand their methods of expression.

They left Chateaubelair about 6 o'clock in the morning, and in ordinary course must have reached the edge of the crater between 10 and 11 A.M. They had no reason to suspect that an eruption was imminent till they were just below the summit, when they heard rumbling noises, and felt the hill shaking. The smell of sulphur (sulphuretted hydrogen) from the lake was much stronger than usual, and on looking down into the crater they saw that the water was much discoloured, in places red, in other places milky, but elsewhere bluish-green, as usual. "One part milk, one part blood, rest all same as before." The red colour was probably due to the stirring up of deposits of reddish sand, which would be produced by the weathering of the red ash beds in the crater wall; the milky water owed its colour to the deposits of fine precipitated sulphur which must have gathered on the lake bottom since the last eruption. They all state that the level of the lake was almost the same as before, that no overflow had taken place, and that within the crater the bush was withered but not burnt.

All agreed that the water was boiling and steaming strongly, especially in the centre of the lake. Some say they saw stones being thrown up, but several witnesses maintained that they saw a stone in the centre of the lake, and it was around this that the steam was rising. The last statement is very important, but its meaning will be better discussed when the history of the eruption has been traced a little further.

Another woman travelling by the same path, but in the opposite direction, passed the crater about mid-day. She also saw the water discoloured. It smelt strongly of sulphur, and was boiling out in the centre of the lake, not at the sides. As she was

descending on the leeward side and was near the "half-way tree," a well-known landmark, she heard a loud report, and saw a rush of steam, whereupon she dropped her basket and ran. This must have been the explosion which took place about 12.30. She saw no stones on the path either when ascending or descending.

By Tuesday evening everyone in Chateaubelair and on the leeward side of the mountain was well aware that the Soufrière was in eruption. Boats had left Wallibu in the afternoon and gone up along the coast as far as De Volet's and Campobello, carrying the news. Windsor Forest Estate was vacated on Tuesday night, and it was only the scarcity of boats which prevented all the inhabitants north of Chateaubelair from at once taking refuge in the villages to the south.

But at Fancy and Owia, no trustworthy intimation of impending danger had yet been obtained. The crater cannot be seen from this side, and the mountain was capped with cloud. People were somewhat anxious on account of the frequency of earthquakes during the previous weeks, and also because word had reached St. Vincent that Montagne Pelée, in Martinique, was in a state of activity. But so far nothing had been seen or heard which indicated that an eruption of the Soufrière was actually in progress.

On the windward side, in Georgetown and the estates on the Carib Country, there was almost as little apprehension. Telephonic messages had been sent from Kingstown to police headquarters at Georgetown that the people at Chateaubelair had seen steam arising from the crater. But a dense trade-wind cloud covered the mountain, the rumbling sounds were mistaken for thunder, and in the West Indies there is always so much untrustworthy news in circulation that it is not difficult to understand that this unexpected information was received with scepticism. It was believed to be a scare due to the news from Martinique.

The Forenoon of Wednesday, May 7th.

Chateaubelair.—On Wednesday morning day dawned on a scene of great excitement in Chateaubelair. "The usually quiet village resembled a hive of angry bees."* Just about 6 A.M. (at sunrise) there was a large outburst of steam, and as the forenoon wore on, the violence of the activity increased. Work was at a standstill; the magnificence of the eruption absorbed all attention. The last stragglers were coming in from the estates and villages on the leeward side, where few had been brave enough to remain overnight, and most of these, before midday, were convinced that it was advisable to take to flight.

During this time, Mr. T. M. McDONALD, of Richmond Vale, and Mr. MATHES, who was his guest, were making constant observations of the eruption. From the

* Captain CALDER, 'Century Magazine,' vol. lxiv., p. 634, August, 1902.

verandah of Richmond Vale House the upper part of the Soufrière was clearly seen. The notes taken by Mr. McDONALD in his pocket-book have since been published,* and form a most valuable contemporary record of the progress of events.

"No further observation was noted at Richmond Vale House till shortly after 6 A.M. on the 7th, when a discharge took place with the usual column of thick vapour, but beneath this was a much shorter column of almost dense black, and of a heavier nature, as it quickly subsided back into the crater. This was the first appearance noted of what was probably solid matter being erupted, the white vapour being no doubt vapour of water only. At about 7.4 A.M. an enormous high column of white vapour was ejected, and it may be here mentioned that these tall columns rose in a very short space of time—say about a minute—to heights of about 30,000 feet and over, by comparison seven or eight times the height of the mountain (nearly 4000 feet). Outbursts took place now at shorter intervals, and at about 10.30 A.M. the eruption became continuous, enormous volumes of vapour reaching to a very great height.

"11.10 A.M. At this time there was thunder and lightning, showers of black and heavy material could now be seen thrown outwards and falling downwards from the column of whitish vapour, associated with loud noises and more violent outbursts. From the commencement the Old Crater seemed to be the scene of activity, but at times it seemed as though some of the discharges proceeded from what is known as the New Crater, a little north-eastwards from Chateaubelair. The area of the escape of vapour seemed now to be extending in a direction corresponding with Morne Ronde."

By means of Mr. McDONALD's notes and of narratives which were given us by Captain CALDER, Dr. BRANCH, Dr. HUGHES, and others who were in Chateaubelair that forenoon, and by Mrs. KELLY, and Mr. ROBERTSON (who was in Wallibu till 12.30 P.M.), it is easy to arrive at a fairly complete idea of what was going on. The eruption was now assuming an acute phase, and the explosions followed one another with increasing rapidity and violence. At first, the steam clouds emitted from the crater followed one another at intervals of an hour or less, and between the outbursts the crater could be seen gently steaming; but about 10.30 A.M. they had so increased in number that an enormous column of vapour was rising from the throat of the volcano, and spreading out in a great mushroom-shaped cloud above the summit of the mountain. Lightnings played incessantly through the steam-cloud, accompanied by sharp peals of thunder. Several of the observers thought that they could distinguish between the thunder and the noises of the mountain, the latter being not so sharp, but lower and more prolonged. The noises were not as yet very violent, neither were they continuous. At the base of the cloud the red glare was often noticed, and some thought they saw flames.

* 'Century Magazine,' vol. lxiv., pp. 639-642, 1902. 'The Sentry,' Newspaper, Kingstown, St. Vincent, May 16th, 1902. See Appendix II.

Mr. McDONALD estimates the height to which the steam shot upwards at 30,000 to 40,000 feet. Mr. DARRELL considered that a few hours later it was about 8 miles. A more reliable estimate is probably that of Major HODDER, R.E., who observed the cloud from the military establishment at Morne Fortuné, above Castries, St. Lucia, in the middle of that afternoon, and found its altitude 12° above the artificial horizon. As the distance apart of these two stations is about 27 miles, this gives a height of $5\frac{1}{2}$ miles.

Fine ash was raining all around the mountain. At Wallibu the sugar mill was working, but they stopped it at 9 o'clock owing to the steady showers of dust. No large stones had yet fallen, but at the base of the pillar of steam, dark, heavy matter was seen projected during the greater outbursts and falling back into the crater. It was obvious that the emissions proceeded mostly, if not entirely, from the old or larger crater. At mid-day the hill was still green, at least on the lower slopes, though the bush around the summit had been withered and blasted by the heat. About 1 o'clock in the afternoon the epoch of maximum activity supervened. The noises were now much louder, and, though not continuous, they accompanied all the great outbursts, which followed one another in rapid succession. Steam was seen ascending from the valleys in thin streaks at first, but afterwards in dense clouds which obscured the view of the mountain. Mr. McDONALD mentions that at 1 o'clock he saw stones being projected to windward, and again at 1.30 P.M., and in great numbers at 1.50. They were carried up by the steam column to heights of thousands of feet, and showed "tails of fine black matter." "Jets of fine black matter," "showers of blackish material," and "dense black upheavals" were also seen, especially at 12.25 P.M.

On Wednesday morning the fish sellers who had crossed to Georgetown on Tuesday, started at day-break to return. Nothing had been seen on the windward side to make them suspect what was going on at the crater. As they went up the path, however, near the "River Bed" (about a mile above Lot 14), they noticed where stones had fallen and indented the earth. This was early in the forenoon, probably between 8 and 9 o'clock. They pushed on to near the summit of the hill, though they heard noises and saw cracks in the earth, for they little knew the dangers they were facing.

The top of the mountain was covered with mist, and the foremost of them followed the path up to the base of the summit cone. Some went up to quite near the lip of the crater, or possibly even to the actual edge. What they saw there was enough to dismay the stoutest hearts. The lake they could hardly have seen, for the whole of the crater depression was filled with dense steam. The bush within the crater was scorched and withered. There was a strong sulphurous smell, and rumblings within the crater mingled with the hissing sound of the steam jets. The lip of the crater near the old rest house was covered with "soft sulphur," that is to say, a coating of mud, probably sulphurous, from the bottom of the lake. Some said

they saw the water of the lake, but this is more than doubtful, others that they saw "where the water run over." The top of the hill was in a continual tremor. They turned and fled back along the path, and, meeting others ascending, told them what they had seen and all returned together. As they passed the estates by the roadside they spread the news that the Soufrière was in eruption, and the lake had boiled over. They were received with incredulity, and when they came to Georgetown they were scoffed at as fools and cowards.

Discharge of the Crater Lake.—It is certain that the lake was boiling, though at its usual level, and in no way much altered on Tuesday at mid-day. That afternoon it was reported at Chateaubelair that the water was at the top of the lip and running over. This statement we were not able to confirm by evidence collected from eye-witnesses, but no serious discharge can have taken place that day or the following night. Mr. ROBERTSON was at Wallibu, and the valleys on the windward side had houses standing on the banks of the streams and full of people. The lake was a quarter of a square mile in area and at least 500 feet deep. Its discharge would have produced tremendous floods of boiling water, much greater than the heaviest of tropical rains could ever have occasioned, and such floods could not possibly have taken place without being noticed. The natural channel of escape for the lake was over the lower or southern lip and down the valleys of these rivers, and we may be sure the bulk of the water had not been discharged before Tuesday morning.

It is in every way probable that the first discharges of the crater lake took place during the night of Tuesday. Mr. MORGAN, of Chateaubelair, told us that early on Wednesday morning large trees were seen floating in the mouth of the bay on which that village stands, "as if there had been heavy floods in the Larikai River." They were uprooted and had lost their branches, but were not burnt, and there had been no heavy showers during the night. Probably gushes of water had flowed down that valley or the ravine of the Rozeau Dry River, and swept before them some of the trees growing on the slopes. We must remember, also, that the fish sellers who were near the lip of the crater on Wednesday, at about 9 o'clock, all agreed in stating that they saw where "the water run over," and that the lip of the crater was covered with soft mud through which they hesitated to walk to where the path descends to leeward. About 11 o'clock, Mrs. KELLY, of Wallibu, who was in a boat off the mouth of the Wallibu River, saw "an upheaval of vapour coming from the Wallibu Dry River, just at the beach." This is the first mention of hot water in any of the stream courses.

Of the first explosion after day-break, that at 6 A.M., Mr. McDONALD states that below "the usual tall column of white vapour" there was "a much shorter column of dense black stuff which seemed heavier, as it quickly subsided back into the crater. This was the first appearance noted of probable solid matter being ejected." It may have been a column of mud and muddy water mixed with stones. The fish sellers, a couple of hours later, found stones lying on the road leading up from the windward

side, but if the crater still contained the lake, there can be little doubt that the larger outbursts would project water and mud to considerable heights.

At 11.10 he notes that "the area of escape of vapour seemed now to be extended in a direction corresponding with Morne Ronde." In all probability there was an overflow of steaming water on the south-west edge of the crater. It was just about this time that Mrs. KELLY saw steam in the Wallibu Dry River near the sea.

At 12.25 "Small vents seen forming on slope near old road and facing Richmond Vale. Jets of vapour being emitted seemingly from them, then a more violent outburst, which appears to be extending the crater towards the left (westwards); dark, blacker upheavals, as if the side of the crater toward Morne Ronde broke away and enlarged in that direction." The floor of the crater was now rising, and the boiling water of the lake was being discharged into the streams on the south-west side of the mountain. As the hot water poured down the valleys clouds of steam arose, and this suggested that a fissure had opened and vents formed in the ravines.

Just about this time (12.30 P.M.) Mr. ROBERTSON was leaving Wallibu. He had hardly got on board his boat on the beach when one of them called out, "Wallibu River coming down on fire." Mr. ROBERTSON turned and saw a raging flood of hot water tearing down the valley. He estimated its height at 30 to 40 feet, and states that he never had seen so high a flood in the river before. There had been no rain that morning at Wallibu, or, so far as he knew, on the leeward side of the mountain, to account for this, and he concluded, naturally, that it was the boiling water of the crater lake pouring down the stream.

No doubt similar torrents coursed down the other streams on this side of the island, as the Wallibu Dry River, the Rozeau Dry River, and perhaps also the Larikai. On the windward side a little after 12 o'clock the Rabaka River came down in floods of boiling water.*

In the light of these facts, Mr. McDONALD's subsequent notes may be readily interpreted.

"12.35. It seemed as if slope to left of old road up Soufrière had formed into fissure, as vapour was issuing from small vents, and at 12.40 these fissures were unmistakable, and discharges from crater were extended to windward.

"12.50. Enormous outburst through vent or front of mountain as far as could be ascertained, the mountain being largely enveloped in vapour, &c."

This last was probably the greatest discharge, and practically completed the emptying of the lake, but again at 1.25 and 1.33 P.M. he mentions volumes of vapour arising from the slopes of the hill. What was happening in the Wallibu and other rivers on the leeward side of the Soufrière at that time we cannot say, as no one who

* Mr. E. O. HOVEY was told that "a wall of water and mud 50 or more feet high (they compared it with the height of a factory chimney) came out of the upper reaches of the river (the Rabaka Dry River) and went out to sea." "Martinique and St. Vincent: a Preliminary Report," 'Bulletin American Museum of Natural History,' vol. 16, p. 341.

was there has survived to tell the tale. If then, as we believe, the lake still occupied the crater till about 1 o'clock on the Wednesday afternoon, one very curious fact requires to be explained. There can be no doubt that the reflection of fire was seen in the steam cloud at 5.20 P.M. on Tuesday and repeatedly after that. All witnesses agree on this point. It could not have been due to the burning of the sulphuretted hydrogen gas, as this gives a colourless flame, and after passing up through boiling water would require contact with something incandescent to ignite it. On the other hand, it may have been, and very probably was in part, the burning of the withered bush which clothed the interior walls of the crater. Many describe the fire, seen after the great outburst at midnight on Tuesday, as "like fire running through a cane brake." On Tuesday morning the bush in the crater was green; on Wednesday morning it was burnt, and it is in every way likely that some of the "flames" were due to bush fires. They may have been ignited by flashes of lightning or by incandescent stones which had fallen, sufficiently hot to set fire to the timber. In the latter case it is necessary to suppose that a cone was built up within the crater sufficiently high to rise above the surface of the lake, and that in the centre of this cone there was a minor crater from which the main explosions proceeded, and in which a surface of molten rock was exposed.

The configuration of the crater floor beneath the waters of the lake is not fully known, and it is by no means improbable that such a cone existed before this eruption, but concealed from view. Descriptions of the crater as it was before the eruption of 1812, state that within the lake there was a conical hill several hundred feet high (see p. 462). From the evidence of the fish sellers it is clear that steam was rising only from the centre of the lake, and this would indicate the presence of an orifice there, comparatively near the surface. Moreover, they stated that on Tuesday stones were being cast up with the puffs of steam, and this would inevitably tend to build a cone around the outlet. In this regard the evidence of a little boy who accompanied these women may be of some importance. He stated that on Tuesday morning he looked into the crater and saw the lake at its usual level, but discoloured and boiling at the centre. Where the water boiled he "saw a stone floating," and the water boiled only when it touched the stone. Though stringently cross-examined, he insisted that there was a floating stone on the surface of the water. This may have been the summit of an interior cone just projecting at that time above the water level. No red reflection was seen till at any rate six hours later, and by that time the lake may have somewhat diminished, or the cone have been built up to such a height as to rise freely above its surface.

According to the soundings published by Mr. FOSTER HUGGINS, since we left St. Vincent, the depth of the depression increased on all sides towards the centre away from the shore. The Admiralty chart shows the lake to have been about half a mile in diameter, and as Mr. HUGGINS gives the greatest distance from the shore on the north-east and west, sides to which he carried a chain of consecutive soundings

as 200 yards, and on the north as 20 yards, there remains a considerable part of the central part of the lake of which the depth and configuration are not known. No trace of any interior cone, however, can be found in Mr. HUGGINS's soundings, especially as he states that the depth in the centre was about 87 fathoms, and if there was none it certainly seems unlikely that comparatively unimportant outbursts like those of Tuesday, May 6, 1902, should have built up a new cone not less than 500 feet in height. In that case it may be that the ruddy light was the reflection of the surface of molten lava exposed in cracks and fissures in the sides of the crater. It must be remembered that not more than one-third of the depth of the depression was occupied by water, and the level of the lake on Tuesday forenoon did not differ greatly from the normal. Mr. WHYMPER's* description of the crater of Cotopaxi as he saw it, is of great interest in this respect.—“ We saw an amphitheatre 2300 feet in diameter from north to south, and 1650 feet across from east to west, with a rugged and irregular crest, notched and cracked; surrounded by cliffs, by perpendicular and even overhanging precipices—some bearing snow, and others apparently encrusted with sulphur. Cavernous recesses belched forth smoke; the sides of cracks and chasms no more than half-way down shone with ruddy light; and so it continued on all sides, right down to the bottom, probably 1200 feet below us”

The Northern and Eastern sides of the Mountain.—Apprehension and anxiety prevailed among the inhabitants of the north shore of the island during the forenoon of Wednesday. About 10 or 11 o'clock boats arrived to take away some of the people, and they brought news that from Chateaubelair the Soufrière had been seen giving off clouds of steam, and that a red glow had been visible on the summit during the previous night. About 11 o'clock thick dark grey clouds were noticed over the summit of the Soufrière. As seen from Fancy “they continued to increase and ascend higher in the air and assume the form of flowers bursting into bloom, the dark ashen-grey clouds being interspersed with streaks of silver. Simultaneously there were earthquakes, severe rumbling noises, and flashes of lightning. The volumes of smoke seemed to take a north-easterly course, and as the day was calm their progress in that direction was not impeded.”†

Mr. DUN, of Owia, reports that he heard noises at 10.15 A.M., 12.30 P.M., and 12.45 P.M. The first was like the rumbling of a heavy goods train, but the second sounded like a great explosion, and left no doubt in his mind of the reality of the eruption. They were accompanied by earthquakes, and about 11 o'clock a little rain fell containing fine particles of ash.

On the windward side the night of Tuesday had passed without anything having been seen which indicated the proximity of danger. On Wednesday morning it became widely known that it was rumoured from Chateaubelair that the Soufrière was in eruption, but in the absence of positive evidence a general scepticism prevailed.

* E. WHYMPER, ‘Travels amongst the Great Andes of the Equator,’ p. 152. 1892.

† Narrative furnished by Mr. CUBBIN.

The fish sellers who crossed the island on Tuesday morning had reported that the crater lake was boiling, and through the night rumbling noises had been heard and earth-tremors felt in several places. The black labourers, among whom evil tidings spread with a marvellous rapidity, were far from confident, and though sugar making was started as usual on several estates, and there was no cessation of work anywhere, it was thought advisable by some of the managers to send a party of white men up the mountain to see what was really happening there. This appears to have been determined on after the receipt of a message from Mr. PORTER, the proprietor of most of the estates in this quarter, at Kingstown, that the leeward side had been evacuated, and great anxiety was felt in Chateaubelair. The party started from Turema and Orange Hill, and rode up the path to the back of Lot 14. There they met the fish sellers returning from the summit after their ineffectual attempt to cross to the leeward country. This must have been about 11 o'clock, and fine ash was now falling around the lower slopes of the hill, so, partly for this reason, and partly from the information given by the fish sellers, they returned to Orange Hill, and reported that it was only too true that the Soufrière, after its long quiescence, was once more in eruption.

About 11 o'clock rain began to fall containing particles of ash, and the noises from the mountain became louder, more frequent, and more threatening. On some of the estates, work was then stopped, and many of the labourers took flight to Georgetown. Others continued to work till near 1 o'clock, when there was a sharp fall of gravelly stones; this put an end to all sugar-making, and people fled to their houses, or began to hide in the cellars, or the rooms of the larger residences in which the managers lived. Smoke could now be seen ascending in vast columns from the crater. Many tried to escape to Georgetown, but when they got to Rabaka they found the Dry River there pouring down in high flood, and the water was so hot it was impossible to wade across. There is no bridge over this river, so most of these refugees gathered at Rabaka House, but some returned to their own dwellings.

In Georgetown it was not till about 11 A.M. that it became known with certainty that a catastrophe was impending. Mr. J. W. CLARKE, teacher in the Government school, has given us an excellent account of his experiences that morning, from which we take the following :—

“ Previous to the above-mentioned date (7th May) there were signs noticed in some places that the crater was in action, but at Georgetown there was no sign observed till the very day, in fact, one can almost say the very hour. On the morning of the 7th, reports of the Soufrière's activity were being questioned for the simple reason above stated. Then women and children went to perform their various regular duties; everything appeared as usual. I kept my regular 7 to 9 private school. Shortly after 9 A.M., sounds resembling distant thunder were heard, but it was not until 10.15 A.M. when particular notice was taken of the sounds, from their long-continued detonation. I was engaged marking my school register of attendances from 10.15 to 10.30 A.M., during which time the noise increased. At about 10.40 there was a distinct flash of lightning seen from the direction of the mountain, followed by a crackling peal of thunder. At 10.45 I got a little uneasy, and just at the same moment I got a message from

Mrs. BALLANTYNE, asking me to send home her daughter, who was then at school. I hesitated for a few minutes, and, when in consideration of what should be done, another stroke of lightning followed by heavy thunder was seen. I then complied with the request, and, feeling apprehensive of danger, I at once went to the manager of the school, Mr. H. B. ISAACS, and suggested that the school should be dismissed, and the children sent home till matters cleared up. After a few minutes' consideration, the suggestion was granted, and I hurried back, and, in a few words, dismissed the school.

"It was now about 11 A.M., and the rumbling noise still continued. About 11.15 drops of water containing sulphurous matter fell, and that was the first direct sign which told me of the disturbance of the Soufrière. The water continued falling only in drops here and there. At this stage, very minute particles of dust or ashes fell, but were only observable on white material. My boots at this time were besmeared with the sulphurous matter mentioned, as I kept walking from one place to another. At about 12 noon I was standing under the galvanised roof of the Grey's store when small pebbles, about a pea grain size, commenced to fall. This falling of pebbles continued for about an hour, during which time several others along with myself gathered pebbles of the same.

"It was about 1.30 P.M. when smoke was distinctly seen issuing from the crater, and volume after volume rose, only to ascend higher than the former; the clouds of smoke got blacker and thicker, and each mass seemed to travel faster than the first.

"At about 2 P.M., pebbles of a larger size commenced falling, and it was becoming injudicious to move about."

One of the most remarkable features of this eruption is the suddenness with which it broke out. At Chateaubelair steam was seen ascending from the crater on Tuesday forenoon, and the inhabitants had at least 24 hours' notice before the volcanic activity assumed a dangerous phase about noon of Wednesday. But everywhere else around the mountain there was no certainty till about 11 o'clock on Wednesday morning—or only 3 hours before the climax was reached, and the great black cloud swept from the crater to the sea, burning and suffocating those in its path. Had the leeward side of the hill not been clear of mist, so that a view of the crater was obtained by those dwelling there, the loss of life would certainly have been much greater than it was, for the noises would have been mistaken for thunder, as they were at Georgetown.

That even at Lot 14, 3 miles from the crater, the reality of the eruption was doubtful till near mid-day, proves that the outbursts seen from Chateaubelair up to that time were comparatively unimportant, and consisted only of steam and hot water, with a little fine ash and a few showers of stones. They constitute the preliminary stage of the eruption. Thereafter the activity rapidly increased in violence. About mid-day the crater lake was discharged, and this showed that the upward pressure of the lava was overcoming the resistance, and forcing a path to the surface. The throat of the crater was being cleared. Vastly greater steam clouds now shot up into the air, and the noises of explosions, hitherto comparatively few, became numerous and loud, while showers of hot stones with trails of sparks began to fall upon the slopes of the cone. Between 12 o'clock and 1 o'clock P.M. the hill was still quite green up to near the summit. No serious damage had yet been done to the vegetation, and at Wallibu and Lot 14 only a thin film of fine grey ash had as yet fallen, just sufficient to give the vegetation a dusty appearance. At Wallibu, before

Mr. ROBERTSON left at 12.30, a few stones had fallen several inches across, and as he rowed away he noticed that some were floating on the surface of the water.

The Climax of the Eruption. The Descent of the Great Black Cloud.

The climax of the eruption was now at hand. It came with terrible suddenness. With laconic brevity, Mr. McDONALD records it as follows :—

“ 1.55. Rumbling. Large black outburst with showers of stones all to windward, and enormously increased activity over the whole area.

“ A terrific huge reddish and purplish curtain advancing up to and over Richmond Estate.

“ At this stage left Richmond Vale House and hurried into and pushed off boat a few minutes after 2 P.M. Saw vapour as we rowed hard across Chateaubelair Bay coming down to sea level past Richmond Point. Sea peppered all round with stones, one of which—about a cubic inch—fell inside the boat, in which were eleven persons.

“ The huge curtain referred to was advancing after the racing boat, which never seemed likely to get out of the range of it, or the falling stones, which latter varied from the size of one's fist downward

“ The lightning and thunder at this time were terrific, and there were noises inland.

“ Everything seemed to point to a general break up, both on land and sea.”

This was the outburst of the great black cloud, which, charged with immense quantities of red-hot dust, poured from the crater and swept down the valleys to the sea.

It will be noticed that Mr. McDONALD describes it as reddish and purplish. He does not enter into particulars as to its form, except that he states that it was like a curtain. All who have seen the side of this cloud use exactly the same term in describing it. It resembled a curtain hanging in folds, black, dense, solid, and well-defined. The cloud swept out to sea over Richmond ; its southern margin was over the headland on the south side of the mouth of the Richmond River. Richmond Vale House stands in the next valley to the south of Richmond Estate, and it was spared and but little damaged, while Richmond was wiped out and destroyed. Mr. McDONALD's boat was perhaps half a mile south of the edge of the black cloud. Stones fell about it, but there does not appear to have been great darkness for some time after, and he does not speak of suffocating vapours or dust, or of any very great heat.

It is interesting to compare with Mr. McDONALD's description the statements made to us by some black and coloured men, who were just a little further north than Mr. McDONALD, and were caught in the edge of the cloud. They had been at Richmond to remove their personal belongings to Chateaubelair, and their boat was returning when the black cloud swooped down.

The sea was perfectly calm and the day clear, though there had been a few drops of rain in the forenoon. The boat had just rounded the point south of Richmond River, and was on the north side of Chateaubelair Bay. The cloud struck them like

a strong breeze, though, being under the shelter of a high spur of land, they did not feel it much. Still it came over the water with a strong ripple and a hissing sound, due to the hot sand falling into the sea and making it steam. In a moment it was pitch dark and intensely hot and stifling. The cloud was highly sulphurous, and this irritated their throats and nostrils, making them cough. The heat was terrible and the suffocating feeling very painful. They threw themselves into the sea to escape burning by the hot sand. It does not appear that the surface of the water was boiling as it was in some other cases. They all dived, and when they returned to the surface the air was still unfit to breathe and the heat intense. So they continued to dive repeatedly, but when they came up again the air was almost as bad as before. How long this lasted they cannot tell, but they thought it might have been several minutes. At last they were all utterly exhausted and could have held out no longer; then they felt the air clearing, the heat diminished, though still very great, and they clung to the boat for a few minutes before they were able to get into it again.

One man was not so good a swimmer as the others, and his strength was soon exhausted. He held on to the gunwale of the boat and took the risk of burning rather than of drowning. He described to us the insufferable heat and the sulphurous smell, and he was rapidly becoming unconscious when the air cleared. The sea around was hissing, and it was so dark that two men hanging on to the boat, side by side, could not see one another, even though they could touch. There was a continuous loud noise, but a person speaking in an ordinary tone of voice could easily make himself heard. This is a curious fact, for the report of this explosion was heard at Barbados. But Mr. McDONALD confirms the statement. He noticed particularly that when he gave orders to his men to launch the boat and leave the shore, he did not require to exert himself in the least to make them hear him.

While this man remained clinging to the gunwale the hot sand rained upon him. His woolly head was wet and the sand, was cooled by contact with it, but it gathered above the lobes of his ears, and there the heat was sufficient to dry his skin and produce painful burns. He told us that when the cloud had passed there was enough sand on his scalp to fill his hat twice over (the hat was an ordinary round straw with flat brim). It formed a layer 2 or 3 inches deep. The ash fell dry, there was no rain and no scalding mud. When they got back into the boat they found it nearly filled with fine ash and stones, and these were so hot that had the boat not been leaky and partly filled with water it might have taken fire. Yet these men were only in the outer edge of the cloud. A few hundred yards further north this cloud deposited in some places 40 feet of red-hot sand in a few minutes. It was for this reason also that they experienced no great shock when the cloud struck them, and did not feel it pass like a strong blast, for Mr. McDONALD states, a cloud came down with a high velocity. His boat was going at perhaps 8 or 10 miles an hour, impelled by the frantic exertions

of a crew fleeing from a dreadful death. But the cloud was travelling at least twice or thrice as fast. The curtain-like lateral margin of the dense black cloud was almost stationary, or consisted of gentle eddies at the side of a rushing torrent.

Another man, who was caught in this cloud and survived, gave us a very interesting narrative which confirms those we have been considering. He was coming south from Campobello on the north shore in a boat with several others. That morning he had gone north from Morne Ronde to rescue his family, and when he got to Campobello the people there knew nothing of the eruption. His party left at 1 o'clock, and a little later they passed Windsor Forest. This was a grazing estate, and he saw that all the cattle were gathered on the beach, running to and fro and bellowing with terror. A few minutes later he heard the sound of a great explosion and saw a huge black mass pouring out of the Wallibu and Larikai Valleys to the south of him. Terrified, he started to return, but at Baleine another similar cloud was rushing down the ravines on the mountain side to the north. No course remained open except to stand right out to sea.

Small stones began to fall in the boat. Then he was enveloped in dense darkness and ash fell, at first wet but afterwards dry and quite cold. By this time they were several miles out from the shore. Another boat was quite near him when the darkness descended. It was never seen again, but with its occupants was totally lost. He thinks it was filled with sand and sunk, for the downpour of ash and stones was so heavy that they had to keep constantly bailing it out. "It rained as fast as if three men were throwing in sand with shovels."

He rowed right out to sea, and that night the tide took him to quite near St. Lucia. When it turned, it carried him back again, and next morning he landed near Chateaubelair.

This narrative proves that the black cloud swept over the north-west side of the mountain, and that here also it poured down the valleys almost like a torrent of water. It was much less dense in this quarter than at Wallibu, and when it had passed a few miles from the shore, though laden with dust and stones, it was quite cold, and was moving so slowly that it did not overturn his boat or raise the sea sufficiently to make it dangerous. He did not mention any great smell of sulphur, but stated that the lightning was terrific, and there was a continuous rumble in the cloud "like the rolling of a barrel."

We have also the reports of several competent observers who saw the cloud from a distance of several miles as it rolled out over the sea past Chateaubelair. The Rev. Mr. DARRELL, of Kingstown, in a brief account of the eruption, which was printed in Kingstown on May 12, writes as follows:—

"We were rapidly proceeding to our destination when an immense cloud, dark, dense, and apparently thick with volcanic material, descended over our pathway, impeding our progress and warning us to proceed no further. This mighty bank of sulphurous vapour and smoke assumed at one time the shape of a gigantic promontory, then a collection of twirling, revolving cloud-whorls, turning with rapid

velocity, now assuming the shape of gigantic cauliflowers, then efflorescing into beautiful flower shapes some dark, some effulgent, some bronze, others pearly white, and all brilliantly illumined by electric flashes. Darkness, however, soon fell upon us. The sulphurous air was laden with fine dust that fell thickly upon and around us, discolouring the sea; a black rain began to fall, followed by another rain of favilla, lapilli, and scoriæ. The electric flashes were marvellously rapid in their motions, and numerous beyond all computation. These, with the thundering noise of the mountain, mingled with the dismal roar of the lava, the shocks of earthquake, the falling of stones, the enormous quantity of material ejected from the belching craters, producing a darkness as dense as a starless night, together with the plutonic energy of the mountain, growing greater and greater every moment, combined to make up a scene of horrors."

Dr. CHRISTIAN BRANCH, of Kingstown, was in the same boat, and gives the following description of the scene:—

"We did not go far, for the first point we rounded disclosed a horrid black, solid wall of smoke jutting into the sea, about two or three miles from us. It looked like a promontory of solid land, but it rolled and tumbled and spread itself out, until, when we could last see it sometime later, it must have extended quite 8 miles over the sea to the west. It was evident we could not go through it, and if it overtook us, as it seemed likely to do, we would be lost in darkness, even if nothing poisonous was in it.

"The island to the north of us and north-east was now covered with a mighty black pall of smoke, perhaps two miles deep, and the smoke column was now a vast shapeless blackness. Then began the most gorgeous display of lightning we could conceive. All around us and above, so near, that several times I saw it between us and the cliffs not 200 yards off. It was still bright daylight with us, but the whole atmosphere quivered and shimmered with wavy lines intersecting each other like trellice work. We were encircled in a bristling ring of fiery bayonets. It was too stupendous to terrify; one could only marvel and feel nervous. A few stones plumped in the sea around us, and then fell pretty thickly. They were light pebbles for the most part, and only these fell in the boat. A nasty shower of mud followed the lightning, and then a long shower of gritty sand. After this a fog of fine dust descended, and it got darker and darker, until we could with great difficulty see the shore and points along which we steered."

On the windward side of the island the black cloud descended in probably no less volume than to leeward. Its main current flowed down the valley of the Rabaka Dry River as, at Owia on the north-west side of the island, and at Georgetown, which stands well to the south of the valley, its action, though traceable, was not devastating or lethal.

In the Carib country work in the fields had stopped, and in the sugar works, though they were full of people, no sugar making was going on. Everyone was watching the progress of the eruption in mingled fear and admiration. Small stones began to fall after mid-day, and about half past one in some places there were showers of hot scalding mud. The cattle, horses and mules were mostly out grazing, but nearly all the people had gathered into the more substantial buildings, the managers' houses or the stores and cellars attached to the sugar works, though many were in their huts in the villages adjoining the estates. Some had been struck with falling stones, but as yet probably no one was killed, and but few injured.

Then, with a loud roar, at 2 o'clock the great convulsion came. Those who were in the open air saw the huge black cloud rolling down the mountain in globular,

surging masses. They fled into the houses and shut the doors. Onward it rushed with a loud rumbling noise and filled with lightnings. Any who were in the open air perished at once. Many of the negroes' huts were so densely crowded with people, that there was hardly standing room. At Lot 14 the manager and his wife and family had shut themselves up in the rum cellar below the house, and firmly closed all doors and windows. They had a terrible experience, but they survived. All of those in the house itself or in the negro village were killed. It will be understood that as the tropical houses are so built as to secure free entrance of air, it is almost impossible to close them up securely, and the suffocating blast reached the interior and stifled all who were there. All the animals in the fields also perished.

At Rabaka many who were prevented from fleeing to Georgetown by the floods of hot water in the river had gathered in the manager's house. In one large room there were fifty people. When they saw the dark cloud coming, they firmly shut all the windows—fortunately they were substantial and well-glazed—and everyone in this room was saved. They felt the heat most intense. It was quite dry, and there was a very strong and irritating odour of sulphur. Some fainted, but all survived. The overseer told us that from the window he saw the black cloud rolling on towards them; when it reached the house there was darkness, and a sharp fall of stones on the roof. The cloud rolled down upon the sea below the house, and when it struck the water it was "filled with fire." It seemed then to rebound from the surface of the sea and return towards the building, and at this time the wall of a ruined sugar store was knocked down, probably by lightning, as nothing else was overturned. It was when the cloud returned from the sea that the suffocating feeling was experienced. Perhaps this was because it took a little time for the noxious gases to penetrate to the interior of the house. Practically all who were in the negroes' huts or in the open air perished.

At Orange Hill there was a large substantial stone-built rum cellar, which, by the orders of the manager, was left open to afford a refuge for any who wished to avail themselves of it. About seventy crowded together there. The windows were not shut, but they were small and faced the sea, so that the blast did not directly strike them. One man stood by the door holding it ajar, to admit any who fled from the huts in the village. Forty were in the cellar, and all were saved. Thirty were in the passage leading into the cellar, and they were all killed.* None of those survived who remained in the labourers' huts, or were fleeing to and fro about the yard in abject terror. Many shut themselves up in a store with a galvanized iron roof. All died, and were found buried in sand with the roof collapsed and fallen upon them. In the

* Mr. E. O. HOVEY states that 132 persons were saved unharmed in this cellar. ("Martinique and St. Vincent: a Preliminary Report." 'Bull. Amer. Mus. Nat. Hist.,' vol. 16, p. 344.) This is an example of how divergent are the statements of different witnesses of the catastrophe. Our informant was caretaker of the estate in June, 1902, and he was an occupant of the cellar during the afternoon of the eruption. (See Appendix III., p. 547.)

under story of the manager's house thirty people died. The manager himself, Mr. FRASER, was found dead on his verandah; his wife's body was lying at the foot of the steps leading up to it. They seem to have been overcome as they were returning from the cellar where they had at first taken refuge. We were told that Mr. FRASER complained that the densely packed crowd of negroes made the atmosphere unbearable, and returned to his house to get some fresh air.

At Turema the fatal cloud did its deadly work quite as effectively. All who were in the manager's house, estimated to number thirty-five, were killed. One woman survived for three days, and was found by the first search parties who went out from Georgetown. She begged for water to drink; they gave her some, and returned to make arrangements to have her taken to the relief hospital, where she died shortly after her arrival. In the "Great House," or Mansion House, four were killed in the kitchen, where the windows and doors had not been effectively closed; two shut themselves up in another room, closing all apertures as thoroughly as possible, and they survived. All who were in the villages or fields perished, without exception.

In Overland village the loss of life was terrible; hundreds were killed. In one small shop by the roadside, in a room perhaps 15 feet square, eighty-seven bodies were found. When we saw this place the ash around was dotted with little hummocks, under each of which lay a heap of bodies, but everything was decently interred, and contrasted strongly with the charnel heaps of bleaching skeletons we saw later on in St. Pierre. One man whom we interviewed had lived in this village; in his house seven died, but four or five survived. When they saw the black cloud coming down, they shut up all doors and windows as tightly as possible. As the hot cloud approached, it was red at first, but changed into black before it passed overhead; the heat was dreadful, and the lightnings very vivid. The air smelt very strongly of sulphur, and their throats were dry and parched. Some burst into spasmodic coughing from the irritant sulphurous acid and fine dust in the air. Many cried out for water, but in a few seconds the suffocating feeling prevented articulation. Then several threw up their hands and fell dead. Others collapsed, but lingered in some cases for an hour or more. Those who survived state that in a few seconds it would have been all over with them. But the air began to clear, there was a slight breeze off the sea, and the windows on that side were thrown open, and, with a sense of great thankfulness, they inhaled again deep breaths of cold pure air. Here, as at Rabaka, those who were watching the cloud state that as it struck the surface of the sea it flashed with fire. One man showed us where the back of his fore arms had been severely burned by hot mud or ash which came down with the black cloud. The parts of his body covered by clothes were protected, but his shirt sleeves were rolled up, and his arms below the elbows were bare.

At Orange Hill, Turema, and Lot 14, there were large herds of cattle, horses, and mules. Every animal on these estates perished. Some were suffocated or burnt extensively; others had apparently been struck by lightning. They were all in the

fields or in open pens, and not one survived. It does not appear, however, that before the great outburst, which took place at 2 o'clock, they had shown any peculiar restlessness as if they were aware of the impending doom. At Windsor Forest the cattle had that forenoon been in a state of trepidation, and had fled to the shore, where they ran up and down, bellowing loudly. Probably this was because earth tremors and subterranean noises were more common and more pronounced there than in the Carib Country. At Wallibu, about 12 o'clock, there was a loud sharp noise accompanying an outburst of steam, and the earth shook. The house-dog ran out into the open air howling with terror. On the windward estates, however, it seems that the dogs remained at the houses and hid themselves, but did not run away or endeavour to escape.

Over the northern shore of the island the dark cloud descended also, but there its velocity was less, and the devastation it produced much less considerable than in the valleys on the south of the mountain.

The Fancy estate lies almost due north of the crater, and about 3 miles distant from it. It is actually nearer the focus of eruption than either Wallibu or Lot 14, though those two have suffered much greater damage. Mr. CUBBIN, who was the local teacher, has supplied us with some interesting notes regarding the events of that afternoon. About 2 o'clock there was a fall of stones, which increased in severity till the manager, Mr. BEACH, considered it desirable to collect all the people on the estate in a large iron-roofed main building. Dark clouds were then seen pouring out over the sea on each side, and soon they broadened till only a narrow passage was visible between them to the north. "The mass of falling *débris* seemed to be closing in upon us, and in a short time it fell, enveloping the building and almost suffocating the inmates; during this time there was almost total darkness. In a few minutes there was a glimmering light, as at dusk. The houses in the village, or most of them, were now observed to have been demolished; this was caused by the falling *débris* and by lightning, and most of the people who remained in the village were either dead or fearfully burnt. Many of them died by next morning."

The estates building was of stone with galvanised iron roof. The windows were all closed, but the door was open at the time the cloud descended, and was shut by the force of the blast. All who were in the building were saved, but about forty were killed in the village. Those who escaped were badly burnt, mostly on the hands, the feet, and the face, but others also on the parts protected by clothes, though their clothing was not scorched or ignited. The dust in the cloud was not red-hot, and consequently did not set anything on fire, the burns being apparently due to steam and other gases.

Much of the damage done on the Fancy estate was apparently due to lightning. Next morning, the shingles from the roofs of the houses, and the galvanised roof of a store, were found dislodged and carried some distance away. The cloud came with a blast, but no one believed that this was sufficient to produce these effects. Moreover, the

materials were scattered irregularly, some being found on the beach below the houses, others on the opposite side towards the hill. These facts all point to the capricious action of lightning from the cloud.

Similarly, at Turema and at Wallibu, the factory chimneys were knocked down. As at both places many of the trees are still standing, and as at Turema the houses are comparatively little damaged, there can be no doubt that the chimneys were not overturned by the blast, and it is generally believed to have been the effects of lightning. It is probable also that some of the huts which were set on fire were really ignited in this way.

The first reports of the catastrophe stated that the deaths were practically all the result of lightning. This is certainly not the case, but it is sufficient to show how rapid were the fatal effects as a rule. On the other hand, there can be no doubt that the lightnings were the cause of many fatalities. We were told by one man, who was looking out of the window of the rum cellar at Orange Hill, that he saw a woman starting to run across the yard to the building from one of the huts. That instant there was a bright flash, and she fell dead. The corpses of some of the animals which perished in the fields gave evidence of having been struck by lightning, and everywhere on the devastated estates it is easy to find trees which show the effects of the same agency.

At Owia, according to Mr. EFFINGHAM DUN, small and large pebbles were falling about 1 o'clock, and from 1.30 to 1.45 there was a rain of hot liquid mud. At this time there was a nauseating odour of sulphuretted hydrogen, but it only lasted about half an hour. It now became very dark, and from about 2 o'clock to 3 o'clock the heat was almost suffocating, "and I had to throw water about the house to make breathing possible." At Owia the damage done was comparatively slight. The crops were injured but not buried, and none of the inhabitants were killed.

That the black cloud surmounted the rampart of the Somma wall, which rises to the north of the crater and poured down the northern slopes of the mountain, is most clearly proved by the evidence of the occupants of the boat already mentioned, which at about 2 o'clock was off Windsor Forest on its way from Campobello to Chateaubelair. They saw a dense, impenetrable mass streaming down the Larikai and other valleys to the south of them, and turned to retrace their course, but immediately afterwards a similar cloud was seen descending the valley at Grand Baleine, so being caught between two fires, they had nothing left but to stand directly out to sea. The Grand Baleine Valley is the largest on the north-west quadrant of the hill, and the main volume of the cloud seems to have coursed along it like a fluid. Between this valley and the estates of Fancy and Owia a series of ridges intervenes, and these apparently sheltered the north-eastern corner of the island and deflected the main force of the blast. In this quarter the destruction was less complete than in any other section of the hill.

At Georgetown no lives were lost, and it seems certain that the deadly black cloud

did not pass over the town. A little after 2 o'clock the inhabitants heard a very loud sound proceeding from the crater. This was described to us as "a long, loud, mournful, unearthly, death-like roar." The mountain at that time was emitting an enormous column of steam, which expanded and spread out laterally as it ascended in the air, and red-hot stones were tumbling down the slopes, giving out trails of sparks. A heavy fall of scoria and stones followed the outburst. Darkness then set in fairly rapidly, though by no means instantaneously, and the rain of ash began. It continued through the whole evening and the early part of the ensuing night.

The first estate north of Georgetown is Mount Bentinck, and here also no lives were lost, though the fields were buried in ashes. Langley Park stands half-a-mile to the north of Mount Bentinck, and many were killed there. In the main building some thirty or forty bodies were found. The house was being cleared of ashes when we visited it, and as they had stuck to the surface of the walls, forming a grey layer like fresh cement, it would seem that on the edge of the cloud the dust was in some places moist and adherent. This was also the case in Fancy, where on the morning of Wednesday it was seen that the walls of the houses were plastered over to a depth of half-an-inch with fine wet ashes.

It is clear that the black cloud passed over Langley Park, and that the fatalities which took place there are to be ascribed to its action. Mount Bentinck and Georgetown escaped, and this shows how sharply defined was the lateral margin of that mass of deadly vapours and dust. It shows also how harmless was the rain of ashes from above, as no one was killed south of the limits of the cloud, though a few were injured by falling stones.

The Rain of Ashes.

The history of the later stages of the eruption which followed the descent of the great black cloud cannot be given with the same fulness and detail as that of the earlier phases. Darkness now covered the mountain and much of the surrounding country, and little could be seen except the flashes of lightning and the occasional fall of red-hot stones. The inhabitants had shut themselves up in their houses, and many were engaged in succouring the wounded and dying, so far as that was possible. Others had hid themselves in inner rooms and cellars in momentary expectation of a sudden and painful death. No man was sure of his life for a moment. Everyone believed that most of his neighbours and friends had perished. Many dreaded that the sea would invade the land in great tidal waves. The earth rocked and shook continuously, and those who dwelt in the more substantial stone-houses were afraid that they would fall on them. The air and sky were filled with lightnings, which quivered and played incessantly. Many houses were already on fire, and trees and buildings were frequently struck. Fine ash and lapilli were raining down upon the roofs of the houses, which were rattling under this bombardment. Every now and

then a fall of bigger stones would crash with a loud noise on the wooden or iron roofs, and not unfrequently these were perforated by large stones which landed in the rooms among the frightened survivors. Those who ventured abroad protected their heads with pillows or pieces of wood, or even with tubs. The ground was covered with a layer of ashes which, though warm, could without difficulty be walked upon. The air was charged with fine falling dust, which irritated the nasal passages, giving rise to coughing and sneezing. A few remarked to us that immediately after the great outburst which took place about 2 o'clock in the afternoon, the pressure of the air seemed to be increased, and the effect on the tympanic membranes of their ears was such that the sound of the roaring of the mountain was at times acutely painful. As we shall see later, there can be no doubt that the eruption of the great black cloud occasioned a sudden rise of barometric pressure. Dr. DUNBAR HUGHES, of Barrualli, was one of the observers who noted this phenomenon.

Above all resounded the roaring of the mountain, and for about two and a half hours in the afternoon the noise was terrible. Even at Kingstown it was so loud that it resembled no sound with which the observers were familiar. Some compared it to the discharge of an enormous gun, except that it was continuous and not intermittent, and we were reminded that among the Caribs there were old traditions regarding the "Great Gun of the Soufrière." Most people described it, however, as having a long, drawn-out, weird, unearthly character, recalling the roar of a wounded animal in intense pain. It is a curious fact that most observers in St. Vincent state that it did not seem to them to be made up of distinct detonations or reports, or if so, these were so numerous as to be blended together without intermission. The sound, however, rose and fell, being at times distinctly much louder than at others. At the same time in Barbados, St. Lucia, Trinidad and elsewhere, the noises from the volcano were compared to the reports of distant cannonading, and the intermittent nature of the sound was one of its distinctive features.

The air also was laden with sulphurous fumes. Especially was this noticeable during the passage overhead of the great black cloud, when the abundance of sulphurous acid (and also sulphuretted hydrogen) was so great as to produce an intense feeling of irritation and suffocation. Thereafter it seems to have been much less, but continued more or less through the whole night. The sulphuretted hydrogen attacked silver articles, and it was noticed in several cases that the silver bracelets on the arms of the coolie women turned rapidly black. At Kingstown the first fall of ash was accompanied by a sulphurous odour; after that, however, it was much less marked. This was also the case at Barbados, where the sulphurous smell of the dust was one of the reasons which led to an early and general recognition of its volcanic character.

The intense heat, however, was even more oppressive than the sulphurous vapours. Within the precincts of the dark cloud it was terrible. The sufferers cried for water, till the scorching of their throats prevented articulation, and they fell groaning to

the ground. But in all the district round the volcano, during the earlier part of the afternoon, the inhabitants felt that the temperature of the air was unusually high, and rendered exertion difficult. Most of them also remarked that their throats were parched and dry, and their thirst excessive. This may have been the effect not only of the high temperature of the air, but also of the fine dust which irritated the mucous membranes. It seems quite clear that the sudden discharge of enormous masses of incandescent sand into the atmosphere was sufficient to raise its temperature all over the northern part of St. Vincent during a period of at least a couple of hours. It is unfortunate that we have no readings of the thermometer which would enable us to ascertain how far these effects proceeded.

Rain does not appear to have fallen in abundance anywhere immediately around the hill that afternoon and night while the eruption was at its maximum or drawing gradually to a close. During the morning and forenoon there were local showers, especially on the windward side, but, though they were heavy, they were certainly not general. As already stated, the climax of the eruption was heralded in some places by showers of hot mud, or wet, hot ashes, which scalded not a few people. But on the leeward side the day was essentially dry, and at Kingstown, though the ash at first fell wet, there were no rains of any consequence. It is not a little remarkable that the discharge of such enormous quantities of water vapour into the air should not have been accompanied by condensation or precipitation on a large scale; but, whatever may be the reasons for it, the fact remains that the great eruption of the Soufrière was essentially a dry eruption.

When the great black cloud was seen rushing out to sea past Chateaubelair, even those whom courage, or a sense of duty, or helplessness, had led to linger in the village, were struck with one impulse to escape. We have some graphic narratives of that flight, which give us a picture of the demoralisation of the black population and the terrors of the eruption, so vivid as to be worth reproducing here. Dr. DUNBAR HUGHES and Captain CALDER, who had both gone there by order of the Administrator, were the last to leave, and Captain CALDER's account of his escape is as follows* :—

“When the black people realised their danger most of them grew madly excited, and in a few minutes everything in the shape of a boat or canoe pushed off from the shore, weighted down to a dangerous degree with human freight, each one excitedly urging on the others. I could then have left with the police in our boat, but with three or four hundred refugees on the shore I quickly determined that our duty was to remain.

“While I was speaking to the people in the street, the excitement and danger were increased by hot, half-melted stones falling from the enveloping cloud. I ordered everyone in the streets to leave the town at once, and, to prevent injury by falling stones, I directed them to take old boards and shingles from the dilapidated houses and cover their heads. Stones up to half a pound in weight were now falling, while the sulphurous fumes and fine light dust rendered breathing difficult. So, with at least three hundred refugees in front, we started out of the Chateaubelair valley, accompanied by the

* ‘Century Magazine,’ August 1902, p. 636.

prayers of some, the excited yelling of others, and the feeling of despair of nearly all. Men, women, and children of all ages scurried up the steep hill as fast as possible, mothers urging on their young children hardly able to crawl, old men imploring the assistance of the younger and stronger, each helping and encouraging the other, clearly showing the brotherhood a common danger engenders.

"One poor woman, with a brood of at least eight, was kept behind by the inability of the youngest two to keep up the pace. Her agonised cry for help I can never forget, nor the thankful smile I got when I picked them up, one in each arm.

"By this time the dense volume of sulphurous cloud, which chased us like a death pall, began to overtake us, and it was hard indeed to get the people to continue struggling on. As the darkness settled over us, a storm of lightning and thunder broke over our heads, and so near were the flashes that one thought that each surely must strike the people on the road, especially as the dry grass on the hill-sides was ignited. It would indeed be difficult to be more uncertain of another minute's life than on that hill-side that dark afternoon.

"As we gained the summit of the next mountain, the poisonous dusty cloud was held in check by a steady breeze coming in the opposite direction, but for which the death-roll by suffocation must have been appalling. I pushed on for 9 miles, until I got an opportunity of communicating with Kingstown, when I learned that sulphurous dust and ashes, accompanied by semi-fused stone, had fallen there. The stones measured on the average at least an inch in diameter.

"When about 4 miles from Chateaubelair, thinking the danger from falling stones had passed, I removed the board I had tied over my head, and, as a result of my want of caution, I was struck down, and remained in a semi-conscious state for over half an hour.

"It is impossible fitly to describe that awful trek through a continual blaze of lightning, driven, as it were, before that deadly and enveloping cloud of sulphurous dust and ashes. The awful grumbling and rumbling of the volcano continued throughout the night, and as the morning dawned, the deep green of the young arrowroot and cane plants had given place to a smooth leaden colour of dust, several inches deep, not a single green leaf of any description being visible."

Mr. T. M. McDONALD left Chateaubelair by boat about the same time as Dr. DUNBAR HUGHES and Captain CALDER started out along the road leading to the south. Two versions of his invaluable diary have appeared, one in the 'Sentry' newspaper of Kingstown, St. Vincent, on May 16th, 1902, the other in the 'Century Magazine' of August, 1902, pp. 638-642. The latter is somewhat amplified, probably by the incorporation of notes given by word of mouth to the reporter. They differ in several respects, but both are obviously copies of the brief notes taken in his pocket book at the time, and these notes we had the privilege of discussing with him on several occasions. On the whole, the version of the 'Sentry' adheres most closely to the original.* His statements are in almost perfect accordance with those of the observers already cited. He does not mention, however, the indraught or steady breeze which, advancing in an opposite direction, held the great black cloud in check. Not a few of the residents of Chateaubelair whom we interviewed when we were there had remarked that an opposing wind sprung up from south or south-east, and regarded it as an important factor in the preservation of the village.

Of the existence of a reverse current or indraught there can be no doubt. That it

* As this edition of the 'Sentry' newspaper was out of print in a few days, a second edition was struck off, which also was soon exhausted. We have considered it advisable to print it in the form of an appendix to this report. (See Appendix II, p. 544.)

was strong or persistent is quite unlikely. Some refugees who fled in boats from Chateaubelair landed near Rosebank and fled precipitately along the road, leaving their boats unfastened in any way. So calm was the afternoon and the ensuing night that these boats were found next morning floating within a few yards of the place where they were deserted. Hence it is clear that the indraught cannot have been vigorous. That any ordinary current of wind could have checked or altered the course of the great black cloud may be regarded as in the highest degree improbable. When it swept past Richmond to the north of Chateaubelair it was travelling with the velocity of a strong breeze or gale, and was so laden with dust that its weight must have been enormous. We may regard it rather as similar to the inrush which takes place when an express train passes at high speed through a railway station—a consequence of the movement of a large mass with a considerable velocity through the air. It may have drawn clear fresh cool air up to the edges of the black cloud, and in this way have mitigated the heat or prevented the lateral diffusion of the foul gases it contained. Possibly it may even have saved the lives of some. But it was a consequence of the great down-rush of dust and gases, and not in any real sense an opposing and countervailing agency. When at a later period we witnessed an eruption of Montagne Pelée the black cloud passed close to us, but although we watched for it carefully, neither of us observed any vigorous opposing indraught of air.

It would seem that immediately around the volcano darkness was absolute while the great dark cloud passed overhead, but that, at least in the district to the south of Chateaubelair and of Georgetown, though it was so dark that lamps had to be lighted in the houses, yet the fugitives could find their way along the roads without much difficulty. Dr. CHRISTIAN BRANCH states that when his boat entered Kingstown about 6 o'clock that evening, the light was still sufficiently good to enable the boatmen to make out clearly the headlands they passed as they rowed southwards along the coast. He says that the darkness was never absolute outside the great black cloud which rolled out of the volcano. Dr. DUNBAR HUGHES, Captain CALDER, and Mr. T. M. McDONALD seem to have been able to travel along the narrow, steep, winding, and in some places dangerous road to Wallibu without any very great inconvenience on account of the darkness. They were able to recognise the people whom they passed, and even to see the features of the land some short distance away. In Georgetown between 2 and 3 o'clock in the afternoon it was very dark, but still people were able to go from one house to another. The danger was rather from the bombs and large pieces of stone which were falling at intervals. During the whole night it was possible to pass through the village with a lantern or even by the light of the innumerable flashes of lightning which shot across the sky. Even in the villages in the Carib Country there is evidence to show that after the black cloud passed, and before night set in, there was a glimmering light sufficient to enable the terrified survivors to flee from one house to another. Within the houses the darkness was greater, and lamps or matches had to be used to furnish light. It is also probable that the

darkness was most intense between 2 and 3 o'clock, shortly after the great outburst, and that as the afternoon wore on the light somewhat improved.

Kingstown.—At Kingstown, and in the south end of the island, it was generally known on Wednesday forenoon that the Soufrière had resumed activity, but there were no physical signs to indicate that the eruption was in progress till about 12.30 in the afternoon. Then the loud noises proceeding from the volcano attracted attention, and about 2 p.m. it was seen that a vast cloud of steam was ascending in the northern sky, till its apparent height was twice as great as that of Mount St. Andrew, which overlooks the town.* It was somewhat pointed or tongue-shaped, but broadened out as it rose, and consisted of bulging, globular, rolling masses which were constantly changing in form and increasing in number. The noises from the mountain were continuous and very loud, with a roaring, long drawn-out character which baffles description.

About half past 2 o'clock grey pebbles of a pumiceous character began to fall, and at the same time the atmosphere was charged with sulphurous odours. Some of the stones were almost the size of a hen's egg, but they did little damage in the town except to the large-leaved trees, some of which had their foliage perforated or torn to shreds; many of them were covered with a thin layer of wet mud. Then smaller stones began to fall "like a sharp shower of hail," and these were followed by little pellets of fine moist ash "about the size of small sago." Dry ash followed, and this formed the bulk of the shower, becoming somewhat finer, lighter in colour, and less abundant as the evening wore on. It had a sulphurous smell, though some compared the odour rather to that of guano. Mr. POWELL, of the Botanic Gardens, informed us that the foliage of the more tender plants was blighted and turned yellow, and all the forward bread-fruit dropped, not from the weight, but from the nature of the ash. The plants with stout thick leaves suffered comparatively little damage, and in fact it was noticed that a few days afterwards their foliage was cleaner, greener, and more free from insects than before, so that there was reason to believe that the ash had insecticidal properties.

Lamps had to be lighted in Kingstown about 4 o'clock in the afternoon, and although the darkness was not intense before nightfall, the air was so full of falling dust that the appearances resembled those of a very thick, dark city fog. During the night the noises continued, but apparently no further showers of stones occurred. The lightnings were very vivid and frequent; the air warm, close, and stuffy. The fine dust penetrated into every corner of the houses, and covered the furniture and all the objects in the rooms with a grey film. It had also an irritating effect on the nasal passage and respiratory organs, and on the conjunctiva, especially of those persons who were obliged to be in the open air.

Through the whole of St. Vincent earthquakes and tremors were very numerous

* See despatch from Mr. CAMERON, Administrator of St. Vincent. Blue Book, 'Correspondence on the Volcanic Eruptions in St. Vincent and Martinique, in May, 1902,' p. 25.

during Wednesday afternoon and evening. At Georgetown and in the Carib Country they were so frequent that several observers stated that they gave up the attempt to keep count of them. The fugitives who were making for Kingstown by sea experienced curious sensations as if the boat were seized by something which held it and shook it vigorously. At Wallibu, Barrualli, Warrawarrou, Kingstown, and on the windward coast, all observers noticed that the houses rocked and shook with an almost continuous vibration. They describe the motion as undulatory rather than resembling the sharp shock which characterises the majority of the earthquakes which they experience in the island. Little damage was done so far as we could ascertain. In Chateaubelair one or two of the houses showed cracks in the walls, others had shingles dislodged from their roofs; but no chimneys were cast down, and even in the Carib Country and at Wallibu the demolition of the factory chimneys was ascribed rather to the lightning than to the earthquakes. It was noticed in several cases, however, that, when the first heavy rains came, the roofs were unusually leaky, as if they had been strained by the rocking motion of the ground. The noises, the earthquakes, the dust in the air, the sulphurous odours, and the darkness all combined to terrify the animals at work on the farms, so that they had to be unharnessed early in the afternoon.

The Phenomena observed in the Adjacent Islands, &c.

The Grenadines.—In the Island of Bequia, one of the Grenadines, about 8 miles south of St. Vincent, sounds as of cannonading were heard, rising to a continuous roar, about 2.15 in the afternoon. The sky was unusually cloudy, and darkness set in about 5.30. A rain of fine, impalpable ash continued all the afternoon, and, presumably, most of the night. At first it was slightly moist, and formed clots or droplets, but afterwards it was quite dry. These particulars are taken from notes supplied by the Rev. Mr. DUFFUS, the Anglican rector. He states that the “sea rose in Bequia Harbour about 2 feet 6 inches.” It is a curious fact that while sea waves were noticed here and in St. Lucia and Barbados, no one seems to have observed them in St. Vincent. If they took place there, they must certainly have been quite inconsiderable, as many people were launching boats or landing from them along the leeward coast from Chateaubelair southwards, and any sudden change in the level of the sea could hardly have escaped remark. There are many houses in these districts which stand very little above sea-level, and the incursion of a sea wave would have left traces too obvious to be overlooked.

St. Lucia.—A most interesting account of the phenomena observed by him in St. Lucia, has been sent to us by Major HODDER, R.E., who was in Castries during the eruptions. From this we extract the following :—

“About this time (the 7th May) we heard that the volcano in St. Vincent was becoming active, so we observed the sky closely in this direction. About 12.30 P.M. on the 7th, I noticed the sky becoming

yellowish in that quarter, and this increased to a darker and more coppery colour, till at 4 P.M. it began to assume the appearance of a London fog. The western (leeward) side of the smoke-cloud remained vertical and fixed exactly in one position, and it could easily be seen to ascend. A good deal of ordinary cloud passed at this time, so the eastern edge was ill-defined, but we could still see the western edge distinctly. This pillar of smoke rose to 12° above the horizon (artificial). Its breadth was about 15° of arc.

“About 6 P.M. we began to see the flashes of lightning play in the dense cloud. They were at the rate of about 3 per minute. They were not like ordinary lightning, but much shorter, and seemed to travel slower. In addition to the “lightning” there were great flashes of a redder colour (the “lightnings” appeared a yellowish-white). The flashes were at intervals of about 5 minutes. The display was much more violent than that of Martinique on the 5th instant; the smoke-cloud was at least double the height, and I concluded that the eruption had been on a vastly greater scale. No detonations were heard, but an earthquake was felt at about 2.45 P.M. by nearly everyone I have spoken to on the Morne, but I did not notice it myself. Next day, so much cloud had come up and rain that we could not see the smoke-cloud; during this time the display of lightning and flashing diminished to, say, one-fifth of its intensity.”

Further particulars supplied at a later date by Major HODDER and by Mr. GERALD DEVAUX, of the Cul de Sac Factory, St. Lucia, enable us to amplify the above account in some important respects.

Detonations were distinctly though faintly heard on the 7th May coming from St. Vincent. They were loudest about 2 P.M., and lasted from about half-past one to five o'clock, and resembled distant thunder, or the sound of the discharge of big guns. There seems to have been no considerable fall of ashes and no darkness, but only a slight haze in the atmosphere. The amount of dust which fell was so slight as to form only a film, the thickness of which was too small to be measured. The ash was dry, and was best seen on the surfaces of the leaves, where, it formed a fine impalpable powder.

Mr. DEVAUX adds, also, that “at the time the detonations were most distinct, the sea (at 3 o'clock) receded from the beach five times in half an hour, about 25 feet each time.”

Barbados.—As in the eruption of 1812, the island of Barbados, which lies almost 100 miles to windward of St. Vincent, received more of the dust emitted by the volcano than any of the neighbouring Windward Islands. In the month of May there appears to be a strong and persistent upper current of air, flowing in an east or north-easterly direction over the Caribbees, the return trade or upper anti-trade wind. The depth of the trade-wind stream at that time and place is not very accurately known: none of the mountains of the Windward Islands is sufficiently high to overtop it. It must be at least 10,000 feet deep, as the steam puffs which rose from Montagne Pelée, in ordinary circumstances, were invariably carried bodily to leeward. But on the 11th July we were in Fort de France, and saw there the tongue-shaped rolling steam cloud which accompanies the more important eruptions, and this rose to such a height that it was entirely in the upper anti-trade current, and floated away to the east or north-east. This is sufficient to show that in

the passage of so large a quantity of dust to the east, over Barbados, we have only a particularly clear example of the operation of causes which are constantly in operation in the region in question.

In Barbados there is a large and very intelligent white population. They were all acquainted with the historical records of the fall of ash in 1812, and great interest was taken in the re-appearance of this phenomenon. We have in consequence a greater mass of exact and reliable information from this island than from any other. The rate of fall of the dust, the thickness of the deposit, its total amount over the whole island, the variations in its character during the duration of the shower, its chemical and mineralogical composition, have all been carefully investigated, together with a great number of other less obvious but hardly less important particulars. We are greatly indebted to Dr. MORRIS, C.M.G., of the Imperial Department of Agriculture for the West Indies, for the kindness with which he placed at our disposal the results of observations made by himself, by the officials of his department, and by many of his friends resident in Barbados.

Mr. W. G. FREEMAN, of the Imperial Department of Agriculture of the West Indies, gives the following account of his observations :—*

“ The morning’s paper of the 7th brought the news that the Soufrière, at St. Vincent, was showing signs of activity. The morning passed without any striking phenomena, except that the weather, as on the previous day, was close and unpleasant. About 2.40 P.M., two loud reports were heard, as if heavy cannon had been fired. A man-of-war saluting with unusually heavy pieces was the first idea, but no similar reports following, the possibility of the sounds having something to do with the morning’s telegram presented itself. The more so as in the eruption of 1812 the people here heard sounds which led them to suppose a naval engagement was taking place ; they even put all in readiness to repel the expected French attack.

“ Soon a dusky cloud arose to the north-westward, not like an ordinary rain-cloud, but with curiously thin edges. This crept gradually up *against the wind*, accompanied by sounds like distant thunder, but no lightning. By 4.30 P.M. it was very gloomy, and the sky was completely overcast as far as I could see, except a band to the south-east, which was dazzlingly bright. This strip of light was at length blotted out by the advancing cloud. At 4.30 the first flash of lightning was seen, and at 5.15 dust was falling fairly fast. The dust shower increased in intensity, and at about 7 P.M. was quite heavy, the particles falling with a low hissing sound.

“ No rain fell, and there was practically no wind. During the night sufficient dampness came just to moisten the ash, but not to disturb it in any way.”

Mr. FREEMAN collected the ash which fell on three vessels of known area. This estimate of the mass of the deposit which fell in Barbados, he gives in the following statement :—

“ From the calculated results of a series of observations made in Strathclyde, on the fall of volcanic ‘ash,’ it would seem that, at a low estimate, about 13 ounces fell per square foot between the hours of 5 P.M. on Wednesday and 5 A.M. on Thursday. This perhaps may not appear a large amount ; but look at it from another point of view : 13 ounces per square foot means 117 ounces per square yard, and to express it in familiar terms in an agricultural community, no less than 16·2 *tons per acre*.

* In a letter to Professor J. W. JUDD, F.R.S., dated May 10, 1902.

“Leaving for the while minor units, such as acres, we find that 10,240 tons of volcanic ‘ash’ were rained on to every square mile of this Island during the past 12 hours of darkness. Supposing the fall to have been approximately equal in depth over the whole Island, the almost incredible amount of 1,699,840 tons of solid matter was added to Barbados last night.”

He gives the thickness of the deposit as about $\frac{1}{5}$ th of an inch, and states that while it was falling there was a slight sulphurous odour, not so strong as to be disagreeable.

In the Barbados ‘Agricultural Reporter,’ of Friday, May 9, there appears an interesting account of the rain of dust in the city of Bridgetown, and its effects on the population :—

“The volcanic dust which began to fall here on Wednesday afternoon continued throughout the night and up to early yesterday morning. Borne from St. Vincent in the upper strata of the air, and there suspended, this stuff obscured the sunlight and produced the phenomenal darkness mentioned in our yesterday’s issue. In colour and consistency it resembles Portland cement. At first the fall was slight and the substance was gritty and coarser than that which fell afterwards. By 7 o’clock, however, a powdery dust was falling thick and fast. Those whose business called them out at this hour and later—those who had no special calling out had sought their places of abode at an early hour—got their faces and hands and all their clothing covered with the stuff, and presented a grimy appearance. Care had to be exercised, too, even by those carrying umbrellas, and travelling on the tram-cars, to avoid receiving injury to their eyes and throats. The dust was everywhere, and upon everything. Passing freely through jalousies and other available openings, it rested on all furniture and other appointments in houses and offices. So early as 8 o’clock the city was deserted, and there prevailed a quietude which contrasted strongly with the activity that ordinarily prevails. Terrific peals of thunder, preceded by vivid flashes of weird lightning, helped to increase the depression of spirits caused by the gloom, and gave rise to a feeling that the worst had not yet been experienced. Judging from appearances at this time, one would have concluded that there was going to take place a heavy downpour of rain. As a matter of fact, however, none fell. The grooves of the tramway rails filled rapidly with the dusty precipitation, and it was simply impossible to keep them clear. As a consequence, the cars ran off repeatedly, and ultimately the service had to be stopped sometime before the usual hour. It should be mentioned as a noteworthy fact in connection with the phenomenon of Wednesday evening, as experienced in Bridgetown, that, while a heavy cloud overshadowed the city and the whole western sky, there was a luminous spot on the southern horizon, just at the extremity of the nebulous area, that offered a striking contrast to the prevailing gloom. There the day-god seems to have entrenched himself, and bidden defiance to the powers of darkness. The light was not bright and cheery, but rather suggestive of an angry mood. Somewhere in the vicinity of 6 o’clock P.M. of the same evening a party of Salvation Army preachers attempted to hold a service in the Upper Green, near the Nelson Statue. The usual singing and exhortation was begun by the detachment, but the falling dust was too much for mortal throat and lungs to endure in the open air, even though engaged in religious work. Besides this there were found few so foolhardy as to accept the invitation to combat the forces of nature. So the party had to desist and retire to cover like the common non-combatants. Full details of the occurrence did not come to hand from St. Vincent on Wednesday evening, and people here retired to bed feeling that the next day would bring news of some awful catastrophe. Loud booming sounds had been heard throughout Barbados during Wednesday, and the natural conclusion was that the experiences in the sister colony must have been awesome. From what was told us in a telegram received about 8 o’clock A.M., these calculations were correct, for the condition of affairs is said to have baffled description.

“Yesterday the atmosphere continued murky, and the city presented the appearance of being enveloped

in a heavy mist. The dust which began to fall on the previous afternoon had continued throughout the night, and at daybreak was perceived to lie full half an inch thick on the surface of the ground and on the roofs of houses. The city buildings generally looked dingy, and the general aspect was cheerless and depressing. Gangs of labourers from the city roads and Sanitary Department, under the direction of the Departmental Inspector, and supplied with brooms, shovels, and hand-barrows, were set to work to clean up the streets from an early hour. The dust collected by them was piled in heaps along the roadside and carted away by the scavengers' carts. To ensure as thorough cleansing as possible, hose attached to the hydrants of the water department was employed for washing the surface of the roads and for flushing gutters and drains. But dust continued to be blown from off the roofs of buildings, where it had lodged, and proved a nuisance throughout the day. Various forms of shades were used for protecting the eyes from the dust, some persons using coloured glasses and others using veils. It was a funny sight to see grave city men wearing the sun veils commonly used by ladies. But the object aimed at was the protection of the visual and nasal organs against the inrush of dust, and this the veil did effectually."

Oscillations in the level of the sea were noted in Barbados and Bequia, and St. Lucia. The 'Barbados Advocate' of May 8 contains the following:—

"There was an unusual spring tide yesterday about 3.45 P.M., the Constitution River nearly submerging the reclaimed land known as 'Fort Royal.' After the subsidence of the water, washerwomen were seen running in all directions in search of the clothes which they had spread out in the early morning to be bleached by the sun. Persons of 40 years' standing said they had never before seen such a tide."

JOHN K. KIRKHAM, Harbour Master at Barbados, sends us the following note of observations made by Mr. ASHBY, the Government diver:—

"At 3.10 P.M. the water suddenly rose 2.5 feet in two minutes, and as rapidly fell after one minute; about three minutes later it rose 1.25 feet in three minutes and as rapidly fell, then rose about 0.8 foot in two minutes, and as rapidly fell. The water rose and fell three times in 15 minutes, the rise each time being about one-half what it was before. It was spring tide here that day."

From Barbados also we obtain some interesting facts regarding the changes of barometric pressure which accompanied the outburst of the great black cloud. The Rev. N. B. WATSON, of St. Martin's Vicarage, in the parish of St. Phillips, sends us a very interesting barographic tracing, which shows repeated oscillations of the recording pen attached to the instrument between 1.30 and 6.30 P.M. of that afternoon, and Mr. O'DONNELL, of the American Weather Bureau, states that, on May 7, "a sudden jar occurred in the barograph record at the time of the explosion."

Here also we may mention that in St. Vincent similar rapid changes in barometric pressure were noticed by at least one observer, Mr. EFFINGHAM DUN, of Owia. He writes as follows:—"On May 7 the barometer (mercurial), which had been steady at 29.85, rose within three minutes to 30.12, when it again fell slowly to 29.82, and remained steady at that throughout the night. This is the only change I observed, though I paid great attention to the barometer."

Considerable air-waves appear, in fact, to have been generated by everyone of the greater eruptions, both of the Soufrière and Montagne Pelée. Rapid rise and fall of the barometric column have been noted in Martinique to accompany several of the eruptions there. The outburst at Pelée on the 9th July produced

a distinct vibration of the recording pen of the barograph in the American Weather Bureau, in Rozeau, Dominica, as was shown to us by Mr. HOBBS, who was in charge of the station. He told us also that in the observatory in St. Kitts similar effects had been traceable in the barographs on the days of several other eruptions. In every case the air-pressure was temporarily increased by the appearance of the great black cloud. We have already remarked how it was noted in Chateaubelair, St. Vincent, that during the climax of the eruption a peculiarly painful sensation was felt when loud sounds were heard, and this finds its explanation in the sudden rise of atmospheric pressure on the tympanic membranes.

Along the whole chain of the Antilles, from Trinidad to Santa Cruz and St. Thomas, detonations resembling distant cannon firing were heard on Wednesday afternoon. In Trinidad they were very distinct and their origin was at once suspected. In the late afternoon the sky to the north was very hazy and thick, and in Port of Spain there was a slight fall of fine dust.* In Dominica, Guadeloupe, Antigua, Montserrat, St. Kitts, St. Thomas, and Santa Cruz there were loud distant noises heard, especially between 2 o'clock and 3 o'clock in the afternoon. No ash seems to have fallen in any of these islands, and there were no earthquakes of importance. Tidal waves were observed in Martinique and Guadeloupe. In Demerara there were neither falls of ash nor sounds of detonations, though the latter are said to have been heard in Venezuela.† In Jamaica it was observed that from the 7th May onward the atmosphere was hazy, and at sunset and sunrise there were remarkable colour effects in the sky. Mr. S. T. SCHARSCHMIDT, of Hanbury, Jamaica, reports that a thin film of dust was observed on the iron roofing of houses at that locality. He collected it by means of a wet sponge, and found it identical in character with the ash which fell in Barbados, only very much finer. The self-registering tide gauge at Port Royal, Jamaica, showed no effects of sea waves or sudden rise or fall in the sea level.

Falls of volcanic dust from the Soufrière are recorded over a very wide area, which seems to be elliptical in shape. It is broadest from east to west, as it extends from Jamaica to at least 900 miles east of St. Vincent, a total distance of about 2,000 miles. Its north and south diameter is much shorter, as the probable limits in these directions are the north end of St. Lucia and Port of Spain in Trinidad. In both these places the amount of dust which fell was very small, and there is no reason to believe that the actual area covered was much more extensive than this. Far the greater part of the dust cloud travelled eastwards before the upper anti-trade currents, and the heaviest falls of dust were all in that quarter. Many ships passing through that part of the Atlantic which lies to the east and south-east of Barbados encountered dust clouds during the 7th, 8th, and 9th of May. The most interesting

* Despatch from Captain F. L. CAMPBELL, H.M.S. "Indefatigable," printed in Bluebook: 'Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' p. 70.

† 'Nature,' vol. 66, p. 554, 1902.

instance of this is that of the barque "Jupiter," which was 830 miles E.S.E. of Barbados on May 8th at 2.30 A.M., when the dust fell upon its decks. Many of the records of these dustfalls are unverified,* but in this case there seems to be no reason to doubt the accuracy of the report, as, although we did not see the dust, we met several people in Barbados to whom it had been shown, and they all described it as very similar to that which was lying on the houses of Bridgetown at the time, only much finer and somewhat paler in colour.

The dust which fell in Jamaica on the 11th May is very different in composition and appearance from all the other volcanic dusts which we have seen. Mr. SCHARSCHMIDT has sent us a sample of the material which he collected, and it proves to consist very largely of vegetable matter, pollen, $\delta\gamma$ -cryptogamic spores, hairs and fragments of the cuticle of plants, desmids and diatoms. In addition to these there are mineral fragments which include small grains of quartz, calcite and impalpable inorganic dust. But there are also small, clear, almost colourless, isotropic grains, which show a very irregular fracture, and have no trace of organic structure. When the dust collected is incinerated on platinum foil at as low a temperature as possible, these fragments remain unchanged. We think it probable that they are pieces of fine volcanic glass, as they closely resemble the fine glassy splinters which can be found in the Barbados dust. There is no augite, no hypersthene, and, so far as we can say, no felspar. This is exactly what might have been expected in consequence of the great distance to which the mineral fragments were carried. The heavier would necessarily sink first and only the lighter glass would reach the limits of the area of distribution. None of these glassy fragments contain any microliths; and the proportion of volcanic glass in the dust sent us is exceedingly small; the average size of the grains is .02 millim.

This dust is quite unlike that which was collected on the decks of the "Jupiter" at about the same distance from the Soufrière in an eastern direction, and shows that while Jamaica was near the limits of the dustfall to the west, it had a far greater extension to the east, so that the Soufrière does not stand in the centre of the elongated elliptical area of distribution. This is due to the persistent easterly currents in the upper atmosphere. It may well be the case that falls of dust took place over the sea to the east of St. Vincent for considerably more than 2,000 miles from the island.

Of the total amount of material ejected on the afternoon of Wednesday, May 7,

* Other authenticated records of dustfalls observed on ships which were obtained by the Harbour-master of Barbados, and published in the 'Agricultural News' (Barbados), June 7th, 1902, are as follows:—

"May 7, 8 P.M., schooner 'Violo,' from Demerara, met the dust 70 miles S. of Barbados, 10 P.M., the Norwegian steamer 'Talisman,' from Demerara, 150 miles S.S.E.

"May 8. Hour not stated, barquentine 'Fanny,' from Pernambuco, 250 miles E.

"May 9 (? 8), 4 P.M., ship 'Monrovia,' from Rio Janeiro, 240 miles S.E."

it is impossible to form even an approximate estimate. When we remember that on the island of Barbados, which has an area of only 166 square miles, and is 100 miles distant from St. Vincent, 1,700,000 tons fell, it does not seem improbable that several billions of tons of solid matter were projected into the air in that short space of time.

THE CLOSING SCENES OF THE ERUPTION OF MAY 7TH, 1902.

Thursday and the following Days.

In the north end of St. Vincent the sun rose on scenes of desolation and despair. Three-fourths of the population lay dead in their houses, and many of the survivors were fearfully burnt and suffering dreadful agony. The green and fertile Carib country, which had been covered with rich crops, lay buried beneath several feet of hot and smoking ashes. The Soufrière, which on the previous morning had been a mass of verdure and green forest, was now the leaden hue of the new fallen ash. With the return of light, though the air was still misty with falling dust, many fled and sought refuge in Georgetown and the region to the south of it. Others stayed to succour the wounded or to bury their dead. A weary procession of mangled and injured toiled along the road to Georgetown. Already before dawn some had passed through the village fleeing from Rabaka and Langley Park, and they spread the tidings of death and destruction. Efforts were early made to penetrate the burnt country and help the sufferers. A police-constable went out along the windward coast to ascertain how great the damage had been. The sights he saw were fearful, dead in every village, almost in every house, corpses everywhere along the roadside, dead cattle strewn through the fields, many of the bodies mangled, burnt, and distorted. The injured and terrified survivors, who were unable to make their way along the roads, were lying among the corpses, crying aloud for water to moisten their scorched throats; many had their skin extensively burnt and peeling off their hands and faces. Slate-coloured vapours still ascended from the mountain, every noise struck terror into the minds of the survivors. There is small room for wonder that it was difficult to get men to volunteer to explore that thirsty, burnt, and death-struck country. Yet the officials, the clergy, and the inhabitants of Georgetown nerved themselves to the task, and in a very short time the helpless injured were being gathered into Georgetown and accommodated in temporary hospitals there. The task of burying the dead had perforce to wait. Despairing of assistance, one boy of fourteen buried his father, his mother, and seven brothers and sisters in a trench he dug in the ground outside the house—a fact which throws a lurid light on the sweeping nature of the calamity which had overtaken the inhabitants. It was not till some days had elapsed that the work of burying the dead was finished, so many were they and so long did it take to clear the ashes out of the houses, huts, and yards, and bring the bodies to view. The official

estimate of the number interred is 1295.* It is certainly an under-estimate rather than an over-estimate. The story of the ready response which the British colonies in the West Indies made to the call of assistance and relief is an inspiring one, but hardly falls to be narrated here. From Trinidad, from Jamaica, Barbados, St. Lucia, in fact from all the British islands of the Caribbean Sea, help was sent. The American Government, without delay, despatched the "Dixie" with stores and medical comforts. Lists of subscriptions were opened in England and elsewhere, and money poured in with great rapidity. Nothing was omitted that could be done to save life or mitigate suffering. The efforts of the medical men were crowned with almost unhopd-for success, and comparatively few of those who were able to reach the hospitals and place themselves in the hands of the doctors died of their injuries or burns.

Chateaubelair had been vacated during the afternoon of Wednesday, and its inhabitants were scattered through the villages and houses to the south, but on Thursday morning, as the noises from the mountain had almost ceased, people began to return, as soon as day broke, to see what damage their houses and crops had suffered during the night. From 3 to 5 inches of ashes had fallen in the village, and some of the houses had had their roofs perforated by falling stones. Others had collapsed under the weight of ash that accumulated on them. No buildings, however, had been set on fire, though on the slopes to the south and east of the town in more than one place the grass had been ignited (probably by flashes of lightning). Most of trees had been stripped of their leaves, and the smaller branches had been broken by the falling stones. The air was still murky with dust, and the mountain almost completely veiled in vapours.

Early in the day, Captain CALDER, Mr. McDONALD, Dr. DUNBAR HUGHES, Mr. GENTLE, and several others arrived in a boat from Wallibu, and from notes supplied by them and by others who were residing in the vicinity of the Soufrière, it is possible for us to form a fairly accurate idea of the progress of events during that and the following days.

Columns of vapour so densely charged with fine ashy material as to be slate-grey in colour, ascended from the crater, and, as a gentle wind was blowing from the north, the dust was carried directly over Chateaubelair, which was in consequence covered with a thick mist, in which it was difficult to recognise anyone at a distance even of only a few yards. The whole mountain and the region round it were enveloped in this cloud of falling ash, but about 10 o'clock there was a sharp shower of rain which

* Bluebook : "Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902," p. 65. Despatch from Governor Sir R. B. LLEWELLYN (dated May 23, 1902).

The total number of bodies found dead is	1295
The deaths in hospital from injuries (burns)	70
Missing	200
	1565

somewhat cleared the atmosphere and dissipated the sulphurous fumes. Lightning and thunder accompanied the rain.

About 2.30 P.M. there was a considerable increase of activity, and as the afternoon was fairly clear, it was possible to see that great clouds of smoke were being discharged from the site of the old crater. Captain CALDER notes that at this time "molten lava was coursing down each deep ravine, clouds of white vapour marking its path over the damp earth."* This is the first record of steam eruptions in the valleys after rains due to the action of the water of the streams on the hot ashes through which they flowed. All the afternoon there was thunder and lightning, with possibly also noises from the mountain, and dense clouds, partly of rain, partly of steam mixed with ashes, floated to leeward, obscuring the coast line and the slopes which face Chateaubelair. During the evening there was more rain and thunder, with lightning, which was very vivid, and continued at intervals through the whole night.

On Friday, the 9th May, there were loud noises and rumblings in the early morning, and about 7 o'clock vast masses of dark smoke were ascending from the crater, and what was supposed to be a stream of boiling mud or lava was seen rushing downwards through the valley of the Wallibu to the sea. After this it became very dark in Chateaubelair, the heat was intense, and breathing was difficult. In Georgetown the sound of the eruption was also heard, and there was a shower of small stones, followed by fine ashes. From Kingstown a great column of smoke was seen to shoot upwards in the air above the volcano. The rumbling sound was heard for half an hour or less. Flashes of lightning were visible in the clouds, but not generally over the sky. There was a slight fall of fine dust, and the air was foggy with suspended matter. At Fancy, on the north shore of the island, stones and sand continued to fall for nearly two hours, and during that time there was darkness (though not total).

Although the descriptions of what was seen from Chateaubelair are not very lucid or satisfactory, it is certain that there was an outburst on Friday morning, which may have been accompanied by a manifestation of the black cloud phenomenon on a small scale. The amount of matter discharged was quite inconsiderable compared with that of the great eruption of the 7th, and as the danger-zone was almost completely vacated, no loss of life occurred.

During the whole afternoon of Friday slaty-coloured vapours were emitted by the crater, and fine dust was falling on the leeward parts of the mountain. Showers of rain are recorded as having taken place several times during the day, and there was a good deal of thunder and lightning, with occasional noises from the volcano. From the ravines on the side towards Chateaubelair steam and dark vapours were often seen to ascend. Mr. McDONALD remarks that from some of the valleys discharges of vapour took place, each accompanied by a flash of lightning and a peal of thunder. He observed that from the ravines clouds of steam were rising, and these led to the belief that fissures were formed in the valleys, or streams of lava were flowing down

* 'Century Magazine,' vol. lxiv, August, 1902, p. 637.

them. As a matter of fact, there can be little doubt that the rain occasioned by the showers was working through the hot sand in the old channels of the streams, and that this was the origin of the clouds of vapour.

On Saturday, the 10th May, it was apparent that the energy of the eruption was spent and a state of quiescence was at hand. For some time after daybreak the crater was almost free from discharges, but about half-past 9 o'clock the steam clouds began to arise again, and continued with intermissions during the remainder of the day. It was obvious to all spectators that the worst was over and that the eruption was drawing to a close.

On the 11th, steam still continued to ascend from the crater, and the mountain was veiled in smoke. On the 12th and 13th, at irregular intervals, sluggish discharges of slaty vapours took place, accompanied by low rumbling noises. The column of steam did not rise more than a few hundred feet above the summit. On the 14th the cloud over the crater was still dense but less lofty. There was a slight rain of small pebbles at Richmond Vale, near Chateaubelair, while steam was seen arising from the crater in the evening.

On the 15th, at 9.30 A.M., there was a slight escape of steam, otherwise the mountain remained clear all day.

By this time on the windward side of the island considerable progress had been made with the work of interment. The wounded had been relieved and removed to hospital in Georgetown and Kingstown. Most of the population of the districts which lay to the north of the mountain, and which had not suffered so severely as the region to the south, had been drawn away and had settled temporarily in Georgetown, Kingstown, and other parts of the island. At Owia, Sandy Bay, and Fancy there were still green fields to be seen, and though many of the inhabitants fled, some remained. Elsewhere everything was covered under a sheet of ashes, very fine on the surface but mixed with lapilli and coarser blocks in the layers beneath. It resembled a desert covered with grey or brownish sand, except for the numerous blasted, broken, leafless trees which rose through the covering of ash. The rains had not yet been sufficient to clear out the old stream channels, and they were filled to the level of their banks with an accumulation of sand. The steam arising from the valleys on the hillsides after rain showed that every shower sent down freshets which, working in the hot ashes, were converted into steam clouds and wholly evaporated before they could reach the sea. At Fancy and in the district around Owia the valleys contained comparatively little deposit, and after a few showers they resumed very much of their old appearance. But immensely greater masses of material had lodged in the river courses on the south side of the hill, and it was seven or eight days before any water was able to flow along their whole length to reach the sea. The Rabaka Dry River was dry for several days after the great eruption, and when it began to flow intermittently after very heavy showers the water came down perfectly black and boiling hot.

From Chateaubelair several parties made a voyage up along the coast in boats, and some landed to examine the burnt-out plantations of Wallibu and Richmond. What chiefly attracted attention was the alteration in the configuration of the surface and in the outlines of the coast. The villages of Morne Ronde and Wallibu had disappeared, and the sea had encroached on the land for a width of about 200 yards at the mouth of the Wallibu stream and for a distance of nearly a mile along the coast to the north of this. The buildings of Richmond Plantation and of Wallibu were surrounded by ashes several feet deep. Richmond had been on fire, and all the woodwork of the houses and the furniture was destroyed, but at Wallibu the barrels in the store, the doors, carts, and furniture were still preserved, though covered with black sand. Some consider that the sand-blast was not hot enough when it passed over Richmond to set fire to combustible substances, but that the house was struck by lightning or that a paraffin lamp, which was left burning in one of the rooms, exploded and ignited the furniture. Mr. McDONALD notes that Richmond Village which stood below the plantation house, was seen to be "covered with 30 or 40 feet of ashes, more or less," and that "the general level of all the flat land as far as Frazer's was raised by 40 or 50 feet, and terminated in abrupt almost vertical bluffs at the sea."

At this time the Wallibu stream was perfectly dry and choked with sand, which filled it up almost level across. It was not till seven or eight days had elapsed after the eruption that the water was again seen to make its way to the sea. Frequent discharges of steam were observed in the upper parts of its course. The other valleys which lie to the north of this, the Wallibu Dry River, Rozeau Dry River, and Larikai, were similarly encumbered with deposits of sand, though perhaps not to an equally great extent, and in these also steam explosions took place after rains.

Very interesting and valuable evidence regarding the conditions prevailing at Wallibu and Richmond about this time is afforded by certain photographs, taken by Mr. WILSON, of Kingstown, on the 14th May.* One of them shows the houses of Richmond disroofed and burnt out, surrounded by several feet of ash and mud, out of which rise the leafless, blasted stems of trees. The wooden framework on which the plantation bell was supported is preserved unburnt. This makes it likely that the fire which attacked the houses was the result of an accident and not a direct consequence of the heat of the volcanic blast. On the slopes across the stream which flows by the south of Richmond, the layer of ashes had been thin, and the rain-showers acting on the naked unprotected surface of the mud have cut little furrows and runnels in it. In the foreground of the picture the sheet of ash around the plantation-houses had a smooth, slightly hummocky or rolling character, exactly resembling the effect produced by a considerable fall of snow, or the surface of fine

* These photographs are reproduced in Mr. E. O. HOVEY's paper, "Martinique and St. Vincent: a Preliminary Report upon the Eruptions of 1902," 'Bull. Amer. Museum Nat. Hist.,' vol. 16, Article 26, Plates 41 and 38.

blown sand strewn by the wind. The rains had up to that time been sufficient to erode only on the steeper slopes, while on the more level ground they had sunk into the porous, incoherent ash, and the original surface characters were still intact.

Another photograph, taken at the same time,* shows Wallibu plantation and the fields to the north of it, with the Soufrière in the background, emitting a column of black smoke. The wind-strewn character of the surface is here also visible on the flat grounds, while on the steep ridge which rises behind Wallibu the layer of ashes has been much thinner, and is already furrowed with rain-rills, and, to a considerable extent, has been washed away by the running water.

According to the descriptions given us by those who had occasion to visit the devastated country at this time, it was a shadowless wilderness of sand and blasted vegetation, in which not a drop of water could be found to drink. The heat of the tropical sun was reflected from the bare surface of the sand, making the air intolerably hot, and every breath of wind stirred the fine dust which formed the superficial layer of the deposit, and blew it into eyes, nostrils, and throat. The rains had not yet been sufficient to make the mud coherent or to wash away the finer particles, and the ash lay on the level cane fields with an undulating surface resembling that of blown snow. Rain was urgently needed to restore the blighted trees and enable them to put out fresh leaves.

In Georgetown and Chateaubelair confidence was rapidly restored, and, owing to the influx of the refugees, the villages were crowded with people. Rations were now being served out daily to the destitute, and settlements erected for their accommodation, so that there was a bustle of activity, and the streets were thronged. The burial of the dead was over, and the living had had time to count their losses and were congratulating themselves on their escape. A week had elapsed since the great eruption; there had been no further destruction, though the devastated country was deserted. A few of the inhabitants had been able to return to attend to the interment of their relatives and friends, and to remove the most valued of their personal belongings. As a rule, however, a great aversion to visit the scenes of suffering and death was manifested by the refugees, but they readily adapted themselves to their new surroundings, and when the wounded had recovered from their burns and injuries, they settled down without any great reluctance in the quarters provided for them.

THE ERUPTION OF THE 18TH MAY, 1902.

The volcano sank into a state of quiescence. After the 15th May no further loud noises were heard, and the emissions of steam were on a very small scale, and took place without violence. The ordinary occupations of life were resumed, and the mountain was no longer observed, hour after hour, with an interest quickened by

* See footnote, p. 417.

apprehension. But a rude awakening was at hand. On the evening of Sunday, May 18th, a second eruption took place, less violent and far less destructive than the former one, but still sufficiently vigorous to throw the whole population into a state of terror, and make those near the mountain flee for their lives.

The afternoon of Sunday was beautifully calm and clear, and the inhabitants of Chateaubelair could see the mountain from base to summit. Many were in boats along the leeward coast examining the strangely-altered surface of the plantations of Wallibu and Richmond, and the startling modifications which had taken place in the coast line to the north of the village. Not a single cloud veiled the face of the mountain, and the bare, burnt surface showed up in every detail in the evening light. Sunset was followed by a clear, starry night with bright moonlight and a cloudless sky. In Chateaubelair, Kingstown, and Georgetown many people were enjoying a walk in the cool refreshing night air, when suddenly, about 8.30 P.M., a loud, prolonged, ominous groan burst from the mountain. Some residents in Kingstown compared it to the noise made when a war-ship lets go her anchor in the bay and the cable rattles out through the hawse holes. At the same moment a great cloud of steam shot from the crater and rose to an immense height in the air. As seen from Kingstown, it was pointed, and its height was estimated at many thousand feet. In this great mass of vapour, lightnings incessantly scintillated. They were tortuous and snaky, and did not resemble the clear flashes often seen on a tropical night. The noises in Chateaubelair were deafening; many thought they were as loud as on the afternoon of the 7th May. Soon the village was enveloped in total darkness. The inhabitants sprang out of their houses, and, seizing their children, fled along the road that leads southwards past Petit Bordel. Darkness settled down on the fugitives, a darkness so intense that they stumbled over the roads, groping their way along the ditches, guiding themselves by feeling the objects on the way side. The lightnings, the thick darkness, the roaring noises, and the falling ash, dismayed the stoutest hearts, and many were weeping and lamenting loudly as they hurried through the night. Some lost their children in the gloom, others fell into ditches or over the banks on to the sea-shore. When they reached Rosebank the light was beginning slowly to improve, and many took refuge with friends there. About 10 o'clock the sky cleared, the moon appeared again, the noises ceased, and the eruption was at an end. The earthquakes noticed during this eruption were few and insignificant, and the fall of ash in Chateaubelair was very slight, and consisted mostly of very fine dust or sand.

In Georgetown also the day had been exceedingly fine, and no warning was given of the coming outburst. When the steam-cloud rose with loud noises from the mountain a general exodus took place, but later in the night, when the darkness lifted, many returned. The ash which fell was in the form of fine dust, and amounted only to a very thin film, most readily seen on roofs and leaves and the stone pavements around the houses.

In Kingstown the consternation was intense. Some rushed about the streets in terror. Others fell on their knees and prayed aloud, but many shut themselves in their houses, dreading an incursion of the great black cloud. A small quantity of very fine dust fell in the town, but there was no deep darkness.

It was noticed in Chateaubelair and Georgetown that many of the children complained of a painful feeling in their ears.* This phenomenon was already described in connection with the first eruption.

Bishop SWABY, of Barbados, was in Kingstown that night, and gives some interesting particulars regarding the eruption as observed from that place. He was at supper when he heard†—

“An explosion like that of a bursting shell, followed by a roaring groaning sound which startled the party, and they ran out into the garden and looked in the direction of the Soufrière. They saw an immense column of white vapour ascending high in the air above the mountain, twisting and twirling upon itself till it assumed the shape of a flower, and forked lightning was playing around it incessantly. The curious thing about the lightning was that it embroidered, as it were, the edge of the cloud of vapour, playing around it with a continuous scintillation as of fireworks. The thunder, too, was continuous, but the thing that struck his lordship was the rattling groan of the labouring mountain. Otherwise a deep hush pervaded the region, and served to make the scene more impressive to the beholder. The phenomenon lasted about 15 minutes, and there was an interval till about 10 o'clock, when it came again. The mountain was about 15 miles from Kingstown, where it was viewed, and the watery vapour must have been about 30,000 or 40,000 feet high to have been seen above the mountain as it was.”

It is impossible to say whether or not a black dust cloud descended from the Soufrière down the valleys of the Wallibu and Rabaka on this occasion, as no one was residing in the houses there, and in the darkness it was impossible for those in Georgetown and Chateaubelair to see what was going on. On the whole, it seems practically certain that there was a recurrence of this phenomenon, though on a scale so small that the deposits which it produced could not be compared with those which filled the ravines on the 7th May. Next morning it was seen that a thin layer of freshly deposited ash covered the surface of the mountain. Apparently there was a repetition of what happened in the first eruption, though there could be no comparison between the magnitude of the two outbursts. The premonitory symptoms were entirely wanting in this case, and the preliminary stages so brief and inconspicuous as to have escaped notice. The loud explosions and sudden rush of steam may be regarded as accompanying the outburst of the dust avalanche and black cloud, the subsequent noises as the effect of upward steam explosions projecting dust, scoria, and bombs into the air. Many people told us that in their opinion the noises were as loud as on the 7th May, but certainly they were not heard so far, and the disastrous calamity which had previously overtaken the island had left behind a state of nervous apprehension which led to exaggerated estimates of the magnitude of the subsequent mani-

* ‘Times,’ Kingstown, Thursday, May 22, 1902.

† ‘Barbados Advocate,’ Saturday, May 24, 1902.

festations. There is no evidence that air-waves or sea-waves of any importance accompanied this outburst.

Estimates of the height to which the steam cloud ascended, as seen from Kingstown, are founded on the angular distance of its apex above the horizon as compared with that of the mountains of known altitude behind the town. But these are fallacious, as the steam cloud which forms part of the avalanche of dust shoots obliquely upwards into the air when the dust subsides, and the steam column was not vertical above the crater, but had been projected southwards, so that as it travelled onward it gradually spread over the town, carrying with it the fine dust which fell there during the evening.

From this time forward there is no satisfactory record of any further eruptions from the Soufrière till the end of August, 1902. It is true that in the local and Colonial papers paragraphs may be found describing violent steam discharges from the crater and rains of ash on the surrounding country, but as the result of careful inquiry on the spot we consider that these are erroneous. It was almost a fortnight after this before any attempt was made to ascend the mountain, or even to examine its lower slopes systematically. Rumbling noises were occasionally heard, but these were partly due to landslips and falls of rock from the crater walls. Several trustworthy observers report that they saw steam gently arising from the crater, but most of the reports are based on nothing more than the appearance of the round-topped masses of cloud which drift across the mountain before the trade-wind,* while others are to be referred to outbursts of steam in the valleys, owing to the action of water in the streams on the hot dust filling the ravines.

In this account of the sequence of events in the eruptions of May, 1902, we have relied principally on the evidence collected by ourselves from the statements of eye-witnesses, and on written accounts of the eruption given us by residents in St. Vincent, at the request of His Excellency the Governor of the Windward Islands. In most cases we were able personally to question these witnesses, and to amplify, and in some cases to correct, their statements in this way. It was a month after the first eruptions before we landed in St. Vincent, and the sifting of the evidence proved to be no easy matter. It is astonishing how widely divergent, even in essential points, may be the narratives of two equally competent observers who were in the same room or in the same boat at the time. We are well aware that there are in this Report not a few statements which would be unhesitatingly contradicted by more than one person who was in a good position to form an accurate opinion as regards the actual facts. Under the circumstances we have been guided in all cases by the balance of good evidence for or against any conclusion, but it is vain to hope that we have escaped errors and mis-statements. We have made little use of the newspaper

* For an instance of these untrustworthy reports and some pregnant remarks on the valueless character of much of the newspaper evidence, see Professor JAGGÂR, 'Popular Science Monthly,' August, 1902, p. 353.

reports, which are in many cases highly sensational and grotesquely inaccurate, but a limited number of personal narratives have been published over the signature of persons well known in the islands, which have been of the greatest value to us. Of these we may mention the diary of Mr. T. M. McDONALD;* the notes by Captain CALDER,† by the Rev. Mr. DARRELL,‡ by Bishop SWABY, of Barbados;§ and the account given by Captain FREEMAN, of the "Roddam,"|| and by the first officer of the "Roraima," Mr. ELLERY S. SCOTT,¶ of the eruption which destroyed St. Pierre, in Martinique.

THE RAINS.

For nearly two weeks the weather had been on the whole remarkably dry, and although showers had fallen on several days, they had been light and local. That steam was seen to ascend on certain occasions from the valleys and ravines on the Soufrière proves that there had been a certain rainfall, but it was probably greater on the higher slopes than on the lower grounds, and all over the island the vegetation was parched and dusty, and the fine ash blowing to and fro caused great inconvenience. A general and considerable fall of rain was urgently needed, and it was not long in coming. On the afternoon following the second eruption, a sharp shower fell on the north end of the island, and the Soufrière smoked all over its surface as the water came in contact with the hot sand. Four days later rain fell in torrents. The record of the rain-gauge in the Botanic Gardens at Kingstown is as follows:—

9 A.M. on 23rd to 9 A.M. on 24th	2.43 inches
„ 24th „ 25th	5.16 „
„ 25th „ 26th	0.82 „
23rd	26th
	8.41 inches.

A total of nearly $8\frac{1}{2}$ inches in three days.

In the south end of St. Vincent these heavy rains did nothing but good. The parched and dust-laden vegetation was cleaned, refreshed and invigorated. The more tender plants had lost their buds and young foliage, while the stout leaves of many of the trees had been perforated or torn, or, in some cases, even stripped from the branches by the rain of stones and scoria. The woods and fields rapidly reassumed their green aspect, and the beneficial effect of the rains was great and immediate.

But in the north end of the island, where the fields lay buried beneath a layer of

* 'Sentry,' Kingstown, May 16, 1902. 'Century Magazine,' August, 1902.

† 'Century Magazine,' August, 1902.

‡ Printed in Kingstown, and sold for the benefit of the Relief Fund. Dated May 12, 1902.

§ 'Barbados Advocate,' Saturday, May 24, 1902.

|| 'Pearson's Magazine,' September, 1902.

¶ 'The Cosmopolitan,' July, 1902.

sand and ashes, the results were almost disastrous. The loose materials on the surface were not protected by any covering of vegetation or held together by the roots of growing plants, and every raindrop carried its burden of sand and mud to the rills, which carved innumerable furrows in the fields. The streams were swollen to torrents, so rapidly did the excessive downpour of rain run off the surface of the ground in the absence of growing crops and plants, which in ordinary circumstances would have restrained and checked the violence of the discharge. So much mud was carried into the sea, that for days its waters were turbid and discoloured for several hundred yards from the shore. In many places the ash was washed almost completely off the steeper slopes, even where it had been a foot or more in thickness. Mud avalanches took place on a small scale wherever the gradient was sufficiently steep to allow the moist semi-fluid material to move by its own weight. Behind the plain on which Georgetown stands, there is a stretch of rising ground which overlooks the fields of the Grand Sable estate in bluffs a couple of hundred feet high or less. Currents of mud flowed down upon the arrowroot fields which lie below, and covered them with fans of débris. They even swept across the public road into that part of the village which is known as Browne's Town, overturning several small houses and burying their occupants in the ruin of their huts. In this way two people were killed.

Hitherto the fields of ash which covered the Soufrière had had their original surface characters well preserved, and in their smoothed and wind-strewn aspect had greatly resembled deposits of blown snow. Occasional showers had been sufficient to furrow the steeper slopes, but on the level ground feather-like rain-rills had not yet been developed. But from this time onward that marvellous rain-sculpture of grooves and furrows converging to a central axial stem, which was as astonishing as it was instructive to the geologists who have since visited the volcano, is to be found in all the photographs which have been taken of the scenes of the eruption.

The scientific investigation of the history and consequences of the recent activity of the Soufrière may be said to date from the period of the rains. The United States steamship "Dixie" entered Kingstown harbour on the 23rd, bringing with it a number of newspaper correspondents and a party of American scientific men, including Professor ISRAEL RUSSELL, Professor JAGGAR, Mr. E. O. HOVEY, Mr. CURTIS, and Mr. BORCHGREVINK. They proceeded at once to collect information, and they and Mr. WILSON, of Kingstown, succeeded in obtaining many excellent photographs of the devastated country. These all show the great amount of erosion which had been effected by the tropical deluges of the 25th May. Only the photographs taken by Mr. WILSON some days before the rains indicate the original aspect of the surface. We can rely also on the descriptions given us by those who had been engaged in the work of searching for the wounded or burying the dead, or had visited the Wallibu Valley during the period of quiescence before the second outburst. At Pelée, in Martinique, owing to the repeated discharges of dust, and also

to the absence of any torrential downpours in the months of May and June, there was less erosion, and it was not very difficult to understand what had been the appearance of the fields of ash and sand immediately after the great eruption.

For seven or eight days after the 7th May, the Wallibu river had been choked with hot sand, and no water was seen to reach the sea at its mouth. Thereafter the flow was resumed, and erosion of the sand accumulations began, but for some days was inconsiderable. The first photographs taken after the rains, however, show that a deep and narrow gorge had now been cut in the new ash deposits, and on each side of the channel were terraces marking pauses in the progress of erosion. During the days of downpour there must have been magnificent explosions of steam all along this valley as the river ploughed its way through hot banks of ashes, but we did not obtain any descriptions of these from eye-witnesses. Probably the whole mountain was enveloped in clouds of steam, as in all the valleys the same process was going on, and moreover the water sinking into the sand (which was still burning hot a few inches beneath the surface) must have generated enormous masses of vapour.

The lower part of the Rabaka Dry River, which alone had at this time been examined, was, on the 11th and 12th, dry, but not encumbered with any great accumulation of ash. Some days elapsed before water was seen to flow past Rabaka. It was hot and on more than one occasion it came down in floods of boiling mud. These were, no doubt, due to the uneven surface of the sand deposits farther up the valley forming temporary lakes by blocking the course of the stream, and as the barriers were cut down or swept away before the pressure of the water, these lakes suddenly discharged their contents. From the time of the rains onward this river has been seen smoking all along its upper part, though it is only for brief periods after heavy showers that the water reaches the sea, and it is then always black, turbid, and very hot.

On the north side of the Soufrière the thickness of the ash deposits had been comparatively slight, and the streams were able to reassert themselves, and to resume their flow after a few days. The water was unfit to drink, being tainted with sulphuretted hydrogen and mineral matters in solution, and there was a deficiency of rain-water, so that the steamer "Wear" was sent there with supplies. The inhabitants did not entirely desert the district till after the 18th May, when the last were removed by steamer to Kingstown. It is worth recording that at Baleine a spring issues from the rocks, and this was found by Lieutenant ROBINSON* to be unaltered, and its waters perfectly good and fresh. The heavy rains did no great damage on the north shore of the island, but as the crops and trees were in large part unburnt at Owia and Sandy Bay, they were stimulated by the moisture and put out fresh green leaves.

On the slopes above Sandy Bay and Overland Village the standing timber had been broken, overturned, and stripped of its leaves and branches, and a deposit of

* Blue Book 'On the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' p. 89.

ash had accumulated perhaps a foot or two feet in thickness. The streams here have a short and rapid descent to the sea and flow in deep narrow valleys with steep sides. The heavy rains washed away nearly all the loose sand and the smaller stones, and with them much of the underlying soil. This material was hurried into the sea, which here is comparatively shallow for some distance from the shore. The streams on reaching sea level dropped their load of sand and pebbles, and it gathered in banks along the beach. The strong shore current generated by the steady trade-wind surf which beats against this coast was unable to remove the deposits as rapidly as they accumulated, and in consequence the sea margin receded to some distance from the old shore marks. From Overland Village north to Robin Rock, a distance of nearly a mile, there was a broad beach of black volcanic sand where formerly the waves had washed the base of a lofty cliff. We were assured that there had been no change in the relative levels of land and sea. When we visited this spot it was evident that the action of the waves was gradually reducing the deposit and cutting back the sand banks till they faced the sea in some places in little cliffs some 4 or 6 feet high. It is probable that a thickness of 15 feet or more of sand was laid down along the sea shore in this way. At the wharf, at Rabaka, where the sugar was formerly loaded into boats, it is said that the water is now shallower by about 12 feet. Nevertheless the old rum store stands at the same altitude above the sea level as it did formerly. The breadth of the sand beach was at most about 200 feet, but all the way from Sandy Bay to Colonarie it was evident that much black volcanic sand had recently gathered on the beaches, and that the sea-waves were only slowly distributing it along the shore.

THE GEOLOGICAL EFFECTS OF THE ERUPTIONS.

The South Part of St. Vincent in June, 1902.

When we arrived in St. Vincent, on the 10th June, and landed at Kingstown, it was not without an effort that we could realise that so recently a great eruption had darkened the air with falling dust and scoria. The south end of the island was a scene of tropical beauty—the rugged peaks and narrow valleys being clad in a dense mantle of green vegetation. In Barbados a thin layer of dark mud had been visible on the usually white surface of the limestone roads, the remains of “the dust”; but in Kingstown it required a rather careful search to find traces of the recent deposit. The depth of the fall of ashes had been small—not over half an inch, and the rains had already washed it almost entirely into the sea. As we journeyed along the leeward coast it was not till we reached Chateaubelair that any striking evidence of the volcanic activity was visible. To the south of that point a few torn leaves on the trees, a thin layer of scoriaceous pebbles on the fields, or a roof damaged by a falling stone, were all that we saw to testify to the recent rain of ashes.

In Chateaubelair the depth of the deposit, as measured on the roofs of the houses, a day or two after the eruption, varied from $2\frac{1}{2}$ to about 4 inches at the south end of the village. It increased rapidly as it was traced northwards, and may have been nearly a foot at the north side of Richmond Vale. Some of the stones which fell there were a foot across. At Petit Bordel the largest stones weighed about 10 pounds. At Barrualli and Layu the thickness of the layer of ash was from $\frac{1}{4}$ to $\frac{1}{2}$ inch, and the largest stones were two or three inches in diameter. In Kingstown the deposit was less than half an inch, and pumiceous stones about the size of a hen's egg were found in the streets.

On the windward side of the island, as we drove to Georgetown, ash deposits were first noticed in the fields about Colonarie, where they had been 2 or 3 inches thick, and after passing Black Point we found abundant remains of the ejecta of the volcano everywhere—on roadsides, arrowroot fields, and on the steeper ground inland. Much of the finer material had been washed away by the rains, what was left was mostly the coarser sand, pumiceous scoria, and fragments of the old andesitic rocks of the mountain, at most 3 or 4 inches across. Nearer Georgetown the sheet of ash was thicker, and in the streets and gardens of the houses in the village the old soil was covered to a depth of 1 to 2 feet. The average original thickness of the deposit may have been 18 inches. As the ground was flat, not much had been washed away except the very finest dust from the surface; the ash had been trodden down and was heaped up in places where it had fallen from the roofs or had been thrown out of the houses. Stones over a foot in diameter had fallen in Georgetown. The Anglican Minister, the Rev. Mr. BELL, showed us where one had penetrated the roof of his study. In the churches and houses nearly all the windows were broken, there was hardly a whole pane of glass in the village. The damage had been greatest in the windows facing the sea and not in those looking to north, south, or to west. The materials had evidently been projected to a great height in the air, and, falling through the steady current of the trade-wind, had acquired a westward velocity. They had fallen in a slanting direction, for where a window was protected by an overhanging verandah, the panes of glass in the upper part had often escaped destruction. But the windows looking towards the volcano had also suffered, though in a less degree. Not many large stones were to be seen in the fields, but we were told that they fell for the most part on the afternoon of Wednesday, and were covered over by the subsequent rain of small scoria and fine dust. The larger blocks fell in occasional showers, but it was not noticed that these followed any particularly loud detonations from the volcano. The bananas and plantains, the mangoes, the bread-fruit, the cabbage and cocoanut palms had had their leaves torn off by falling stones. Many were still standing leafless, others were putting out new growth. The crops in the gardens were buried, but in the fields behind the town the rains had washed away more of the material and the arrowroot was reappearing in patches. In the south end of the town and around Grand Sable House torrents of thick brownish mud had flowed over the fields and

roads from the slopes behind. From these last the ash was mostly washed away and deep furrows had been cut in the naked soil.

A few houses were set on fire, perhaps by lightning or possibly by hot falling stones. Others had collapsed under the weight of the ash which gathered on the roofs, while several had been knocked down by streams of mud on the day of the heavy rains.

The difference between the thickness of the ash deposit and the amount of damage done in Georgetown and at Chateaubelair is so striking as to deserve discussion. Though Georgetown suffered most, it is quite a mile more distant from the crater. There is no reason to believe that the black cloud of dust passed over either of these places; the effects observed are entirely due to the rain of ashes during the afternoon and night of the 7th May. Whatever may be the reason, it is clear that the crater projected a larger amount of material to eastward than to westward. This is in accordance with observations made by Mr. McDONALD from Chateaubelair during the forenoon and up till 2 o'clock on Wednesday. He saw many showers of stones "principally to windward." Part of the dust, however, that fell in Georgetown may be due to a slow lateral spreading of the margins of the black cloud after its first velocity and heat had diminished. Chateaubelair is protected by several ridges and spurs which lie between it and the crater, but to the north of Georgetown spreads the level Carib Country. The strength of the trade-wind appears to have been unable to direct most of the material to leeward: it must have been hurled out of the throat of the volcano in an eastward direction against the wind.

All the country north of Black Point (south of Georgetown) on the windward, and from the north side of Chateaubelair on the leeward coast, and of a line passing between these two points and over the summit of Morne Garu, showed on its surface a covering of ashes more or less deep. Within this area there was another less extensive region in which the damage to vegetation had been severe, and everything presented a burnt and scorched appearance. This is the true "devastated country," and its southern border starts about half a mile north of Georgetown, and running along the ridges to the summit of Morne Garu, descends by the spurs on the leeward side to the south side of the mouth of the Richmond River, a mile north of Chateaubelair. It does not include quite the whole of St. Vincent north of this line: as at Owia and Sandy Bay, in the north-eastern corner of the island, there was comparatively little damage done to the vegetation (though the ash gathered to a depth of several inches). The boundary must be represented by a line drawn so as to exclude this quarter. (See map, Plate 39.) In the country included within this line not only was vegetation greatly injured or destroyed, but many of the inhabitants and most of the cattle in the fields were killed. Outside this line no one died directly from injuries received from the volcano. Consequently it marks the limit of the "danger zone."

The Deposits in the Wallibu Valley.

In that great depression which lies on the south side of the Soufrière, between that mountain and the Morne Garu, the ejecta of the recent eruption have accumulated to a far greater depth than elsewhere in the devastated country, and there, also, was the most complete destruction of vegetation, and the greatest loss of life. The western side of the valley is drained principally by the Wallibu River, the eastern by the Rabaka Dry River, and the geological phenomena exhibited along the course of these streams and their tributaries are most interesting and important.

The western valley, that of the Wallibu, is, as already described, unlike the common type of Antillean valleys, in being broad, open, and flat-bottomed, especially in the lower parts. The higher affluents which drain the southern flanks of the Soufrière have that steep-sided character, with drooping taluses of ash alternating with vertical cliffs of lava, which mark the mountain ravines of St. Vincent. But in the lower parts of its course the Wallibu River has a comparatively gentle gradient, and along the coast at its mouth there is a considerable expanse of fairly level ground. This is separated from another similar valley, that of the Wallibu Dry River, on the north, by a flat-topped ridge which rises behind the plantation works of Wallibu. To the north of this, behind the Carib village of Morne Ronde, the mountain rises rapidly and steeply, while on the southern side the rugged mass of Richmond Peak overlooks the plain beneath. To the south, the Wallibu River is separated by a sharp-topped spur from the valley of the Richmond River, which also, in its lower part, has only a slight fall and a comparatively slow current.

The Richmond Valley is carved in an old series of lavas and ash beds which seem to have proceeded from the Morne Garu volcano. They dip gently westwards to the sea. But the valleys of the Wallibu, and Wallibu Dry River have been cut out of that series of ash beds and coarse agglomerates (see Plate 30, fig. 2), well-bedded and without alternations of lava, which constitute the southern part of the Soufrière near its base, and are the record of the most recent phases of its volcanic activity. So soft is this material that on successive visits we could see that the streams flowing in it had deepened their channels by several feet in a couple of days. This may account for these rivers having reached an approximate base level before the others in the island.

From the mouth of the Richmond River to that of the Rozeau Dry River along the coast, and thence up the valleys of the Wallibu and Wallibu Dry Rivers, the surface is covered with a mass of recent ash deposit of irregular and varying thickness (see Plate 25, fig. 2). It is thinnest on the back of the ridge on which stands the dwelling-house of Wallibu—where it attains the depth of four or five feet. Here it was deeply furrowed with rain-rills, which had a pinnate arrangement (see Plate 26). The slopes on each side of this ridge are very steep and on them little ash rested; the rains had already washed the greater part down upon

the level ground beneath. The trees where not uprooted and cast down were beginning to put out fresh green leaves. The wet and sticky black mud hung in festoons on the slope, separated by sharp narrow little rivulets cut by the water (see Plate 28, fig. 1).

Along the sea cliffs and on the banks of the streams excellent sections were exposed, both of the layer of new hot ash, and of the older tuffs on which it rested (see Plate 25, fig. 1). The deposits of the recent eruption varied from 5 feet to about 40 feet in depth, as seen in these sections. They formed irregular hog-backed rounded mounds, the long axes of which were roughly parallel to the direction of the valley, resembling in many ways the drumlins of boulder clay which one sees on the low grounds of Scotland (see Plate 25, fig. 1). These rounded mounds of sand bore no very close relationship in their disposition to the pre-existing features of the topography; the thickest deposits were not in every case in the old stream valleys, but the sand seemed to have been irregularly heaped and not spread out into a sheet with level upper surface.

It was not difficult to see that originally the deposits had had a smooth and rolling character, though the rains had scored them deeply with a converging pattern of stream channels. We saw no evidence, however, of wind-rippled surfaces, or of dunes; it looked rather as if a vast mass had been dumped down suddenly in great irregular mounds, and then the rain of ashes and of stones had smoothed the surface over.

In the upper part of the valley of the Wallibu Dry River the rolling surface of the fields of ash, with rain sculpture everywhere, was very like that seen along the shore, but it was only in the main stream channels that the underlying tuffs were visible, and the thickness of the recent accumulation could be ascertained. It was mostly from 5 to 12 feet deep, and here also it presented little uniformity in this respect.

But the channel of the Wallibu River was on the whole more narrow and steep-sided than that of the Wallibu Dry River. It was from 400 to 600 yards wide and from 200 to 400 feet deep. Here the deposit was of greater thickness and had suffered more severe erosion. Its original depth could not be made out with certainty, but at the time we saw it, it was from 60 to 80 feet thick in some places (see Plate 29, fig. 2).

The river flowing through the masses of sand which had been heaped up in its channel had cut a deep narrow gorge, sometimes with perpendicular walls. More frequently on one or both sides of the stream there was a series of terraces which varied in breadth from a few feet to 20 or 30 yards. As many as six of these could sometimes be seen one above the other (see Plate 29, fig. 1). They had all the marks of river terraces cut by water, the highest were always farthest from the stream, and their surfaces had a gentle slope down the valley. Where sections of these terraces were exposed it was evident that they were eroded out of the thick deposit of new

hot ash. They were bounded towards the river by sharp slopes, from 1 foot to 10 or 12 feet high, by which the surface descended from one terrace to another, and little landslips frequently took place as the loose material dried after rains and tumbled down the bank upon the terrace beneath (see Plate 29, fig. 2). They were not, as a rule, equally numerous and well-developed on both sides of the stream; commonly on one side there was a vertical cliff, while on the other several terraces might be seen. Traced down the stream few of them extended for more than 70 to 100 yards, and they were best seen during the last mile of the river's course, where it runs on the south side of the flat-topped ridge behind Wallibu estate.

The origin of these terraces is not difficult to explain. Immediately after the eruption the valley was more or less completely filled with fine hot ash, mixed with a certain proportion of coarse blocks and bombs. The river when it resumed its flow found it a matter of no great difficulty to wash out or re-arrange this material. But the rains in St. Vincent are intermittent, and take the form of heavy showers, during which the streams come down in flood and frequently are raging torrents. At other times the quantity of water in their channels is very small, though this stream was never quite dry in the month of June when we saw it. A small body of water, black as ink and thick with mud, was always flowing to the sea, steaming strongly and very hot. It came in gushes which rushed down with a wave in front of them (see Plate 23, fig. 2), several inches or a foot in height, and between these the current was very slight and, in fact, almost ceased. It was not hot water but black boiling mud, and the stronger rushes carried with them pebbles several inches in diameter. On one occasion, when we were returning from an ascent of the mountain, we found one of the smaller streams, which we had crossed dryshod in the morning, so swollen by heavy showers that it was flowing in a thick black current 2 or 3 feet deep and several yards across. It was very hot, and we had to cut down some of the bare tree trunks to form a temporary bridge. Not long after we had crossed, this was carried away by the current.

This thick muddy flow constantly bears much material to the sea. The banks of ash on each side frequently slip down, especially when the hot sand dries after rain, and when it is undermined by the stream, and little landslides take place which temporarily dam back the current till the pressure of water increases and sweeps away the obstruction. Hence the intermittent flow and gushes of hot mud. A process of gentle erosion constantly goes on in which the river, unable to deepen its channel rapidly, keeps widening it by undermining the banks on each side and carrying away the material precipitated into the stream.

But after rains the magnitude and velocity of the current are greatly increased, and its cutting power so enhanced that it trenches deep grooves in the ash filling its old channel. According to our observations the increase of depth may amount to several feet in an hour, so soft is the recent ash and so rapid and powerful are the torrents. When the shower is over the flood subsides almost as rapidly as it rose, and the

widening of the gorge by the action of the gently flowing stream of thick mud is resumed, though now at a somewhat lower level. A new terrace is in this way formed inside of the old one which has been partly destroyed.

In the upper part of the Wallibu Valley, at a distance of about a mile from its mouth, it widens out to form a comparatively broad expanse, where the principal tributaries join to form the main stream above Petit Wallibu. Here was a wide plain of new volcanic ash (see Plate 28, fig. 2), which at the time of our visit had as yet been comparatively little eroded, and preserved in some measure its original surface configuration. The streams which cross this flat had each cut its gorge in the deposit, but between these there still lay wide stretches of the recent sand, which had hardly been attacked by erosive agents. The surface, though on the whole level, was rolling and irregular, and bore several pools and one lake of greenish turbid water. As seen from a distance of less than a mile this lake was perhaps 200 yards long, and steamed strongly, the water being apparently quite hot. It was impossible to get near to it, as the gorges cut in the hot smoking ash could not be crossed in safety. The swelling, hummocky, gently-rounded features which had originally characterised the ash-deposits, had in this area not yet been masked by the effects of erosion as in the stream valley lower down.

The ash which had gathered in the valleys of the Wallibu and Wallibu Dry Rivers was a fine sand, nearly black when wet, but, when dry and hot, brownish-grey or buff in colour. In the month of June the coating of fine grey dust which had at first formed the surface layer had been washed away, and the mass of the material was about as coarse as an ordinary sea sand or a little coarser. In it lay many lapilli, scoriaceous, yellowish, with white shining crystals of plagioclase and dark pyroxenes. Large rounded bombs were less frequent, but many could be found varying from a few inches up to 3 feet in diameter. They were rounded or flattened, with rough tuberculate or nodular outer surfaces, and when broken up were black in the interior, vitreous and vesicular, with scattered crystals of pyroxene, plagioclase, and olivine. Ejected blocks of the weathered andesites and reddened decomposed andesitic tuffs, which constitute the walls of the crater, were quite as numerous as the fresh spongy vitreous bombs, and entirely distinct from them in character and appearance. About half a mile north-west of Wallibu works were some ejected blocks, roughly cubical and measuring 5 feet by 4 feet by 4 feet. As they lay upon a layer of the new ash they could not have fallen from the cliffs behind, from which also they were several hundred yards distant. In the same place were bombs over 2 feet across.

Many of these ejected blocks when struck with the hammer broke with great ease, and large, flat pieces commonly flaked off their surfaces. They had been intensely heated before being projected from the crater, and rapidly cooled in their passage through the air, and it seemed as if concentric cracks or planes of weakness had been produced by the contraction of the chilled surface on the hot interior, recalling in

many ways the perlitic structure shown by certain igneous glasses, and the spheroidal structure of weathered dolerites.

Another type of rock fairly well represented among the blocks imbedded in the ash in the Wallibu was a granular, holo-crystalline aggregate of felspar, brown hornblende, and olivine in very varying proportions. They appeared to be agglomerations of the first minerals to crystallise out of the igneous magma within the crater, but a minute description of these, and a discussion of their origin, may be deferred till they have been more completely investigated.

The most striking feature of the Wallibu deposits was the scarcity of coarse material. Certainly over 90 per cent. of the whole could only be described as a volcanic sand, and often the fragments above a couple of inches in diameter did not form over 3 per cent. of the mass. These statements are based not on the appearance of the surface, where the coarser and heavier materials had been concentrated by the action of rain in washing away the finer stuff, but on the vertical sections afforded by the cliffs, where the ash could be inspected in the condition in which it fell. There were not a few places where we found it impossible to collect any number of stones large enough to yield good hand specimens. Where the larger blocks and bombs did occur they were often numerous, forming fields of stones, and in each area they seemed to be, as a rule, of the same kind, as if the materials had fallen together like the rain of sparks from a rocket. In one place we might find a cluster of bombs, in another of ejected blocks, and so on, and each different part of the area yielded distinct types of rock.

The vertical sections of the new ash did not show any very distinct stratification, nor were the larger blocks more abundant near the lower surface. Most of the material was as little stratified as a recent blown sand. At the base lay the charred crops of sugar cane, not burnt up or destroyed, but rotting where exposed to the atmosphere under the attack of saprophytic fungi, which gave out a stench resembling that of guano.

The ash itself was very hot. Probably it was not quite red-hot, though as we were never on it by night, we cannot say how it would have looked in the dark. But if a walking-stick were thrust several inches into it and withdrawn after a minute or two, the brass ferrule was too hot to handle. The surface when wet was cool, firm, and good to walk upon, exactly resembling a fine cinder path. After showers, however, it soon dried, and the sides and bottoms of the little rain-rills became sufficiently hot to burn the naked feet of our coolie porters. From cracks in the ash, through which water had percolated to the interior, steam constantly ascended in little jets, which deposited thin yellow and greenish incrustations. One day, as we were being rowed to the landing-place at the mouth of the Wallibu River, a landslip took place in the dry, yellowish ash on the top of the low sea-cliff. The stump of a tree, which had been entombed in the ash, was in this way exposed. It was charred but covered with a film of grey dust, and as soon as the air touched it, the wood burst

into flame and sent up curling wreaths of blue smoke. After a few hours without rain, the surface of the fields of ash ceased to steam, and the haze in the atmosphere gradually cleared. Then all along the valley the blue smoke of burning wood rose from the banks of ashes beside the river.

Where the deposit was thick the interior was certainly not much below a dull red heat, even at the time when we were there. A month after the eruption, it still gave out a dry, stuffy smell, recalling hot lime freshly raked from the kiln.

It may be remarked that in the ash there was much charred wood, mostly the remains of the trunks and the stouter branches of trees. They were mere remnants which had lost every vestige of their original form, and all the bark and the smaller branches had vanished. With the erosion of the deposits in the Wallibu, much of this burnt wood was swept down to the sea and floated about. The villagers diligently gathered it in their boats and stored it up to be used for the fires on which they cook their food.

Although the volcanic sand was very hot, it was a bad conductor. The surface had been cooled by the rains, and only after some hours' drought did it even feel hot to the hand except in the rain runnels. But the water did not penetrate far, and at a depth of a foot the ash was quite dry. The sugar cane, which was exposed in the bottom of the rain rills in the cane fields around Wallibu Works, had probably been covered by the hot ashes for several days, yet it was not destroyed, only charred on the surface. The wet earth beneath had cooled the under layer of the deposit.

The intense heat of the ash explains the extraordinary steam explosions which took place along the streams after heavy rains. Then, from the River Wallibu, immense clouds ascended which might quite fairly be compared to those emitted by Vulcano or Stromboli during an eruption. Often, when the mornings were wet, we watched them from the windows of Sea View Cottage, near Chateaubelair, rising in great masses over the ridge which lies to the north-east of the village (see Plate 23, fig. 1). They drifted to leeward before the trade-wind, strewing the waves of the sea with dust. Great globular, turgescient pillars of steam would shoot up in a few seconds to heights of about 2000 feet, their surfaces covered with rounded swelling excrescences, expanding and multiplying as they rose, and when their upward velocity was spent, they floated slowly away before the wind. They were exactly like the cauliflower clouds which used to rise from the fissure on the southern side of Pelée. Often at their base they were grey, but, as they ascended, their margins and upper surfaces would change to brilliant white as the sun illuminated them. Half an hour after the rain was over they ceased to appear, or were very much diminished in size and number.

We had more than one opportunity of seeing their origin on the Wallibu River, though after heavy rains it was not possible to approach very near the banks. But even when the water was low, steam puffs would go up at intervals. They invariably occurred where the stream was washing out the base of a cliff on the outside of its

bends. The loose sand, undermined in this way, would tumble down, and at the contact of the hot ash with the water a great column of steam would shoot into the air. Very little noise attended these outbursts; as a rule, only a low, faint rumble. In dry weather the river of steaming mud flowed quietly along its channel, effecting little erosion, and few explosions were to be seen. But when it came down in flood, the fine ash was ploughed out like snow, and enormous outbursts followed one another in rapid succession.

It is not desirable to be caught in a tropical downpour of rain on the Soufrière, as the streams rise so rapidly that they cannot be crossed, the more so as they are filled with boiling mud. But on one occasion, when we were descending from the crater, a short but heavy shower overtook us. The Rozeau Dry River lies to the west of the trail. It had been dry when we were going up; as we came down it was filled with a thundering inky torrent, 30 feet broad, roaring along at 20 miles an hour. The new ash had already been very thoroughly cleaned out of the upper portion of the valley, but down below there were still remains of the hot sand deposit, and as the torrent spread down the stream, it carved into the banks of ash, which seemed to melt away before its attack. As the water touched the hot sand, pillars of black mud capped with a great club-shaped cloud of steam towered into the air. It was a marvellous and beautiful spectacle, the column of water curving outward at its apex, and dropping down like a fountain, while over it played the delicate feathery steam cloud, the height of which we estimated at 700 to 800 feet. It might well be compared to a geyser of boiling mud, and the contrast between the black base and the pearly apex heightened its beauty.

The Carib Country and the Valley of the Rabaka Dry River.

On the windward side of the island, in the Carib Country, and the ravine of the Rabaka Dry River, the phenomena already described in the valley of the Wallibu and Wallibu Dry River, on the leeward side, are repeated in all their essential features, though with a few not unimportant differences. The topographical conditions are here considerably simpler than on the western side, as in the Carib Country we have a broad triangular plain, the base of which rests on the sea shore, and extends from south of Georgetown to Overland Village, while the apex is placed some distance above Lot 14 estate (see Plate 2I, fig. 1). It slopes gently to the sea coast, and though its surface shows traces of old eroded terraces, masked with an accumulation of loose surface deposits, it is crossed by no prominent ridges, but forms a smooth expanse, which is entirely under cultivation, and before the eruption was considered the best agricultural district in the whole island. Across this sloping plain several streams flow in shallow valleys, the most important being the Rabaka Dry River.

The conditions already described as prevailing at Georgetown are typical of the

whole Carib Country. It is buried under a sheet of ashes, the depth of which to the south of that village is only a few inches, but increases as it is traced northward to Rabaka, Waterloo, and Orange Hill, where it is often 3 feet. At Overland the deposit again thins out to less than 1 foot, and from thence to Victoria Village must probably have been at first less than this, though the powerful action of the rains on the steep hill slopes in this quarter had removed the greater part of it before our arrival. Traced inland, this sheet of sand and scoria gradually thickens, and above Lot 14, on the flat ground known as the Mahoe, it was probably in many places 5 feet deep. Over the whole of this country the rains had caused immense erosion, and everywhere the surface was sculptured with furrows and runnels (see Plate 21, fig 2). Where these united, they had formed considerable streams, which had cut downwards into the soft red earth which underlay the new ash, and sometimes in the midst of a level cultivated field a gully 20 feet wide and 12 feet deep would testify to the rapid and intense erosion which these transient torrents had effected.

The original surface configuration of the fields of ash had been greatly modified before the middle of June, when we were there, but it was clear that they had at first been characterised by a smoothed and strewn appearance resembling that of a fall of snow. They were covered with rain-rills, the arrangement of which depended to a considerable extent on the slope of the ground, for where that was slight, the tributary furrows united to form a main channel at comparatively high angles, and were often tortuous; but where the slope was high the rills were straight and nearly parallel. For yards they would run side by side, separated only by a narrow ridge, a foot or more in height, and they united to form larger channels only where the gradient diminished (see Plate 28, fig. 1).

The finer dust had disappeared from the surface, and only the coarser sand, lapilli, bombs, and ejected blocks remained. On the whole there seemed to be a greater abundance of coarse material and large stones than in the Wallibu Valley. This may have been a consequence of the greater rainfall on the windward side of the mountain, which had removed the finer ingredients more completely, and it must also be borne in mind that two rainy weeks had elapsed since we had explored the valley of the Wallibu. Still, an impression was left on our minds that bombs, ejected blocks and large lapilli were commoner here than on the leeward side.* This did not, however, efface that distinctive character of the ejecta of the Soufrière already noted, for still the fine material greatly preponderated, and the new ash might best be described as a sand and dust deposit. Bombs, 2 or 3 feet in diameter, were not uncommon, and the different kinds of rock found at Wallibu were all present here also.

In the Carib Country there were none of those round backed ridges of sand—30 or 40 feet deep in their centres—which were seen on the Wallibu Dry River. The uniformity in depth of the sheets of ashes and their smoothed upper surfaces

* Mr. E. O. HOVEY is of the same opinion: "Martinique and St. Vincent: a Preliminary Report upon the Eruptions of 1902" 'Bull. Amer. Museum Nat. Hist.,' vol. 16, p. 339, 1902.

were far more noticeable here than on the leeward country. There was nothing to show that the material had been dumped down suddenly in irregular heaps by an avalanche; it was more easy to suppose that it had fallen in a steady rain from above and the regular thinning away in all directions outwards from the crater greatly favoured this supposition.

During the days of heavy rain true mud lavas, on a small scale, had flowed down some of the streams. They were thick, pasty currents of black volcanic sand mixed with water, and sometimes had not been sufficiently fluid to reach the sea, as the gradients in the lower parts of the streams are low. At Waterloo a flow of this mud could still be seen to occupy parts of the stream courses. It was 3 feet thick, and consisted mostly of the fine material from the surface of the fields of ash.

In the estates along the shore of the Carib Country the damage done to trees and buildings was by no means so great as might have been expected. The chimney of the factory at Turema and the wall of an old store at Rabaka were knocked down, probably by lightning. The iron roofs of some stores and verandahs and many of the ill-built, trash-roofed, wooden huts of the labourers had collapsed under the weight of the pile of ashes which accumulated on them. Some of the thatched huts had taken fire, it may be from flashes of lightning, or possibly from hot falling stones, for we were told that some stones broke when they struck the ground, and their interior was red hot. The trees had had their branches broken and their leaves stripped by the rain of scoria, but were mostly still living and renewing their foliage. Some, however, had been struck by lightning and their trunks had been split asunder. We were told that many had had their branches weighted down to the ground or broken off the stems by the ashes which gathered on them in the afternoon and night of the 7th of May. But in the managers' buildings, though everything was very dirty from the layer of ashes which had covered all walls and furniture, little had been destroyed. The glass in the windows was often broken and the roofs sometimes perforated by falling stones, but the furniture, the stores in the cellars, the machinery, the pictures, had not been destroyed and were apparently very little damaged.

At Lot 14 the damage had been greater, the village of labourers' huts was ruined, partly by fire, partly by the weight of ashes that gathered on the roofs. But the sugar works and the manager's house were not much more injured. In the house the windows were broken, and a verandah had collapsed, but nothing was burnt, and in the cellar below, in which the manager and his family had taken refuge, everything seemed to be in its normal condition.

The stream known as the Rabaka Dry River heads in a series of tributaries which flow in deep ravines down the slopes on the northern side of the Morne Garu and the south side of the Soufrière. These unite to form a main trunk which descends to the Carib Country, at first through a valley which, though not so precipitous on its sides as is usually the case in the mountain gorges of St. Vincent, is still a deep and well-marked trough, but about half a mile above its mouth the river

enters a shallow open broad channel, which is partly filled with terraced masses of ash. This is known as "the lava bed," and the accumulation is said to be the product of the eruption of 1812. Before that time the river is believed to have flowed more or less continuously, but since then it often dries up for weeks together, and its flow is considerable only after rains. That at Wallibu so much more water should be found than in the larger stream at Rabaka is certainly a curious fact, and one for which it is not possible to offer an explanation without making a thorough study of the geology and physical features of its drainage area—an investigation which could not be undertaken in the present condition of affairs.

In the upper part of the Rabaka Dry River enormous accumulations of hot sand have been piled up exactly like those already described in the Wallibu Valley (see Plate 32). We were told by Mr. SPENCE, who is well acquainted with the district, and visited it along with us, that in some places this deposit could not be less than 200 feet thick. Certainly it had almost completely obliterated the old valley of the river at more than one point, and where formerly there had been a deep trough, there was now a rolling plain of black sand, through which, after rains, a sluggish stream of mud wound its way, but on a dry day the steaming channel was streamless. Here erosion had produced much smaller effects than on the leeward side, and in many places the ash had been practically unattacked, and the original nature of the surface could be well seen (see Plate 34, fig. 1). It was flattened on the whole, but rolling, hummocky, and uneven, so that pools of water had gathered in the hollows, and in these little deposits of alluvium had formed from the fine materials washed into them by the rain. These pools were now mostly dry, and the dark surface of the ash was freely steaming. After a couple of hours of dry weather, spots of grey would appear where the heat had been sufficient to dry the upper layers. Then our porters could with difficulty cross it with bare feet; when wet it was quite cool to walk upon. The deeper parts of this deposit must have been at a very high temperature. Where the hot, grey ash was exposed by landslips in the banks of the stream, it had a stifling stuffy smell. A good deal of charred wood was mingled with the sand, and there were many spongy, vitreous bombs, but angular broken pieces of the old rocks of the hill made up far the greater portion of the coarse material in the mass. As a whole, however, it was an accumulation of sand and dust; bombs and ejected blocks were more numerous than on the Wallibu, but formed only a small percentage of the mass.

In the main valley of the Rabaka Dry River vast quantities of this material had been heaped, but, strange to say, the tributary valleys contained far less of it, and the great pile of ash in the main channel often formed dams across their mouths, behind which lakes of water and of liquid mud had formed (see Plate 34, figs. 1 and 2). On several occasions great and sudden floods of mud had rushed down the stream, and it was clear that they might have originated by the water eroding these dams till they were so weakened that they gave way before the pressure of

the lake of mud behind them, and a large part of the contents of one of the lateral valleys was suddenly discharged into the river. In some of the ravines these lakes of thick, black mud were quite half a mile long, and if heavy rains should cause rapid erosion by the main stream, flows of mud-lava may yet take place on a large scale.* The hot sand in the Rabaka Valley does not readily form mud. It is too hot for the rain to penetrate far below the surface, and the water evaporates rapidly. The mud lakes had been filled with the cold material from the slopes above, which had been washed into the depressions by the heavy rainfall.

The terraces and other features due to steam erosion exhibited on the Wallibu were repeated in all their essential features in the valley of the Rabaka Dry River, only here the process was far less advanced, owing to the smaller body of water which characterised this stream. The new cut gorge was not over 20 or 30 feet deep, the terraces on the whole less numerous and closer together (see Plate 32). Two or three of them could often be seen, 3 or 4 feet apart, and as the valley was broader than that of the Wallibu, the terraces were better preserved and more extensive. But much of the deposit retained its original rolling surface, and only a part of the sheet of sand in the valley had been remodelled by running water. The faces of the terraces and the banks overlooking the stream often slipped and the hot grey ash was then exposed. It showed only a rudimentary and very imperfect stratification, and at a depth of a foot or so the sand was grey and quite dry, while the upper part was moist and of a darker colour.

Only after rains did water occupy the stream channel; usually it was empty, and the moist, stony mud on the bottom continuously steamed. But after a heavy shower great steam explosions rose from the whole upper course of the river, and probably they were no less violent than those on the Wallibu (see Plate 31, fig. 1). As the banks were lower, the landslides of hot ash were smaller, and great rolling sudden steam-jets were not so frequent as on the leeward side. Sometimes the river flowed hot, black, and thick with mud, right down to the sea, and on one occasion we had to ride our horses through the surf, as a current of mud, several inches deep, was flowing in the river, and was too hot for our horses to cross it. This was only after severe rains, as in most cases the water was completely evaporated in passing through the hot sand in the upper part of the channel.

In more than one place in the valley of the Rabaka Dry River we observed large, circular, flat-bottomed depressions, some of which were perhaps 20 yards in diameter, while others were not more than 8 or 10 feet (see Plate 33, fig. 2). At most they were 8 feet deep, and often they occupied the apex of a low flat cone, the diminutive size of which, as compared with the broad, flat central basin, reminded one of the lunar craters. Very commonly the surface of the ash fields around such a crater was covered with scattered blocks in considerable numbers. They were most frequent

* As this proof is being corrected for the press, we learn that such mud-flows have actually taken place. 'Sentry' newspaper, Kingstown, November 28, 1902.

around the larger pits, and from their arrangement it was impossible to avoid the conclusion that they had been emitted from the funnel around which they lay. The sharp cut walls of ash surrounding these depressions showed that they had been blown out of the deposits of ashes by a large steam explosion. Usually they were situated on the course of the stream (we observed none at any distance from the main river or its tributaries), and in one place there were three of them in contact with one another, planted on the river banks. They were incomplete, less than half of the whole circle remaining in each case, as the side next the river had been cut away as the stream deepened its gorge.

We did not see any of these pits in operation, but the method of their origin is probably as follows:—water in some way gets access to the undermost and hottest layers of the ash, probably by the river cutting suddenly into its banks at their base and undermining them. A large part of the bank then at once subsides into the stream, and when the hot ash touches the water large masses of steam are immediately generated, and the explosion lifts the super-incumbent mass and carries it upwards into the air. In some cases many tons of material must have been thus projected into space. The finest dust floated up to great heights, and was wafted away by the wind; many of the stones must have been shot up obliquely and have fallen on the fields of ash which surrounded the focus of eruption. But a very large part of the material subsides again into the funnel-shaped cavity through which the steam explosion ascended. It is, however, not sufficient to fill it up, and the basin-shaped depression is a measure of the amount of material which was borne away by the wind or projected beyond the lip of the funnel. The low, flat cone around the crater is formed by the sand and stones which fell just outside the edge of the cavity. There was no evidence that more than one explosion had taken place from each funnel, but in more than one place a group of pits was seen which suggested that the same process was several times repeated at approximately the same spot.

It is not likely that they were occasioned by water penetrating through cracks on the surface and reaching the hot mass below, for cracks were few and small except where landslides were taking place on the stream banks, and in no case were fissures seen in connection with the pits. Possibly, however, after heavy rains, springs rising in the bottom of the valley may introduce water into the basal layers of the deposit, and in this way steam explosions would be produced with exactly similar effects.

In the valley of the Wallibu Dry River we found a low, flat cone some 30 yards across at its base, and about 10 feet high, with a large flat central crater on its summit, perhaps 15 feet across, circular, some 3 feet deep, with a layer of mud in its interior, and around the lip of this crater were three others, smaller but almost equally perfect. The rains had somewhat destroyed the structure, a little lake had formed in the main crater, and had drained out through a notch on one side. It stood about 15 yards from a deep narrow gully occupied by a small stream, and not far from the base of a high bluff, so that the introduction of water below the surface might

have been due either to the undermining action of the stream, or to the uprise of springs from below. On the whole the latter seemed the more probable explanation. Similar cones, but less perfect, were seen elsewhere in the same valley (see Plate 31, fig. 1, and Plate 28, fig. 2).

The Slopes of the Mountain.

The condition of the southern flanks of the Soufrière presents in most respects a very complete contrast to that of the Wallibu and Rabaka Valleys which lie beneath them, for while in these much deposition of new material has taken place, and erosion, though rapid, has not yet been able to bring the former surface to light, on the higher grounds the accumulation has been slight, and erosion is proceeding at a very rapid rate. On the sharp knife-edged spurs and deep ravines of the south side of the mountain the washing action of the rains has had great effect. Everywhere the surface of the new ash is furrowed with rain-rills, and where the slopes are steep the wet mud has often slipped bodily into the valley bottoms. The deposit must have been several feet deep on the lower part of the hill slopes, but exactly how deep it was can no longer be determined, as only a remnant of the original mass remains, quite insufficient to enable a judgment to be formed regarding its original thickness and disposition.

The sides of the valleys incline at angles averaging 40° (which appears to be about the angle of repose of the taluses of weathered ash), except where the edges of the few lava beds visible in this quarter form vertical cliffs. On the gentler slopes the ash still hangs in festooned masses, the discontinuous remains of a sheet once spread over the whole surface. The rain has cut down to the old soil and laid bare the scorched vegetation, which, as its roots, being surrounded with damp earth, were often not destroyed, is slowly recovering and sending up scattered leaves of fresh green among the dreary wastes of mud.

The rain sculpture furnished a most interesting subject of study, for it presented a very great variety of forms which depended principally on the original slope and configuration of the surface on which it rested. Where the ground was steep the rills were many, narrow, and straight, with small furrows joining them on each side in a manner reminding one of the pinnules of a feather. They ran side by side down the slopes, converging only slowly to unite to form a main channel of a higher order. The intervening ridges were 2 or 3 feet high, and perhaps twice as broad at their base, their summits gently rounded, their sides fluted with minor furrows. Often one of these ridges had collapsed or slid down, and its remains been washed away; and this somewhat diversified the usual regularity of the features. As erosion advanced the furrows broadened, as the loose ash was more easily removed than the old rock beneath the soil, and the ridges between wasted away, getting thinner and thinner.

On the more level ground the sculpture recalled rather the tributaries of a river, tortuous, irregular, converging and frequently uniting to form a large main trunk. A

study of the photographs will give a better idea of the features of these rills than pages of description.

In many places the sides of the valleys were absolutely bare, and not only had the new ash been removed, but with it much of the old soil had also disappeared, as might be seen from the manner in which the roots of the broken trees projected above the surface, and in the lakes of water previously mentioned as forming in the lateral valleys of the Rabaka Dry River, a red mud from the eroded soil was mixed with the black mud furnished by the new volcanic ash. Often we could see areas of many hundred square yards absolutely divested of the covering of black mud, and sometimes the whole side of a valley showed it only in one or two patches (see Plate 34, fig. 2).

Fine sand with little lapilli had formed the greater part of the material which lay upon the slopes. Stones of any size were very few, for when they fell they could not rest unless they buried themselves in the hot sand, but rolled down into the ravines. This was, in fact, one of the dangers of the ascent, as the rains had loosened many stones in the old agglomerates, and when once they were set in motion nothing could stop them till they landed in the stream hundreds of feet below. The ash, as a rule, held together very well when it was wet, but every now and then the pressure of the foot started small landslips, which tumbled down the slopes.

In the deep narrow valleys which score this side of the mountain little of the new ash was left (see Plate 30, fig. 2). At the higher elevations a patch here and there, on the extremities of the bends or behind a projecting bank, was all that could be seen. Further down, flows of the black liquid mud had congealed in the channels, unable to travel further. Once a sharp shower, brief but heavy, overtook us as we were descending. Below us was a place, "the river bed," where the path crossed a tributary of the Rabaka, dry except after rains. As we did not wish to be cut off by a raging torrent, we hurried down to get over the crossing. The stream was pouring in cascades over the rocks in the upper part of its course, but when we reached the "river bed" there was no water there. Looking up the stream we saw a creeping mass of stiff black mud slowly winding its way down the channel like a serpent. The water was so loaded with sand and mud that it almost ceased to flow. Had the rain continued, of course an inky torrent would have rushed down into the lake of mud in the lower end of the valley. Further down, where the ravines entered the plains of the Carib Country, and on the flat ground near Wallibu, some part of the masses of ash originally piled up there still remained (see Plate 31, fig. 2).

We have mentioned the explosions of hot ash and water in the lower part of the Rozeau Dry River. The remains of the deposit were terraced and deeply cut into, a mere shadow of their former selves. So great had been the erosion along the whole course of these steep valleys, that it would be rash to say to what extent they had been encumbered with hot sand immediately after the great eruption. In more than one place there were still 40 feet of ashes in the bottom of the gorges (see Plate 30, fig. 2).

Possibly the original thickness had been twice or thrice as great. The terracing of these deposits was often very fine, and in other places subsidences and landslides had taken place, and the surface of the ash was fissured, broken, and highly irregular. Great stones, 12 feet across and more, lay in this ash, whether ejected during this eruption or fallen from the old agglomerates in the cliffs, we could not say.

On the knife-edges of the spurs the new mass still lay, firm, coherent, forming an excellent pathway, so smooth that one might have thought it had been prepared expressly for the foot (see Plate 35). When the sun shone after heavy rain, the smooth, narrow, winding strips of fine wet ash on these ridges reflected the light, and they seemed like lines of silver traced on the dark-grey background of the hill. On them the rain had little power of erosion, as they formed, in fact, the only tracts of level ground on the whole mountain. The fine dust was washed away, and the coarser sand exposed by the beat of the raindrops. They were, as a rule, from 2 to 5 feet wide; laterally, as the slope increased, little rivulets formed, at first so small that they had almost no cutting power, and on each side of the smooth central line there was a strip fluted with little furrows. This passed in turn into the sculptured rill-marked surfaces of the sides of the valleys. The division line between these surfaces was wonderfully sharp, for as each channel deepened it pushed its head upwards, and the steep upper parts of the erosion curves produced little rapid descents, by which one type of surface graded into another.

At heights above 2500 feet the gradient of the path became somewhat steeper, and as the surface here was covered with 5 feet or more of fine ash, which when wet became a pasty mud, it was more toilsome to climb this slope than any of the lower parts of the hill. The foot sank deeply in the soft, wet ash, which was furrowed with shallow rain grooves, some three or four feet deep, the edges of which slipped away when we endeavoured to cross them. The deep radial valleys which trench the hill sides gradually pass into shallow, wide depressions when traced up to this level, and the oft-quoted comparison of an eroded volcanic cone to be a partially opened umbrella, was very applicable to the Soufrière. The greater steepness of the upper slopes showed also that the mountain possesses the typical profile of ash cones, though this was by no means very evident in a distant view owing to the great erosion, which produced an appearance of rugged irregularity.

This ash on the summit of the cone was mostly a fine black powder, smelling strongly of sulphuretted hydrogen, but in it there was a very large number of ejected blocks and bombs, many of them on the surface, but apparently still more were embedded in its mass, as if the last material to gather on the slopes had contained proportionately fewer of the coarse ingredients. The ejected blocks were often four or five feet in diameter, and the spongy andesitic bombs not seldom two or three feet. Many of these last seemed to have been hot and plastic when they fell and to have stuck together, or, at any rate, the later had moulded themselves on the surfaces of the earlier. Though the fragments of the old lavas and ash beds which had been

torn from the walls of the crater showed in their brittle condition the effects of having been intensely heated and suddenly chilled, they were never fused on their surfaces, and presented no resemblance to the true bombs. Occasionally pieces of much charred wood entombed in the ash were to be seen in the sides of the rain-rills. On the windward side the proportion of large fragments in the ash deposit at this level was especially high, and where much of the finer dust had been washed away, great accumulations of stones remained. That the lower slopes were so bare, while near the summit fine ash remained in considerable sheets, was due to the deeper erosion features at lower levels. The sides of the ravines showed angles of about 40° , the winding knife-edges of the spurs had an average inclination of less than 10° , the summit cone sloped at about 25° to 30° . The amount of erosion of the new ashes was in direct proportion to the steepness of the slope in each case. It must also be remembered that originally the depth of the deposit was greatest immediately around the crater; in some of the shallow valleys it was at least 12 feet, and may have been 20 feet in places. On the valley sides the large blocks and bombs could not rest; they rolled down the steep slopes till they rested in the bottom of the ravines; but on the edge of the crater they had accumulated, partly because more fell there and partly also because the surface slanted at angles sufficiently low to enable them to lodge.

The Crater.

In the month of June it is seldom that the Soufrière is not capped with cloud, and one may consider himself fortunate if, after climbing the hill, he obtains a clear view of the crater. Our first ascent—from the leeward side—was made on a very favourable day, and the first of the party to reach the edge of the crater could see the bottom for a few minutes. Then the mist closed in, and only once again did it lift sufficiently to enable us to discern the lakes in the floor of the depression. Heavy rain followed soon after, and we had to beat a hurried retreat. From the windward side we twice essayed the ascent, and were both times baffled by the mist. Once we persevered and reached the edge of the cliff, but it was impossible to see for more than 30 yards in any direction, and the interior of the crater was a sea of vapour. For 10 days we waited in Georgetown, making daily excursions, but ready to start for the summit at any time when the conditions might seem propitious; but we never had a clear view of the upper part of the hill, and we were informed by the inhabitants that the month of June of this year (1902) had been more than usually unfavourable for our purpose.

The first party to ascend the hill was led by Mr. T. M. McDONALD, of Chateaubelair, and included Professor JAGGAR, Mr. E. O. HOVEY, and Mr. CURTIS. They had a very fine day (31st May), and were able to secure photographs of the north wall of the crater. Professor JAGGAR was so good as to give us full descriptions of what he saw. Mr. McDONALD also conducted our first ascent, and placed all his information

at our disposal. Lieutenant ROBINSON, R.E., who was accompanied by the Rev. Mr. HUCKERBY, climbed the hill on the 3rd June, and from him we obtained a valuable report* as to the state of matters on that date. On the 5th June, Mr. BIDDICK made an ascent. We followed on the 12th June, and since then several parties have reached the edge of the crater. We are indebted to Mr. ARTHUR DARRELL, of Kingstown, for notes made on the appearance of the old and new craters on the 27th June; and in the 'Sentry' (of Kingstown), dated 15th August, a few notes are published on the condition of the crater on the 12th August by Messrs. A., I., and F. RICHARDS, of Kingstown. These ascents were all from Wallibu on the leeward side.

On the windward side ascents have been made by Mr. ANTOINE; Mr. BEACH, Professor JAGGAR,† Mr. HOVEY, Mr. CURTIS, Mr. TAYLOR, and Mr. WILSON, of Kingstown; by Mr. HOVEY and Mr. CURTIS; and by us.

The great convulsions of the spring of this year have effected little alteration in the general outline of the two craters. Both remain somewhat modified indeed, but the changes in them, according to the opinion of those who knew them before, have not been very extensive. Exact measurements could not be made under the circumstances; it is doubtful whether a careful survey was ever executed. Mr. POWELL, Mr. T. M. McDONALD, and Mr. DARRELL consider that the shape of the main crater is more elliptical—it is broader from east to west than it was before, and that on the whole it is larger. This agrees with the statement of inhabitants of Chateaubelair that, as seen from that place, the lip of the crater is slightly lower and somewhat different in profile, even if we allow for the effect of the disappearance of the vegetation which covered it. The small crater has not disappeared, neither is it much enlarged. Apparently it took no part in this eruption; its walls are now bare, and it is reported—we believe by Mr. HOVEY‡—that it contained a slight deposit of black ash in its bottom. The saddle between it and the large crater still stands, though not quite so high as before, and several observers have seen landslides taking place on it, the material tumbling down into the large crater.

When the first party ascended the mountain, they found a little water in the bottom of the main crater. It was boiling vigorously and giving out clouds of steam, especially at the south-east corner, close to the walls.§ Lieutenant ROBINSON, R.E., on the 6th June, reports as follows:—

* Published among the papers in the Blue Book "On the Volcanic Eruptions in St. Vincent and Martinique in May, 1902," p. 89.

† T. A. JAGGAR, "Field Notes of a Geologist on Martinique and St. Vincent," 'Popular Science Monthly,' August 1902, p. 366.

‡ E. O. HOVEY, "Martinique and St. Vincent: a Preliminary Report upon the Eruptions of 1902," 'Bull. Amer. Museum Nat. Hist.,' vol. 16, p. 338.

§ E. O. HOVEY, *loc. cit.*, plate 37, figs. 1 and 2.

The crater was completely filled with a cloud of smoke or steam. After some minutes this cleared away, and the bottom and sides of the crater were distinctly visible. The clearing and filling of the crater with steam was repeated at intervals of about ten minutes. The bottom of the crater contained three small lakes, two of which were of very yellow looking mud, with a low wall of mud and boulders separating them. They were quite quiescent. To the right was a very large mound of earth and stones, which divided them from the third. This third pool was the cause of the steam alluded to above, for by walking some distance to the west along the edge of the crater, it was seen to be bubbling up with excessive fury, and throwing up liquid mud amidst the steam. I estimated these pools to be 1800 feet below the lip of the crater."

When we saw it, matters were in very much the same condition as on the 6th June. Three irregular lakes of greenish, turbid opalescent water occupied the floor of the crater. The two on the west side were not boiling; only a narrow isthmus separated them. In the south-east corner was another lake, the whole of which could not be seen from where we stood. It was boiling vigorously with a hissing noise, and throwing up jets of mud and steam. The steam rose in curling wreaths along the high south-eastern wall. Water was flowing gently from one of the other lakes into this one. The floor of the crater was bare and stony, and on its northern side lay a huge pile of broken rock, which had evidently fallen from the vertical cliff behind it. This mound separated the lake which was boiling from the other two.

Of the dimensions and depth of the crater it was very difficult to form an opinion; the mists trailing over the hill prevented us from obtaining a clear view, and we had only a brief glimpse of the base of the opposite wall. Its summit we did not see. Formerly the southern lip was about 1100 feet above the surface of the lake, and it is now generally agreed that the cavity is deeper than before by several hundred feet. Some good judges consider it to be 2000 feet, others will not admit that it is more than 1400. Probably from 1600 to 1800 feet would be a fair average of the estimates formed. The cliff, which on the north side overlooks the lakes of water, must be over 2000 feet high.*

The south-western wall, on which we stood, sloped downwards at its upper part for perhaps half its whole height. The angle of declivity varied from 40° to 44° , and a layer of wet black ash, deeply furrowed with grooves eroded by the rains, covered the surface. The lower part of this wall was steeper, probably vertical; we could not see it from where we were. We tried to work round the lip of the crater to the windward side, but in the thick wet mud, in which we sank to the knees, walking was very difficult, and as a storm of wind and rain sprung up, we had to give up the attempt.

Professor JAGGAR's photographs show that the north-eastern wall is vertical, a precipice of bare rock. From it great landslides and falls of rock frequently tumble on the floor of the crater with a loud noise.† They have been seen by more than one

* Further modifications have since taken place, owing to the eruptions of September and October, 1902.

† E. O. HOVEY, "Martinique and St. Vincent: a Preliminary Report upon the Eruptions of May 1902," 'Bull. Amer. Museum Nat. Hist.,' vol. 16, p. 337, 1902.

observer. When we ascended from the windward side we reached the edge of this precipice. Our guide had been there with Mr. HOVEY, and he told us that since his previous visit the outline of the cliff had considerably altered. We saw great cracks some yards back from the edge, running parallel to it, and before we descended a large slice of the face broke away and slid down into the abyss. When he ascended with us, Mr. McDONALD remarked that since his previous visit much rock had fallen, and lay at the foot of this great cliff.

Lieutenant ROBINSON found that the southern lip of the crater was 2450 feet above sea level.* We made it at three different places 2690, 2630, 2700. These measurements are by aneroid, and are sufficient to show that on this side the lip is lower than before by at least 300 feet. At the spot where we reached the edge of the crater from the windward side, our aneroid recorded a height of 3050 feet, where formerly the altitude must have been not less than 3400 feet. There can be no doubt that a considerable mass of the crater walls has been blown into the air by the great explosions of the 7th May, and that the crater is somewhat larger, and its rim distinctly lower in consequence.

The amount of water in the crater continually increased during the months of June, July, and August, 1902. When first visited, it contained only one small lake. A few days later Lieutenant ROBINSON found that there were three—as was also the case when we were there.

Mr. ARTHUR L. DARRELL, along with a party from Kingstown and Chateaubelair, ascended the mountain on the 27th June. He has sent us a description of the crater, from which we extract the following:—

“We remained on the summit about three-quarters of an hour, carefully inspecting the localities. I knew both of the craters, the old and the new—as they are popularly called—that were such objects of interest to tourists before the eruption, having visited them as late as last Easter Monday, March 30.

“I noticed that the old crater was changed in shape. Before the eruption it was nearly circular, to-day it is an irregular oval, the extension being towards the Morne Ronde side on the north-west, and on the south-east where the Rest House formerly stood. The hill on which the Rest House formerly stood has been blown away. The northern and eastern sides of the crater are more precipitous than they were. The south-eastern side is not so steep, but shelves from the top to the bottom. In some places the sides are very rugged, being covered with large angular rocks, and in other places with smoother and finer material. The southern edge of the crater is lower than it was before the eruption. As I stood on the south-west edge of the cone, and looked across to the north-east, I noticed that the ‘saddle’ or division ridge between the two craters seemed lower than it was formerly, for I could see over it to the furthest side of what was known as the ‘New Crater,’ which appeared to be more precipitous than formerly, and which presented a dark brick red appearance. Before the eruption, no one standing where I stood, on the edge of the old crater—now lower than before—could see the new crater beyond the ‘saddle.’ I therefore infer that the crest of the ‘saddle’ is considerably lower than it was; otherwise I could not have seen the red-faced further side of the ‘New Crater.’

* Report by Lieutenant ALICK C. ROBINSON, R.E., Blue Book, ‘Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,’ p. 90.

"At the bottom of the old crater, I saw three lakelets. The bottom of the crater consists of broken rock and sand. The depth from the edge of the cone to the surfaces of the lakelets does not seem to be much greater than was the depth from the edge of the old crater before the eruption to the surface of the old lake. The water in the three lakelets is not of the same colour. The northernmost one is of a dark olive-green colour, that of the central one is of a light sea-green colour, and that of the most southerly one is black and muddy. This lakelet is constantly bubbling and boiling, emitting steam with a hissing noise, the vapour rising, at the time of my visit, considerably above the crater's mouth. The side of the crater, at the base of which this black boiling lake lies, is very steep. This bubbling pool cannot be seen by looking over the side of the crater, at the base of which it lies. I had to walk at least a hundred yards to the western edge before I could get a clear view of it. Large stones were seen and heard falling into the boiling lake, which is separated from the middle lakelet by material, from which an apparently ashen-coloured smoke seemed to ascend. In addition to these three lakelets there is a small pool of yellowish mud resting on the top of a mound that lies close to the base of the eastern side of the crater.

"I was anxious to complete my inspection of the mountain by visiting the New Crater so-called, but I was unable to do so owing to the great difficulty of passing to the other side of the crater—the guide being unacquainted with that part of the mountain."

It would seem that during the three weeks which had elapsed since we were there, no considerable change had taken place in the appearance of the crater and of its lakes of water.

The latest account which we have received is one which is given in 'The Sentry,' on 15th August, 1902, in the form of an interview with MESSRS. ADOLPHUS RICHARDS, IVAN RICHARDS, and FRASER RICHARDS, of Kingstown. From this it would appear that there was then only one lake, which was about one-third the extent of that which occupied the crater before the eruptions. They state that steam and pebbles were being intermittently thrown up with a hissing noise from a small opening on the lip of the crater. This looks as if the activity were increasing. Two weeks later another eruption broke out.

The Evidence for the Hot Blast and the Avalanche of Dust.

The statements of the survivors in the Carib Country and of those observers who were in Chateaubelair, place beyond doubt the existence of a great black cloud which swept from the crater down the flanks of the mountain on the fatal afternoon of the 7th May. It remains for us to discuss the bearings of our observations on the deposits of ash and other associated phenomena in the devastated country on the peculiar features which are exhibited by eruptions of this remarkable type. These establish, we believe, the presence also of an avalanche of dust, sand and stones, and of a hot blast, and that they are intimately connected with one another and with the great black cloud.

The Avalanche of Dust, Sand, and Stones.

The distribution of the materials emitted during this eruption on the Soufrière and in the valley at its base, cannot fail to be regarded as very remarkable, and cannot be

accounted for on the theory that the ash was simply rained down from above and gathered where it fell. A rain of ashes certainly took place. It lasted for hours on Wednesday afternoon and night. Hot stones falling from the sky struck many people in the Carib Country who were running from one house to another in the gloom, seeking for their friends, and the noise of the ash clattering on the roofs was at times almost deafening. In this way a sheet of deposit must have been laid down all round the volcano, which thinned out gradually as it was traced further and further from the crater. Over Chateaubelair and Georgetown the black cloud did not pass, and the thickness of the ash is a measure of the violence of the shower. Since 18 inches accumulated in Georgetown, it does not seem unlikely that the 3 feet or more of deposit in the Carib Country, and the 5 feet on the Mahoe, above Lot 14, should be due almost entirely to the hailstorm of sand and scoria which lasted through the night. Over the Carib Country the great black cloud rolled, but we cannot believe that it piled up any considerable sheet of ashes, though we know it was laden with dust, as some of the survivors state that on the floor of the huts a thickness of 1 or 2 inches gathered during the passage of the wave of heat which killed so many people. Similarly, we may allow that the ash which lay on the ridges and slopes of the spurs, on the mountain and on the edge of the crater, which had originally varied in depth from a few feet up to 10 or 12 feet, might have been entirely deposited during the rain of ashes. But this will not account for the extraordinary manner in which the ravines were choked with hot sand. The very high temperature of these masses shows that they did not gradually accumulate, but must have been practically instantaneously piled up in the valleys. So fine is the dust that it would have fallen very slowly had it been projected vertically upwards from the crater, and should in that case have been nearly cold before it reached the surface of the ground. It would probably also have shown traces of stratification, as after each outburst the coarse stones would first fall and the finer material would settle slowly afterwards. But on the banks of the Wallibu and Rabaka Dry River it was rare to see sections of the new ash which exhibited any pronounced stratification. The whole mass—sand, stones, scoria, and burnt timber—seemed to have reached its present position practically simultaneously.

The distribution of the heaped-up ash is also significant. It is principally on the south side of the mountain, as if it had risen in the crater and welled over the lower southern lip. It is found in the Rozeau and Larikai Valleys on the east side, though not in very great quantity. On the west side it lies only in the Rabaka Valley, there being hardly any in the gorges above Orange Hill, Overland Village, and Sandy Bay. Very little was able to surmount the great Somma rampart and reach the valleys above Fancy and Grand Baleine.

Moreover, it did not gather on the slopes and ridges of the spurs that descend to the Wallibu and Rabaka Dry River. Nor did it rest in the upper part of the ravine between the spurs. It is only when it reached the flatter ground below that its

velocity was checked and it could accumulate. Nine-tenths of the mass seemed to have been launched into the upper part of the two valleys above-mentioned, and there it was met by the opposing face of the Morne Garu. It could not ascend the slopes, but was split into two currents, one following the Wallibu and Wallibu Dry Rivers, the other the Rabaka Dry River. Into these it poured, and on the leeward side it coursed along till it reached the shore, for sections of the round-backed mounds it formed are to be seen in the sea cliffs (see Plate 24, fig. 1). In the Rabaka Dry River the great tide of sand came down the main valley, rapidly slowing down, and it had come to rest before it had reached the lowest part of the channel, which begins about a mile above the sea. This contained none of the thick banks of hot sand which obstructed the stream above that point. The deluge of dust and stones had poured down the main valley like a flood, and, gathering there, had obstructed the mouths of all the lateral gorges. It clearly did not drain off the whole surface and down all the stream channels like a fall of rain, but in one well-defined torrent it coursed along the ravine of the Rabaka Dry River. The trees growing on the slopes of the mountain and in the wooded gorges were overwhelmed, caught up, charred, reduced to shapeless fragments, and swept along by the avalanche. How much erosion had been effected by the moving mass we could not tell. There was no very striking evidence of its action, and in the upper parts of the mountain it seems likely that no great changes had been produced. But the surface was plastered with mud where it was not washed and scoured by the tropical rains, and, as we had no knowledge of its configuration before the eruption, it was not easy to form an opinion as to the effects of the avalanche. Further down the sheets of sand covered over and obliterated all traces of the havoc they had wrought as they swept down the valleys.

The geological evidence was not sufficient to demonstrate what form the discharge took, what was its path, and how great was its velocity when it left the crater; but by the time it reached the valleys below it was a rushing torrent of sand, stones, and hot gases, which coursed along the valley bottoms, adapting itself readily to all changes in their configuration, too heavy to surmount any great height, but sweeping over the surface of the minor ridges, and coming gradually to rest in the deep ravines behind them. In the valleys of the Wallibu and Rabaka Dry River it filled the whole channel, and when its energy was spent it lay in banks, with irregular rounded upper surfaces, like glaciers of black sand.* In the more shallow and open valley of the Wallibu Dry River it had not been confined to a narrow and steep-sided channel, but had been free to spread out laterally, and there it took the form of a broad sheet of sand with round-backed ridges, like wreaths of snow, pointing down the valley and diverging with a slightly fan-shaped arrangement.

* The likeness to a glacier is remarked also by Mr. E. O. HOVEY, "St. Vincent and Martinique: a Preliminary Report upon the Eruptions of 1902," 'Bull. Amer. Museum Nat. Hist.,' vol. 16, p. 343, and Plate 39, fig. 1.

The Hot Blast.

Few in St. Vincent saw the avalanche and none survived, but many were struck by the hot blast, and of these a certain number have lived to tell the tale. Its force was nearly spent before it reached the low grounds of the Carib Country, where stood the most populous estates, but there and in Fancy the men who were holding the doors or windows felt the shock, and in one or two cases were knocked down by the impulse of the blast. The houses are built so as to freely admit the air, and the labourers' huts cannot be effectively shut up. Several of the survivors in Overland Village and Orange Hill told us that the sudden rush of the hot gases was like a powerful puff of wind. It is only, however, further up the mountain that it was sufficiently vigorous to leave behind unmistakable effects. At Richmond and at Rabaka it was travelling at perhaps 30 miles an hour, a strong breeze, but not sufficient to do much damage. The trees were broken by the falling stones, and the weight of ash which gathered on the branches and the withered leaves was often sufficient to weigh them down to the ground or tear them off the stems. It was not the violence of the blast which injured the vegetation.

The varying effects of the blast, the rate at which it travelled, and the changes in its velocity as it swept down the hill-sides, are best studied in that tract of flat land above Lot 14, over which the path to the summit passes before it takes to the knife-edges of the spurs (see Plate 31, fig. 2). There the configuration of the surface is comparatively simple; elsewhere the irregularities presented by the ridges and ravines, and their influence in shielding or exposing the standing timber, must constantly be kept in mind, for sometimes in a valley which everywhere else was swept bare of all vegetation, a solitary tree would be left standing behind some ridge, or in a little lateral gully, where it had been sheltered and preserved. But the changes were essentially of the same character on all parts of the hill, though local conditions had had an influence in modifying their intensity.

At Lot 14 the trees were still standing, but had lost many of their branches, and principally those which were on the side towards the crater. This was also to be seen on the lower slopes on the leeward side and on the ridge behind Wallibu (see Plate 26). The trees were erect for the most part, but all branches facing the blast had been stripped; a few of those pointing towards the opposite quarter still remained. The smaller and weaker trees were often bent over and inclined away from the crater. A glance at these leafless twisted stems was sufficient to show that over them a blast had swept, tearing off everything that directly obstructed its path (see Plate 27, fig. 2).

Further up the hill, on both sides, the effects were still more noticeable. Many of the trees were overturned; others still stood, mere trunks without leaves or branches. All that had fallen lay parallel, and pointed down the valleys and the slopes. The smaller branches had disappeared, broken off and swept away, or burnt up by the

heat of the gases. At the upper limit of this flat ground, above Lot 14, few trees were still standing; practically all were broken off at the base, their prostrate stems pointed down the slopes, their leaves and smaller branches had disappeared. Only the stoutest remained erect, mere columnar broken trunks without a branch. Here the destruction was quite as great as that of the most powerful tornadô, the forest had been mown down as if by a mighty sickle (see Plate 35). The sides of the branches and trunks towards the crater were charred and eroded by the hot sand carried by the blast. The timber was green and had not been set on fire, but the bark and wood had been carried away on the weather side to depths of $\frac{1}{4}$ or $\frac{1}{2}$ an inch or more, while on the lee side the parched dry bark still adhered and flaked away readily when touched with the finger nail.

At a higher level still, about 1500 feet, as a rule, everything was overturned, cut down or uprooted, only the trunks—blackened and half destroyed by the consuming blast, lay scattered on the slopes; a single tree here and there had miraculously escaped. Even the large “cotton trees,” 8 to 10 feet in diameter, had been overwhelmed; nothing could resist the violence of the blast (see Plate 34, fig. 2).

Nearer the summit all that was left of the rich forest that had clothed the hill with green, was the scattered, shapeless fragments of burnt wood buried in the stinking black mud which covered all the higher slopes. The destruction was so complete that it could not be said whether it was to be ascribed to the heat of that tide of incandescent ash and superheated gases, or to the velocity with which it swept along the ground. The mud lay thick on these parts of the hill, and it was not often that the old surface could be seen in the bottom of the rain gullies. The buried wood appeared to have been in every case uprooted, but we could not be certain whether the shapeless fragments we saw were the bases of the stems with the roots, or pieces of the trunks broken off. One thing was clear,—that blast of sand-laden gases must have been at least bright red-hot when it welled over the lip of the crater, so completely had everything combustible been reduced to charcoal, when it had escaped entire destruction. And it is also certain that the passage of so enormous a volume of sand and stones over the surface must have wiped out all vegetation, cut down everything standing, and swept up all that could be carried away in its own moving mass.

One of the most interesting features of the hot blast was the rapidity with which its velocity diminished as it swept from the higher slopes down upon the plain. At Lot 14 no buildings were damaged by its force, and the trees had not suffered more than if they had been exposed to an ordinary gale, if we allow for the high temperature of the blast and the dust it carried with it. At a point a mile and a half further up the hill, the destruction was more complete than that effected by a hurricane or tornado, and in so short a distance the velocity must have changed from perhaps 100 miles an hour to 30 or 40 miles an hour. There are many reasons why this should have been the case. One of the most obvious is the change of gradient,

another is the great weight of sand and dust which was kept in suspension by the gases, still another the cooling by expansion of the steam and by contact with the cold surface of the ground, but these will be discussed more fully when we have considered also the evidence afforded by the eruptions of Pelée.

The area over which the hot blast spread was enormously greater than that traversed by the avalanche. The latter poured down the south side of the hill, and along the streams that drain these slopes, but the blast covered the whole hillside and ascended the shoulders of Morne Garu. The trees were broken down on all the higher parts of the mountain, though least on the northern side and above Fancy and Owia. From Point Espagnol to Langley Park the whole upper part of the Carib Country, and the ridges and spurs higher up, were ruined by the blast, the timber for the most part fallen, and the prostrate trunks pointing outward from the crater and down the deep radial valleys. This is true also of the leeward side from Windsor Forest to Richmond Estate.

In short, the effects of the hot blast were shown by all the area over which the great black cloud had swept in the first part of its course, and the correspondence between the region of broken forest and the country covered by the black cloud before it reached the lower grounds is too close to be accidental. When the dust in the cloud was hot and when the velocity of the gases was still very great, the forest had been cut down and the wood charred, eroded and destroyed. When the hot sand had cooled and was subsiding, the hot blast effects diminished and finally disappeared, though still the cloud of dust and asphyxiating gases crept along the ground, and where it passed over the estates on the windward shore, though no longer able to cut down or overturn the trees and structures in its path, brought injury or death to their living occupants. To those who saw it from outside it was a black cloud, or even a purplish and reddish cloud, but those who were overtaken by it felt it a hot blast laden with sand and dust. This was near the shore, or on the sea; only there did any who were caught survive. No one in St. Vincent was in the region of its fiercest energy, but in the city of St. Pierre a similar blast wrought desolation with a completeness and a murderous violence which are now a matter of history.

But the blast is not merely a phase of the great black cloud, it has also a very close relationship to the avalanche of sand. In it the gases greatly preponderated; it had a mobility, a power of surmounting obstacles, a tendency to spread laterally which the avalanche does not seem to have possessed. We may say that the hot blast coursed over the ground like a current of heavy gas; the avalanche resembled a viscous heavy fluid. The blast which swept down into the valleys at the south side of the Soufrière was too heavy to climb vertically the steep face of the Morne Garu. It split into two parts, one of which ascended obliquely over each shoulder of the mountain, moving down the forest and strewing the trunks of the trees before it in such a way as to mark the direction of its path. The avalanche, on the other hand, when it reached these valleys turned almost at a right angle to its previous course and

flowed along them, clinging to the ravines, but did not ascend the opposing slopes. Except in this case there is little evidence that the blast had any power of climbing; it seems, in fact, to have been so heavy, so weighted down with dust that it flowed along the depressions of the hill sides almost like a torrent of water. This is shown by the evidence of those who saw the black cloud pouring in an inky mass down the gorges on the leeward side of the hill. The Carib Country is too level and open to modify in any way the course of this current, but at Chateaubelair there are several well-marked ridges between the village and Wallibu, and these certainly directed and deflected the path of the blast. At the north side of Richmond Vale Estate there is a spur, some 500 or 600 feet high, running down to the sea, and on the side of this ridge next the crater the destruction of vegetation has been very great; on the south side, a week or two after the eruption, everything was as green as before. It is in every way probable that these ridges saved the village, and a careful examination of the difference in the appearances presented by the country to the north and to the south side of each of them convinced us that they had intercepted the violence of the blast, and protected the region behind them. The moving mass of sand and gases was too heavy to rise freely in the air when it passed over Richmond, but clung to the surface of the ground, and lost much of its energy and dropped most of its solid matter before it could surmount an obstruction. The blast was, in fact, only the lighter and upper portion of the avalanche of dust.

The rampart of the Somma wall, which faces the crater on its northern side, undoubtedly protected the country behind it, for the denser portion of the cloud was too much loaded with dust to climb this ridge. Its main force poured over the lower south lip of the crater with the avalanche of sand. On the north side the deposits are comparatively thin, and it seems that most of the material that emerged from the crater was deflected by the Somma and lodged in the valley of the Larikai. A black cloud certainly descended on Grand Baleine, but in surmounting the intervening summit ridge of the hill it had lost by far the greater part of its burden of ash and therewith most of its violence.

The Subsidence at Wallibu and Morne Ronde.

More than one-half of the ash which gathered on the slopes of the Soufrière has already been washed into the sea. When this rainy season is over little will remain on the higher ground, and the underlying soil will be in large measure also removed. So rapid is erosion under the conditions that prevail on the mountain, that when the eruptions cease it will soon be difficult to find traces of the new ash deposits on the higher grounds, and when tropical nature again spreads a thick mantle of vegetation over the naked surface, it will cover over and obliterate most of the effects of the eruptions of this year. The thick ash deposits in the valleys will long remain, but even these will disappear at last before the persistent, restless action of the streams.

On the devastated estates a new soil will form, and agriculture will go on as before. The changes in the crater will remain, but they are not of great magnitude, and should the volcano sink into repose, a new lake may gather not unlike the old one, and the rocky walls will be again covered with green forest.

In addition to the widening of the crater and the obstruction of the valleys, the only important alteration in the geography of St. Vincent which has been occasioned by this eruption is the disappearance of a narrow strip of land along the leeward coast. It extends from Wallibu to Morne Ronde, and had a maximum breadth of perhaps 200 yards. Between these two points there formerly stretched a low flat beach on which ran the public road. Two villages stood on this beach, one at Wallibu, the other at Morne Ronde, and both have totally disappeared. The road is gone, and the bluff which stood behind it has now receded for several yards, and presents a clean-cut section to the sea. At first the water washed its base, but the soft loose ash is constantly tumbling down, and a narrow beach had formed when we were there. It was not safe to explore this cliff very closely, for the hot dry ash above was frequently slipping every here and there, and masses of many tons were being precipitated on the shore (see Plate 25, fig. 1).

Mr. ROBERTSON, of Wallibu, told us that 64 acres of that estate which lay between the bluff and the sea have vanished, and the chimney of the works, formerly about 200 yards from the shore, is now quite near the edge of the cliffs. The subsidence runs all along the coast to a little south of Morne Ronde Point, a distance of nearly a mile, and although the face of the cliff has been somewhat modified by slipping and by the erosive action of the rivers and the sea before we arrived, it is clear that originally it was a nearly straight line.

There can be no doubt that the low beach below the bluff was mostly a talus of material fallen from the cliffs or brought down by the rivers and spread along the shore. The submarine slopes off Wallibu are very steep, depths of 120 fathoms being found in less than half a mile from the land. This means an angle of 16° , and as at Morne Ronde and along this shore the earthquakes attendant on the eruptions were more severe than anywhere else in the island, it is probable that this loose talus slid down into deeper water. Then the cliff behind the beach, being composed of soft, incoherent ash, would crumble away and break down, as it was doing when we were there.*

At the mouths of several of the streams further north along this shore there were, before the eruption, little patches of flat ground on which, in some cases, stood the houses of peasant cultivators. These consisted of matter brought down by the torrents which flow in the deep ravines on this side of the mountain, and deposited in the shape of alluvial fans when the current of the streams was checked on reaching the sea.

* See also T. A. JAGGAR, "Field Notes of a Geologist in Martinique and St. Vincent," 'Popular Science Monthly,' vol. 61, p. 363, August, 1902.

Mr. T. M. McDONALD, who knows every feature of this coast intimately, pointed out to us that in most cases they had disappeared with the houses which were planted on them. The rocky cliffs behind them still stand, bearing the remains of the trees which previously grew there. Only the wedged-shaped deltas have vanished. There is no trace of any fracture or dislocation, and the marks cut by the tide on this shore show that there has been no change of level, or, at any rate, none of more than a few inches. Everything points to the slipping of loose, gravelly and sandy deposits piled on the steep submarine slope which characterises this coast.

There is no difficulty in believing that a similar process has gone on at Wallibu and Morne Ronde, and this will explain most of the facts observed. Mr. ROBERTSON, of Wallibu, left for Chateaubelair on Wednesday, the 7th May, about 12.30 P.M. As he went down from his house to the boat on the sea shore he noticed that the flat beach was sinking, "dropping down like stuff from a cart," and the bluff behind was tumbling down on the flat below. The impression formed on his mind was that the soft, incoherent accumulations were settling down under the concussions and earthquakes produced by the eruptions. No one saw this part of the shore again till several days had elapsed, and then it was in essentially the same condition as when we visited it, except that there was no beach below the cliff.

Allowance must also be made for the effects of the blast which rushed out from the valleys of the Wallibu and Wallibu Dry Rivers. It was strong enough to blow down many of the trees on the ridge above Wallibu, and must have raised the sea in powerful waves. Captain FREEMAN, of the "Roddam," states that in St. Pierre harbour the waves occasioned by the blast were tempestuous.* But here the wind was off the shore, and may have really produced very little effect.

There is, however, a certain amount of evidence which points to the conclusion that the inner boundary of this depressed tract may be a fault line. It certainly has a very straight trend, and, looking at it on the map, one is at once struck with this peculiarity. In this respect it differs entirely from the smaller subsidences at the mouths of the streams at Larikai and Trois Loups. Another fact of great interest is the rapidity with which the water deepens off the new shore. Mr. P. FOSTER HUGGINS,† of Chateaubelair, found that at 50 feet from the beach the depth is $7\frac{1}{2}$ fathoms, at 100 feet 18 fathoms, a submarine slope of 45° . This is a gradient much higher than is usual along this side of the island, and may be due to the presence of a fracture, on the seaward side of which there has been depression.

Formerly, this coast was known as Hot Waters, because the water obtained by digging pits in the sand was warm. This may indicate the existence of a fissure up which hot water rose, but all the inhabitants of the villages on the beach whom we

* Captain FREEMAN, "The Awful Doom of St. Pierre." See 'Pearson's Magazine,' September, 1902, p. 316.

† "An Account of the Eruptions of the St. Vincent Soufrière," p. 17. By P. FOSTER HUGGINS, St. Vincent, 1902.

cross-questioned said there were no hot springs, and that the warm water was merely surface water.

There is no evidence of any submarine cliff, and had there been any at first, it must now have been masked by the material dislodged by landslides, or eroded by the sea from the cliffs and the vast quantities of new ash which have been deposited all along this coast by the rivers which flow down the mountain. At present, we can only say that the existence of a fault along the shore from Wallibu to Morne Ronde remains an open question.

PREVIOUS ERUPTIONS OF THE SOUFRIÈRE.

The Eruption of 1718.

The earliest historical account of an eruption in St. Vincent is that which appears in the 'Weekly Journal,'* or 'Saturday's Post' (otherwise known as 'Mist's Journal'), of July 5, 1718. It occupies the principal place in that number, and was considered so important that the usual letters from correspondents were crowded out to make room for it.

It begins as follows :—

"We have a piece of public news this time of such consequence, and so necessary for all our readers to be fully acquainted with it, that our friends who have written several letters to us, which otherwise deserve publishing, must excuse us for this week.

"This relates to the entire desolation of the Island of St. Vincent, in the West Indies, by the immediate hand of Nature, directed by Providence, and in a manner astonishing to all the world, the like of which never happened since the creation, or, at least not since the destruction of the earth by the water in the general deluge.

"Our accounts of this comes from so many hands, and several places, that it would be impossible to bring the letters all separately into this journal; and when we had done so, or attempted to do so, would have the story confused, and the world not perfectly informed. We have therefore thought it better to give the substance of this amazing accident in one collection, making together as full and as distinct account of the whole, as we believe is possible to come at by any intelligence whatsoever; and at the close of this account we shall give some probable guesses at the natural cause of so terrible an operation. The relation is as follows, viz. :—

"An account of the island of St. Vincent, in the West Indies, and of its entire destruction on the 26th March last, with some rational suggestions concerning the causes and manner of it.

"The island of St. Vincent is the most populous of any possessed by the Carribeans, its altitude is 16 degrees north from the line. Those who have seen the island Ferre or Fietre, one of the Canaries, affirm that this is much of the same figure. It may be about 8 leagues in length and 6 in breadth. There are in it several high mountains and very fruitful plains, if they were cultivated. The Carribeans have many fair villages, where they live pleasantly, and without any disturbance; and though they have a jealousy of the strangers, yet do they not deny them the bread of the country, which is cassava,

* Reprinted in 'Daniel Defoe: his Life, and recently discovered Miscellaneous Writings.' By WILLIAM LEE. Vol. 2, p. 52. 1869.

water, fruits, and all other provisions growing in their country, if they want them, taking in exchange wedges, hooks, and other implements of iron, which they much esteem.

“On the 24th March a French sloop arrived at Martinico, that passed by the island of St. Vincent the 22nd, that, as the master reported, he bought some fish of some of the savages who inhabited there, and who came off to him in their canoes. He says that all was safe, and in very good condition there, for anything he perceived, only that some of his seamen report that since the disaster, that one of the Indians told them that they had been terribly frightened with earthquakes for some time, and with flashes of fire like lightning, which did not come out of the clouds as usual, but out of the earth, and that they had felt these earthquakes for a month past, to their very great amazement.

“On the 27th in the morning the air was darkened in a dreadful manner, which darkness by all accounts seems to have extended over all the colonies and islands which were within 100 miles of the place, but was perceived to be more or less dark as those islands were further or nearer from the place.

“But that which is most remarkable of all is, that at some of the islands, and at Martinico in particular, a dreadful flash of lightning, as they called it, was seen on the 26th about 11 o'clock at night. This flash, which they called lightning, we shall account for in the following part of the relation.

“It is to be observed in the next place, that as there were several ships, or other vessels at sea, in several ports among the islands, some of these had a more terrible sight of this thing than others; particularly they write that in one sloop which is come into Martinico, the men are so terrified still, and were so amazed at what they saw and heard, that they appeared perfectly stupified, and gave little or no account. Others are come into other ports so horribly frightened, that they scarce retain their senses; others give confused accounts, and so more or less distinct as they were nearer or farther from the place. The sum of what may be gathered from them all is this:

“That they saw in the night that terrible flash of fire, that after that they heard innumerable clashes of thunder; some say it was thunder they heard, others that it were cannon only, that the noise was a thousand times as loud as thunder or cannon, considering that it appeared to be at a great distance from them.

“That the next morning, when the day began to break, the air looked dismally, viz., all overhead was a deep impenetrable darkness, but below, all around the edge of the horizon it looked as if the heavens were all on fire.* As the day came on still the darkness increased till it was far darker than it had been in any part of the night before, and as they thought the cloud descended upon them, the darkness still increased after this, viz., in the afternoon they were surprised with the falling of something upon them as thick as smoke, but fine as dust, and yet solid as sand; this fell thicker and faster as they were nearer or farther off, some ships had it 9 inches, others a foot thick upon their decks; the island of Martinico is covered with it at about 1 to 9 inches thick; at Barbados it is frightful, even to St. Christophers it exceeded 4 inches; it is fallen over the whole extent of the Island of Hispaniola, and there is no doubt but it has been seen on the continent of New Spain, about the point of Guiana, and the mouth of the River Orinoco, all of which will perhaps be accounted for in some measure in the following narrative.

“This continued falling for two or three days and nights successively, and it was impossible for any man to find out or so much as guess at the meaning of it, or of any natural cause to produce it, till the whole came to discover itself, but all people stood amazed at the cause, and several letters were sent to England of it, from Barbados in particular, as of a strange, miraculous shower of sand, of which we gave an account in our Journal of the 20th past. The first news that was given of the whole thing was by some vessels that were under sail on the night of the 26th, belonging to Martinico, by which we had the following particulars: that on the said 26th, about midnight, the whole island of St. Vincent rose up into the air with a most dreadful eruption of fire from underneath the earth, and an inconceivable noise in

* Compare with this the descriptions of the appearance of the sky immediately before the fall of dust in Barbados on the 7th of May, 1902, as given on p. 408 and p. 409.

the air at its rising up; that it was not only blown up, but blown out of the very sea with a dreadful force, as it were torn up by the roots, or blown up from the foundations of the earth.

"That the terror was inexpressible, and cannot be represented by words; that the noise of the bursting of the earth at first is not possible to be described, that the force of the blow or blast is such, and the whole body of the island was raised so furiously that the earth was entirely separated into small particles like dust; and as it rose to an immense height so it spread itself to an incredible distance, and fell light and gradually, like a small but thick mist. This part, we suppose, must be occasioned by the force of the blow, effectually separating the parts, otherwise they would have fallen with a violence of motion proportioned to the weight of the whole, the particles pressing one another, whereas now every grain was loose and independent in the air, and fell no faster than it was pressed by its own weight, as in a shower of snow or rain.

"The more solid parts of this land, which were lifted up by this blast, and supposed to be of stone, slate, or clay, or such solid matter as would not dissipate or separate in the air like the rest, being lifted also to an immense height, and their plunging by a mighty force received by their own weight into the sea, must of necessity make a noise or blow equal to that of the loudest cannon, and perhaps to thunder itself; and these we think to be the several reports or blows which were heard even to St. Christopher's Island (which is a vast distance from that of St. Vincent), and of which the people in these islands, as well as in the ships, heard about a thousand or twelve hundred distinct blows or reports, and supposed it to be the noise of guns.

"As soon as it was understood by the inhabitants in the other islands what it was, *that is to say*, that it was an eruption of the earth at the island of St. Vincent or thereabouts, sloops, barks, and other small vessels came from all parts to see how it was, to enquire into the damage suffered, and to get an account of the particulars; but how astonished must these enquirers be, when meeting from all parts upon the same errand, they may be supposed to go cruising about to find the island; some examining up their books to cast up the length they had sailed, some blaming their own negligence for not keeping a right reckoning, some their men for mistaking their distance, others taking observations to know the latitude they were in; at last all concluding, as it really was, to their great confusion, that the said island was no more, that there appeared no remains except three little rocks, no, not any tokens that such an island had been there, but that, on the contrary, in the place of it the sea was excessive deep, and no bottom to found at 200 fathoms.

"As this is an event so wonderful as no history can give us an account of the like, so it cannot be unpleasant to our readers to consider briefly some natural causes which may be assigned for it."

The writer then proceeds to discuss the possible theoretical explanations of this remarkable phenomenon. He rejects the suggestion that it was due to an earthquake, owing to the fact that the island was blown into the air. Two other causes are considered, the sudden admission of air through cracks produced by earthquakes to vast bodies of sulphurous and nitrous gases in the caverns and hollows of the earth, producing a violent explosion, and the ingress of water through fissures to the subterranean fires. He is rather inclined to adopt the latter hypothesis.

The writer of this article was probably no other than DANIEL DEFOE, the author of "Robinson Crusoe," who was at that time editor of this Journal, and the hand of that master of romance may be traced in the paragraph which suggests that "sloops, barques, and other vessels came from all parts to see how it was," and joined in a futile search for the remains of the island which was now sunk "full-fathom deep." This embellishment of the narrative apparently evoked protest* and contra-

* 'Daniel Defoe: his Life and Miscellaneous Writings.' By WILLIAM LEE, vol. 1, p. 280.

diction, for in the same Journal of August 2, 1718, he returns to the subject and says: "They pretend to tell us a strange story, viz., that the island of St. Vincent is found again, and is turned into a volcano, or burning mountain, but we must acknowledge, we do not believe one word of it."

On account of its obvious exaggeration this record has been received by many with incredulity. SHEPHARD, in his "History of St. Vincent," does not mention it, and it is generally held that the eruption of 1812 is the first of which there is historical evidence. Mr. HUGGINS, in his "Account of the Eruptions of the St. Vincent Soufrière," discredits the story altogether.*

'It is supposed by some that St. Vincent was the scene of an eruption in the year 1718, which produced what is called the old crater of the Soufrière, and, because some seamen that year heard and saw signs of volcanic disturbance having taken place somewhere in these seas, it has been concluded that St. Vincent was the place. This, however, is doubtful, and it is certain from the researches of Major PATRICK CRICHTON, of this island, that the Caribs did not even possess a tradition of such an occurrence, and their language, I believe, did not contain any word expressive of such an event, which, it must be supposed, would have left some trace behind, uneffaceable within a century. The Caribs, it is true, looked upon the mountain with dread, but only as the abode of a vengeful spirit hiding himself in the clouds."

The reasons he advances, however, are somewhat discounted by Mr. ANDERSON's statement in 1784.†

"The most remarkable of these mountains is one that terminates the north-west end of the island and the highest in it, and has always been mentioned to have had volcanic eruptions in it. The traditions of the oldest inhabitants in the island, and the ravines at the bottom, seem to me to vindicate the assertion."

It was well known, he says, to be a volcano, and this was the main reason which led him to make the ascent.

If an eruption took place in 1718, there must have been persons living in St. Vincent in 1784 who had witnessed it. But it is not mentioned by Mr. ANDERSON, and apparently he was in ignorance of its occurrence. This is in itself strange, but it is not inexplicable.

In 1718 the island was practically entirely in the hands of the Caribs, who, according to DEFOE's account, were not unwilling to trade with passing vessels, but resented the intrusion of white settlers. SHEPHARD‡ states in his history that St. Vincent was first colonised in 1719 by a party from Martinique, who had been invited to take up land by one of the two tribes into which the Caribs were divided. In 1723 the DUKE OF MONTAGUE sent an English expedition to take possession of the island, but they found the natives hostile and the French colonists already in possession, and in consequence they gave up the attempt. Gradually, however, the English established themselves alongside of the French settlers, and for a time the

* P. FOSTER HUGGINS, 'An Account of the Eruptions of the St. Vincent Soufrière,' 1902, p. 3.

† "An Account of Morne Garou, a Mountain in the Island of St. Vincent, with a Description of the Volcano on its Summit." By Mr. JAMES ANDERSON ('Phil. Trans.,' vol. 75, p. 16, 1785).

‡ 'An Historical Account of the Island of St. Vincent.' By CHARLES SHEPHARD, p. 23.

island was neutral or debateable ground ; but in 1779 the French, with the aid of the Caribs, obtained the supremacy. In terms of the Peace of Versailles in 1783, the island was handed over to the British on the 1st January, 1784. Mr. ANDERSON'S ascent of the Soufrière took place in March of that year, and it is evident that at that time the Caribs still maintained their hostile attitude, for one of the dangers to which he considered himself exposed was the possibility of being cut off by some of their roving bands. In 1793 war broke out again between the Caribs and the English, and lasted for several years.

Mr. ANDERSON seems to have never met the Caribs of the north end of the island, and relied principally on the planters, who had not long been settled in that quarter, as it was considered a reserve for the natives. His porters were negro slaves. He may never have heard of the eruption of 1718, or if he did so, may have considered the evidence for it unsatisfactory.

The anonymous writer of the account of the eruption of 1812 distinctly refers to a previous eruption in the words, "A century had now elapsed since the last convulsion of the mountain" (see p. 463).

The internal evidence, also, is very strongly in favour of DEFOE'S narrative having been founded on fact. The first news to reach England was that in Barbados the startling phenomenon of a rain of ashes had caused great alarm, and it had been preceded by sounds like distant cannonading. Similar noises were heard all over the Caribbean Sea from Antigua to Trinidad. The rain of dust had been heaviest near St. Vincent and in Barbados. Around the Soufrière the inhabitants had for a month previously been terrified by the frequency of the earthquakes. Then the mountain burst with a tremendous noise, hot stones and sand rained down for hours, and there was complete darkness. Those who were in ships off the island saw a flash of fire, followed by loud crashing noises. The eruption was sudden in its outburst, and lasted only for a day or two.

These are, in fact, the distinctive features of the eruptions of 1812 and 1902, as observed by those at some distance from the volcano, and the whole account resembles in many respects very closely some of the narratives which were published in the spring of this year. If DEFOE was able so exactly to predict the circumstances of future eruptions of the Soufrière, he must have had even more of the creative spirit of imagination than he has hitherto been credited with.

HUMBOLDT made several references to this eruption, though it is not exactly known from what sources he drew his information. He seems to have had no doubt of its having occurred.*

If the statements regarding the area over which ash fell, from Hispaniola (Hayti) to the mouth of the Orinoco, and the depth of the layer in Barbados and Martinique, are even approximately true, this was the greatest and most violent of all

* HUMBOLDT, 'Personal Narrative,' English Translation by Mrs. WILLIAMS, vol. 4, p. 26.

the eruptions of the Soufrière since the colonisation of the West Indies. We should hesitate, however, to place much reliance on them.

The Condition of the Crater in 1784.

In the 'Philosophical Transactions of the Royal Society' for the year 1785 there is an account of what was apparently the first visit made by any white man to the crater of the Soufrière. At that time the name "Morne Garu" was applied also to this mountain; only after it was known that at the summit there was a cone emitting sulphurous vapours was it replaced by its present name "La Soufrière." In the early part of the 19th century the volcano was often referred to under both names, or under either. At present it seems that only the term Soufrière is used. Morne Garu is now the name of the mountain mass which lies immediately to the south. Mr. ANDERSON* in his ascent met with enormous difficulties, and had to hew a path through the forest for most of the way. For three days he was baffled, but on the fourth, with Mr. FRASER and some negro slaves, he reached the crater's rim, and, descending into the interior, explored it thoroughly. Probably, had he had Carib guides, he would have been shown a much easier way.

"March 4 being the day I had fixed to finish my excursion, about 4 in the morning I left the house of Mr. FRASER, who, out of curiosity, agreed to accompany me, of which I was very glad, as he was a sensible young man, and with the assistance of two negroes we pursued our journey. We found very little obstruction in our way up until we got to the place where I returned, and there, for about a quarter of a mile, we had considerable difficulty to clear our way through grass and ferns. After we came within a quarter of a mile from the top, we found ourselves in another climate all at once, the air very cold, and the vegetable productions changed; here was nothing but barrenness over the whole summit of the mountain. On the confines of the grassy region and the barren I found some beautiful plants. Moss grows here in such plenty that I frequently sunk up to my knees in it. This is the only place in the West Indies that produced any moss that I have seen. About noon we gained the top of the peak I had directed my course to before, when, in an instant, we were surprised with one of the grandest and most awful scenes I had ever beheld. I was struck with it amazingly, as I could not have conceived such a very large and so singularly-formed an excavation. It is situated on the centre of the mountain, and where the various ridges unite. Its diameter is something more than a mile, and its circumference to appearance a perfect circle. Its depth from the surrounding margin is above a quarter of a mile, and it narrows a little, but very regularly, to the bottom. Its sides are very smooth, and for the most part covered with short moss, except towards the mouth, where there are a number of small holes and rents. This is the only place where it is possible to go down to the bottom; it is exceedingly dangerous, owing to the numberless small chasms. On the west side is a section of red rock like granite, cut very smooth, and of the same declivity with the other parts. All the rest of the surrounding sides seems to be composed of sand, that looks to have undergone the action of intense fire. It has a crust, quite smooth, of about an inch thick, and hard almost as rock, after breaking through which, you find nothing but loose sand. In the centre of the bottom is a burning mountain, of about a mile in

* "An Account of Morne Garou, a Mountain in the Island of St. Vincent, with a Description of the Volcano on its Summit." By Mr. JAMES ANDERSON ('Phil. Trans.,' vol. 75, p. 16, 1785).

circumference, of a conic form, but quite level. On the summit, out of the centre of the top, arises another mount, eight or ten feet high, a perfect cone; from its apex issues a column of smoke. It is composed of large masses of red *granite-like* rock of various sizes and shapes, which appear to have been split into their present magnitudes by some terrible convulsion of nature, and are piled up very regular. From most parts of the mountain issue great quantities of smoke, especially on the north side, which appears to be burning from top to bottom, and the heat is so intense, that it is impossible to go upon it. Going round the base is very dangerous, as large masses of rock are constantly splitting with the heat and tumbling to the bottom. At the bottom, on the north side, is a very large rock split in two; each of these halves, which are separated to a considerable distance from each other, is rent in all directions, and, from the crevices, issue efflorescences of a glossy appearance, which taste like vitriol, and also beautiful crystallisations of sulphur. On all parts of the mountain are great quantities of sulphur in all states; also alum, vitriol, and other minerals. From the external appearance of this mountain, I imagine it has only begun to burn lately, as on several parts of it I saw small shrubs and grass, which looked as if they had been scorched and burnt. There are several holes on the south, from which issues smoke, seemingly broken out lately, as the bushes round are but lately burnt. On two opposite sides of the burning mountain, east and west, reaching from its base to that of the side of the crater, are two lakes of water, about a stone's throw in breadth; they appear to be deep in the middle, their bottom to be covered with a clay-like substance. The water seems pleasant to the taste, and is of a chalybeate nature. I suppose these lakes receive great increase if they are not entirely supported by the rain that tumbles down the side of the crater. I observed on the north side of the bottom traces of beds of rivers, that to appearance run great quantities of water at times to both these lakes. By the stones at their edges, I could perceive that either absorption or evaporation, or perhaps both, go on fast. The greater part of the bottom of the crater, except the mountain and two lakes, is very level. On the south part are several shrubs and small trees. There are many stones in it that seem to be impregnated with minerals. I saw several pieces of pumice-stone. I also found many stones about the size of a man's fist, rough, on one side blue, which appearance, I imagine, they have got from heat and being in contact with some mineral. These stones are scattered over the whole mountain; one or two I have sent you, with some others.

"After I had got up from the bottom of the crater, I could not help viewing it with admiration, from its wonderful structure and regularity. Here I found an excavation cut through the mountain and rocks to an amazing depth, and with as much regularity and proportion of its constituent parts as if it had been planned by the hand of the most skilful mathematician. I wished much to remain on the mountain all night, to examine its several ridges with more attention next day; but I could not prevail on my companion to stay, and therefore thought it advisable to accompany him."

From this description it will be seen that in 1784 the volcano was in a solfataric condition, emitting much steam and sulphuretted hydrogen. Its crater contained only two small lakes, and had a low interior cone. The depth of the cavity is given as a quarter of a mile, but this must be exaggerated, for the next account of it, in 1812, states that it is only 500 feet deep, yet, apparently, not a single important alteration had taken place during the years that intervened.

The Eruption of 1812.

Many accounts of the eruption of the year 1812 have appeared in works on the West Indies, and in treatises on volcanology. They appear to be for the most part abridged from the original papers which, on May 7th, 1813, were ordered by the House of Commons to be printed, and appeared subsequently as an official paper. They have

this year been reprinted as an appendix to the Blue Book on the volcanic eruptions in St. Vincent and Martinique, in May, 1902, where they will be found in full.

Of these papers the most important is an anonymous account which was published in the 'Evening News,' June 30th, 1812.

As it is the only contemporary history of the disaster, we reproduce it *in extenso*:—

"Description of the Eruption of the Soufrière Mountain, on Thursday night, April 30th, 1812, in the Island of St. Vincent.

"The Soufrière Mountain, the most northerly of the lofty chain running through the centre of this island, and the highest of the whole, as computed by the most accurate survey that has yet been taken, had for some time past indicated much disquietude; and from the extraordinary frequency and violence of earthquakes, which are calculated to have exceeded 200 within the last year, portended some great movement or eruption. The apprehension, however, was not so immediate as to restrain curiosity, or to prevent repeated visits to the crater, which of late had been more numerous than at any former period, even up to Sunday last, April 26th, when some gentlemen ascended it, and remained there for some time. Nothing unusual was then remarked, or any external difference observed, except rather a stronger emission of smoke from the interstices of the conical hill, at the bottom of the crater. To those who have not visited this romantic and wonderful spot, a slight description of it, as it lately stood, is previously necessary and indispensable to form any conception of it, and to the better understanding the account which follows; for no one living can expect to see it again in the perfection and beauty in which it was on Sunday, the 26th instant.

"About 2,000 feet from the level of the sea (calculating from conjecture), on the south side of the mountain, and rather more than two-thirds of its height, opens an immense circular chasm, somewhat exceeding half a mile in diameter, and between 400 and 500 feet in depth. Exactly in the centre of this capacious bowl rose a conical hill, about 250 or 300 feet in height, and about 200 in diameter, richly covered and variegated with shrubs, brushwood, and vines above half-way up, and for the remainder powdered over with virgin sulphur to the top. From the fissures in the cone and interstices of the rocks a thin, white smoke was constantly emitted, occasionally tinged with a slight, bluish flame. The precipitous sides of this magnificent amphitheatre were fringed with various evergreens and aromatic shrubs, flowers, and many alpine plants. On the north and south sides of the base of the cone were two pieces of water, one perfectly pure and tasteless, the other strongly impregnated with sulphur and alum. This lonely and beautiful spot was rendered more enchanting by the singularly melodious notes of a bird, an inhabitant of these upper solitudes, and altogether unknown to the other parts of the island; hence principally called or supposed to be invisible; though it certainly has been seen, and is a species of the merle.

"A century has now elapsed since the last convulsion of the mountain, or since any other element had disturbed the serenity of this wilderness than those which are common to the tropical tempest; it apparently slumbered in primeval solitude and tranquillity, and from the luxuriant vegetation and growth of the forest, which covered its sides from the base nearly to the summit, seemed to discountenance the fact and falsify the records of the ancient volcano. Such was the majestic, peaceful Soufrière on April 27th, but we trod on '*ignem repositam cineri doloso*,'* and our imaginary safety was soon to be confounded by the sudden danger of devastation. Just as the plantation bells rang 12 at noon on Monday, the 27th, an abrupt and dreadful crash from the mountain, with a severe concussion of the earth and tremulous noise in the air, alarmed all around it. The resurrection of this fiery furnace was proclaimed in a moment by a vast column of thick, black, ropey smoke, like that of an immense glass-house, bursting forth at once, and

* "*Ignes suppositos cineri doloso*."—HORACE, II, Od. I, 7.

mounting to the sky, showering down sand, with gritty calcined particles of earth and favilla mixed, on all below. This driven before the wind towards Wallibu and Morne Ronde, darkened the air like a cataract of rain, and covered the ridges, woods, and cane pieces with light grey-coloured ashes, resembling snow when a little sublimed by dust. As the eruption increased this continual shower expanded, destroying every appearance of vegetation. At night a very considerable degree of ignition was observed on the lips of the crater, but it was not asserted that there was as yet any visible ascension of flame. The same awful scene presented itself on Tuesday, the fall of favilla and calcined pebbles still increasing, and the compact, pitchy column from the crater rising perpendicularly to an immense height, with a noise at intervals like the muttering of distant thunder. On Wednesday, the 29th, all these menacing symptoms of horror and combustion still gathered more thick and terrific for miles around the dismal and half obscured mountain. The prodigious column shot up with quicker motion, dilating as it rose, like a balloon. The sun appeared in total eclipse, and shed a meridian twilight over us that aggravated the wintry gloom of the scene, now completely powdered over with falling particles. It was evident that the crisis was as yet to come; that the burning fluid was struggling for a vent, and labouring to throw off the superincumbent strata and obstructions which suppressed the ignivomous torrent. At night it was manifest that it had greatly disengaged itself from its burden by the appearance of fire flashing now and then, flaking above the mouth of the crater.

“On Thursday, the memorable April 30th, the reflection of the rising sun on this majestic body of curling vapour was sublime beyond imagination, and comparison of the glaciers of the Andes or Cordilleras with it can but feebly convey an idea of the fleecy whiteness and brilliancy of this awful column of intermingled and wreathed smoke and clouds. It afterwards assumed a more sulphureous cast, like what we call thunder-clouds, and in the course of the day a ferruginous and sanguine appearance, with much livelier action in the ascent, a more extensive dilation, as if almost freed from every obstruction. After noon the noise was incessant, and resembled the approach of thunder still nearer and nearer, with a vibration that affected the feelings and hearing; as yet there was no convulsive motion or sensible earthquake. Terror and consternation now seized all beholders; the Charraibs settled at Morne Ronde, at the foot of the Soufrière, abandoned their houses with their live stock and everything they possessed, and fled precipitately towards town; the negroes became confused, forsook their work, looked up to the mountain and, as it shook, trembled with the dread of what they could neither understand nor describe. The birds fell to the ground overpowered with showers of favilla, unable to keep themselves on the wing; the cattle were starving for want of food, as not a blade of grass or a leaf was now to be found. The sea was much discoloured, but in nowise uncommonly agitated; it is remarkable that throughout the whole of this violent disturbance of the earth, it continued quite passive, and did not at any time sympathise with the agitation of the land. About 4 o'clock P.M. the noise became more alarming, and just before sunset the clouds reflected a bright copper colour, suffused with fire. Scarcely had the day closed, when the flame burst at length pyramidically from the crater through the mass of smoke; the rolling of the thunder became more awful and deafening; electric flashes quickly succeeded, attended with loud claps, and now, indeed, the hurly-burly began. Those only who have witnessed such a sight can form any idea of the magnificence and variety of the lightning and electric flashes: some forked zig-zag, playing across the perpendicular column from the crater; others shooting upwards from the mouth, like rockets of the most dazzling lustre; others like shells, with their trailing fires flying in different parabolas, with the most vivid scintillations from the dark, sanguine column, which now seemed inflexible and immovable by the wind. Shortly after 7 P.M. the mighty cauldron was seen to simmer, and the ebullition of lava to break out on the north-west side. This, immediately after boiling over the orifice and flowing a short way, was opposed by the activity of a higher point of land, over which it was impelled by the immense tide of liquefied fire that drove it on, forming the figure of V in grand illumination. Sometimes when the ebullition slackened, or was insufficient to urge it over the obstructing hill, it recoiled back, like a reflux billow from the rock, and then again rushed forward, impelled by fresh supplies, and scaling every obstacle, carrying rocks and woods

together in its course down the slope of the mountain, until it precipitated itself down some vast ravine, concealed from our sight by the intervening ridges of Morne Ronde. Vast globular bodies of fire were seen projected from the fiery furnace, and bursting, fell back into it, or over it on the surrounding bushes, which were instantly set in flames. About four hours from the lava boiling over the crater it reached the sea, as we could observe from the reflection of the fire and the electric flashes attending it. About half-past one another stream of lava was seen descending to the eastward towards Rabaka. The thundering noise of the mountain, and the vibration of sound that had been so formidable hitherto, now mingled in the sullen, monotonous roar of the rolling lava, became so terrible that dismay was almost turned into despair. At this time the first earthquake was felt; this was followed by showers of cinders, that fell with the hissing noise of hail, during two hours; at 3 o'clock a rolling on the roofs of the houses indicated a fall of stones, which soon thickened, and at length descended in a rain of intermingled fire that threatened at once the fate of Pompeii or Herculaneum. The crackling and coruscations from the crater at this period exceeded all that had yet passed; the eyes were struck with momentary blindness, and the ears stunned with the glomeration of sounds. People sought shelter in cellars, under rocks, or anywhere, for everywhere was nearly the same; and the miserable negroes, flying from their huts, were knocked down or wounded, many killed in the open air. Several houses were set on fire; the estates situated in the immediate vicinity seemed doomed to destruction. Had the stones that fell been proportionately heavy to their size, not a living creature could have escaped without death; these having undergone a thorough fusion, they were divested of their natural gravity and fell almost as light as *pumex*, though in some places as large as a man's head. This dreadful rain of stones and fire lasted upwards of an hour, and was again succeeded by cinders from 3 till 6 o'clock in the morning; earthquake followed earthquake almost momentarily, or rather the whole of this part of the island was in a state of continued oscillation, not agitated by shocks, vertical or horizontal, but undulated like water shaken in a bowl.

"The break of day, if such it could be called, was truly terrific. Darkness was only visible at 8 o'clock, and the birth of May dawned like the Day of Judgment. A chaotic gloom enveloped the mountain, and an impenetrable haze hung over the sea, with black, sulphurous clouds of a sulphurous cast. The whole island was covered with favilla, cinders, scoriæ, and broken masses of volcanic matter. It was not until the afternoon the muttering noise of the mountain sunk gradually into a solemn yet suspicious silence. Such were the particulars of this sublime and tremendous scene, from commencement to catastrophe: to describe the effects is, if possible, a more difficult and truly most distressing task."

In SHEPHARD'S 'Historical Account of the Island of St. Vincent' (London, 1831) there is an account of this eruption, which is obviously very largely extracted from that given above. It is valuable, however, in that it gives some further particulars as to the subsequent history of the mountain, and the distribution and amount of the damage incurred:—

"The volcano still, however, burned, and on June 9th it again gave alarming signs of activity, but nothing more occurred than the throwing up of a quantity of stones and ashes, which fell back into the abyss from whence they came. All the former beauty of the Soufrière was, of course, destroyed; the conical mount disappeared, and an extensive lake of yellow-coloured water, whose agitated waves perpetually threw up vast quantities of black sand, supplied its place. A new crater was formed on the north-east of the original one, and the face of the mountain was entirely changed. Many of the adjoining ravines were filled up, particularly Wallibu and Duvalle's. In the former the river was absorbed for some years; but the gradual accumulation of water burst through the sandy barrier, and carried away many negro houses in its progress; 32 slaves, belonging to Wallibu estate, were washed into the sea by the torrent. At Duvalle's, the former settlement of the Carib chief, a sugar plantation had been established by Messrs.

Thesiger and Calvelly; the works, situated in a valley, were entirely covered by the sand and ashes, and some hogsheads of sugar remain there at present calcined to a cinder. The Rabaka River was also filled up, and its stream seldom reaches the sea except in cases of heavy rains. It was at first feared that the island would be rendered barren by the ashes, which lay on its surface to a considerable depth, but they did not prove so injurious as was supposed. The great danger was famine; but the neighbouring colonies of Barbados, Demerara, and Dominica, with a generous promptitude, hastened to supply the island with provisions, and a Committee was appointed by the Council and Assembly for the purpose of purchasing supplies. An investigation of the losses sustained was also made, and a petition presented to the Prince Regent praying for relief, which was most favourably received, and, on the case being laid before Parliament, the sum of £25,000 was voted for the relief of the sufferers. It is a wonderful circumstance, although the air was perfectly calm during the eruption, that Barbados, which is 80 miles to the windward, was covered several inches deep with the ashes, and the inhabitants, on the last day of the eruption, were terrified by the approach of utter darkness, which continued for four hours and a half, and then slowly decreased. There also, and in several other islands, the troops were under arms, supposing, from the continued noise, that the hostile fleets were engaging."

SHEPARD was for some years resident in St. Vincent, and must have been acquainted with many who had witnessed the eruption. To MONTGOMERY MARTIN'S 'History of the West Indies' he contributed an account of this eruption, with much other matter regarding the island of St. Vincent.

In the Blue Book* there appear also several letters and other papers which, as they were written within a short time after the outburst, give valuable particulars as to its history and consequences.

It would appear that immediately before the eruption of 1812 the crater was in practically the same condition as when seen by Mr. ANDERSON in 1784. The interior cone was still standing, and at its base were the two small lakes of water—one bitter and sulphurous, the other fresh. From apertures in this cone gases were still being emitted. It was a "Soufrière" giving out steam and sulphuretted hydrogen, like many others in the Caribbees, and from the action of the sulphurous vapours no vegetation grew on the upper part of this cone, but its base and the encircling walls of the crater were covered with low bush. On this inner cone a coating of sulphur had been deposited, and the acids generated by the oxidation of the sulphurous vapours had attacked the rocks, giving rise to aluminous salts which, with the precipitated sulphur, contaminated the water of one of the lakes.

The eruption began about noon on April 27th with the emission of a great cloud of black smoke, accompanied by a trembling of the earth and a loud noise. For three days steam continued to ascend in a great column from the crater, and fine dust with lapilli and scoriæ rained down on the slopes of the mountain, covering everything with a deposit of grey ash, and injuring all the vegetation, but causing no loss of life. One is irresistibly reminded of the earlier stages of the eruption of Pelée in May, 1902,

* Blue Book: 'Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' pp. 91 *et seq.*

when for several days steam rose from the mountain, and the streets of St. Pierre were powdered over with light grey ashes.

On the morning of April 30th the violence of the eruption was obviously increasing, and the Caribs of Morne Ronde deserted their houses. The volume of steam was becoming larger and larger, and in the afternoon the noises from the mountain became louder and louder, but as yet earthquakes had not occurred in any number. As night fell the climax was at hand, and the "lava" welled over the edge of the crater, one stream flowing down to the north-west—probably from the new crater—while another was seen shortly afterwards to descend into the valleys of the Wallibu and Rabaka. The noises from the crater were now deafening, and a rain of cinders followed, which lasted, with intermissions, till 6 o'clock next morning. The earthquakes were numerous and violent, but the sea remained calm all night, and there were no tidal waves.

The account first published in the 'Evening News' states clearly that lava was seen to flow down to the north of Morne Ronde, and took four hours to reach the sea, but there is much reason to believe that in this case historical accuracy has been sacrificed to literary effect, and that many of the graphic touches in the picture are to be ascribed to the action of a powerful imagination. According to SHEPHARD another smaller outburst followed on June 8th, but did no damage.

The extent of country devastated by this eruption is sufficiently indicated by the List of Estates, to which a share of the money voted by Parliament was assigned :—

*Appropriation of the Sum of £25,000 granted to the Sufferers by the Volcanic Eruption by 53rd Geo. III,
cap. 136.*

	Estimated Loss.			Paid.		
	£	s.	d.	£	s.	d.
Robert Sutherland for Rabaka	19,378	0	0	5,300	0	0
John and Lewis Grant for Wallibu	8,261	0	0	3,900	0	0
Charles Thesiger for Duvallie's	7,800	0	0	3,750	0	0
John Cruickshank for Langley Park	8,064	0	0	2,400	0	0
Alex. Cruickshank and A. Cuming—Lot 14	6,974	0	0	2,100	0	0
Thomas Browne for Grand Sable	7,392	0	0	1,580	0	0
John Smith and Alex. Cuming for Rabaka	4,780	0	0	1,200	0	0
William McKenzie for Turama	4,006	0	0	1,140	0	0
Robert Brown for Mount Bentinck	3,718	10	0	793	0	0
Thomas Fraser for Fraser's	1,262	0	0	700	0	0
James Cruickshank for Richmond	3,528	0	0	654	0	0
John Low	378	10	0	200	0	0
Thomas Dickson	577	0	0	200	0	0
Jane Dermot	508	0	0	200	0	0
Jno. W. Carmichael	820	0	0	180	0	0
Henry Haffey	1,100	5	0	166	0	0
Fanny Cruickshank	150	0	0	60	0	0
Thomas Riddock	75	0	0	57	0	0
Alexander Clunes	250	0	0	50	0	0
Henry Charles	103	0	0	50	0	0
				24,680	7	0
Treasury Charges and Commissions				319	13	0
	£79,125	5	0	£25,000	0	0

It would seem that all the leeward side of St. Vincent, from Richmond to Windsor Forest, had suffered severely, and the Carib Country, from Grand Sable to Turema, was covered with 6 to 10 inches of ashes. At Fancy and Owia, on the northern shore, there was a certain amount of damage, but it is quite probable that, as in this eruption, they suffered only to a comparatively slight extent. The list of ruined estates is practically the same as that of May, 1902, and in particular we may note that, while the country around Georgetown was much damaged, no injury was done at Chateaubelair. The depth of ashes which fell on the Windward Estates was only 10 inches, as compared with 3 to 5 feet on this last occasion, and this leads to the belief that on the whole the eruption of 1812 was the less violent of the two.

The agricultural value of the land in the Carib country was hardly affected, as the eruption took place when the crop was over; and the same amount of sugar and rum was manufactured there in 1813 as in 1812.* Duvallie's was completely destroyed; Fraser's (north of Wallibu) recovered only in 1814; and Wallibu in 1814 or 1815.†

The loss of life was also small, and singularly little mention is made of it, as nearly all those killed appear to have been negro slaves, and their value was no doubt reckoned up in the grand total of losses. As at that time the sugar estates were highly prosperous there must have been a large population, so we may safely conclude that a hot asphyxiating blast did not pass over the villages. The injuries are ascribed to hot falling stones, and to the collapse of the roofs of the huts on their occupants. It is said that the number of lives lost was 56.‡ By the collapse of the roof of the Grand Sable house "a gentleman of the name of PHILLIPS" was killed. According to a letter from WILLIAM MACKENZIE, of Turema, very few lives were lost in the Carib Country. § Nine perished at Duvallie's. ||

In a letter from ALEXANDER CRUICKSHANKS, who was part proprietor of Lot 14,¶ we have some further particulars which are of great importance:—"The Rabaka River and the Wallibu have totally disappeared, not one drop of water being left in the channels of the Rabaka. The lava is 50 to 60 feet, and in some 80 feet, above the bed of the river, and in some places on the other side of the island the lava is about 130 feet, covering completely a fall in the Wallibu River, which was 70 feet high, not only to the top, but 50 or 60 feet above the top of it, from which the water formerly fell."

It is also repeatedly stated, in the correspondence printed in the Blue Book, that the Rivers Rabaka and Wallibu have completely dried up, and this was of

* Blue Book, p. 88; also p. 98.

† Blue Book 'On the Eruptions in St. Vincent and Martinique in May, 1902,' p. 87.

‡ P. FOSTER HUGGINS, 'Account of the Eruptions of the St. Vincent Soufrière,' p. 7.

§ Blue Book 'On the Eruptions of St. Vincent and Martinique in May, 1902,' p. 92.

|| Blue Book, p. 95.

¶ Blue Book, p. 92.

great importance, as affecting the supply of water to the sugar works and rum distilleries.

The material which filled up these valleys is called "lava," but there can be no doubt it was sand and ashes. SHEPHARD expressly states that the Wallibu was dammed up with sand, and that some years afterwards "the gradual accumulation of water burst through the sandy barrier and carried away many negro houses in its progress; 32 slaves belonging to Wallibu Estate were washed into the sea by the torrent."

In short, it is clear that after the eruption of 1812 the upper parts of the Wallibu and Rabaka Valleys were in very much the same condition as at present, filled with deep deposits of sand, which obstructed the lateral valleys and caused lakes of water and of mud to collect there. Obviously, on the night of April 30th, in the height of the eruption, an avalanche of sand swept down the mountain and lodged in the ravines on the south side of it. With it there must have been a black cloud and a hot blast, only the eruption was less violent than that of this year, and the black cloud had lost its burden of hot ash, and lifted off the surface of the ground before it reached the Carib Country, so that it mounted into the air over the heads of the inhabitants, and they narrowly escaped a sudden and painful death.

The avalanche of dust must have welled over the southern lip of the main crater, and we learn, from SHEPHARD and other sources, that after the eruption the interior cone had vanished. Major CRICHTON is said to have visited the crater three days after the eruption,* and to have found in it a lake of water. It was then in very much the same condition as before the eruption of this year—that is to say, its depth had increased to about 1600 feet, and it had a highly concave, basin-shaped floor.

It is, of course, possible to argue that this discharge of sand took the form of a mud lava, and not of a dry hot avalanche, but this explanation will not meet the exigencies of the case when we consider the phenomena on the north and west sides of the mountain. About midnight on April 30th, when the epoch of maximum activity supervened, a great outburst was noticed on the north-west side. This marked apparently the formation of the new crater. A current of red-hot matter discharged from it, passed down the valley of Larikai to the north of Morne Ronde. It is also stated (*see* p. 464) that the lava,† "immediately after boiling over the orifice and flowing a short way, was opposed by the activity of a higher point of land, over which it was impelled by the immense tide of liquefied fire that drove it on, forming the figure of V in grand illumination. Sometimes, when the ebullition slackened, or was insufficient to urge it over the obstructing hill, it recoiled back, like a refluent billow, from the rock, and then again rushed forward, impelled by fresh supplies, and, scaling every obstacle, carrying rocks and woods together in its course down the slope of the mountain, until it precipitated itself down some vast ravine concealed

* P. FOSTER HUGGINS, 'An Account of the Eruptions of the St. Vincent Soufrière,' p. 94.

† Blue Book, p. 94.

from our sight by the intervening ridges of Morne Ronde. Vast globular bodies of fire were seen projected from the fiery furnace, and bursting fell back into it, or over it on the surrounding bushes, which were instantly set in flames. About four hours from the lava boiling over the crater it reached the sea, as we could observe from the reflection of the fire and the electric flashes attending it."

Now this looks very circumstantial, but the fact remains that when we were at the mouth of the Larikai this summer we saw no lava in the bed of the stream there. This torrent flows down a tremendous gorge, cut in the old lavas and tuffs of this part of the mountain. On the shore below there had been a small alluvial fan, but, as already mentioned (p. 454), it had slid away and disappeared during the eruption of this year. In the mouth of the gorge there was a low cliff, some 50 feet high or more, capped by a few feet of the new, hot sand. The cliff consisted of a material so exactly similar to the ash that overlay it that, if it had not been for the old burnt soil, it would have been hardly possible to find a line of demarcation between the two deposits. We were at once struck with the similarity, and came to the conclusion that here was the evidence of a dust avalanche eruption previous to that of this year. We could not land, as the bare cliff of volcanic sand was tumbling in frequent landslips into the sea, but we rowed quite close in, and could make out all the more important features of the deposit. It was almost unstratified, and had a few stones scattered through it, but for the most part was a yellow or brownish incoherent sand. It could not possibly have been an alluvial deposit, as it showed so little stratification, and such a torrent as the Larikai could not be supposed to lay down 50 feet of sand in any part of the precipitous ravine. Further up we could see where it had been rolling down enormous boulders during the recent heavy rains. This was the last deposit in the Larikai, and it bore all the marks of a dust avalanche. It lay in the old eroded gorge behind a bend, where a projecting mass of rock had protected it from erosion.

On the north-west corner of the Soufrière lay the estate of Duvallie's (De Volet's), also known as Windsor Forest.* This was at that time a sugar estate, but is now devoted to grazing and cocoa. It is stated to have been "entirely covered with the matter thrown out by the volcano; the sugar works totally covered and not discernible; nine negroes killed, the rest escaped over the mountains and came to town much cut and bruised." †

According to SHEPHARD it was "entirely covered with sand and ashes." This may have been the work of a mud lava, but it is more probable that from the position of the new crater that its emissions were directed principally towards the north side of the hill, and that the black cloud which descended there was even more destructive than in the present year, when most of the valleys in this quarter

* P. FOSTER HUGGINS, 'An Account of the Eruptions of the St. Vincent Soufrière, 1902,' p. 7.

† Blue Book: 'Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' p. 95.

contained deposits of ashes which for some days prevented the flow of the streams. The estate was thereafter abandoned for a number of years.

It is universally believed in St. Vincent that the ejecta of this eruption which gathered in the valley of the Rabaka Dry River greatly modified the volume of the water in this stream and the regularity of its flow, which has hitherto been more or less continuous, like that of the majority of the rivers in the island. Thenceforward it was a "dry river"—that is to say, its capacious channel contained no water except immediately after rain. The bed of the stream is probably 200 yards in width, and was usually occupied only by black sand, mud, and boulders, so that it could be crossed dry shod, but at times it was suddenly filled from bank to bank with a rushing torrent which could not be forded, and, as there was no bridge, the north end of the island on the windward side was then cut off from communication with the rest. So mysterious and unaccountable did those periodical floods seem, that many believed they were due to some strange outflow from the crater lake by means of subterranean passages.*

The explanation usually accepted was of a more simple character†:—

"The Windward slopes of this portion of the range are drained by a channel called the Dry River, which runs through the Carib Country, and which from its peculiarity deserves notice. Before the eruption of 1812, a stream of average size filled this now dry watercourse, and emptied itself into the sea. During the eruption, the channel of the stream was completely filled and choked with scoriæ, rocks, and gravel, underneath which the water now, in ordinary times, disappears some distance before it reaches the coast and finds its way to the sea. In floods, however, the water comes down with singular force and volume, filling the rocky bed, which is 200 yards across (where the highway passes it), from bank to bank. The water is described as advancing in huge waves, like the bore of a tideway. On these occasions it is very destructive, and it has already washed away many acres of cane land from the estate of Langley Park, situated on its bank."

Before we reached St. Vincent the conditions had again been entirely altered by the eruption of this year. It is not possible now for us to be certain exactly what were the causes which so modified the behaviour of this stream about the year 1812. The explanation offered—viz., that the valley above was choked with sand and scoria—is, at any rate, credible, as if these materials were dry and porous they might absorb much of the water before it could reach the lower levels. It also strongly confirms SHEPHARD's statement, that the Rabaka River was filled with sand, and strengthens the evidence in favour of the eruption of 1812 having been characterised by the emission of an avalanche of dust.

For the last mile of its course, before reaching the sea, the Rabaka Dry River flows through what is known as the "lava bed." In this it has cut a wide channel, and on each side it is flanked by two or three well-marked terraces, the traces of former

* P. FOSTER HUGGINS, 'An Account of the Eruptions of the St. Vincent Soufrière,' p. 9.

† 'Historical and Descriptive Sketch of the Colony of St. Vincent, W.I.' (Compiled under the direction of the Commissioner for the Windward Islands, 1891.) By T. B. C. MUSGRAVE, p. 3.

levels at which the stream flowed as it gradually eroded the mass which had obstructed its older and still wider bed. The "lava bed" is composed of black mud, sand, and stones, in which there is also a considerable amount of charred timber. It is essentially a recent formation, which has filled up and obliterated the course of the stream at no very distant date, as it differs considerably in appearance from the rocks of the Carib Country, in which the primitive valley of the Rabaka River had been carved, and has altogether a much newer aspect than the weathered tuffs exposed on the sea cliffs and in the higher slopes and ravines. Many believe that it is a product of the eruption of 1812, but we could find no very conclusive evidence of this, and there is no inherent reason why it may not have been due to some earlier eruption.

It faces the river in low, vertical banks from 6 feet to 20 feet high; its upper surface is perfectly terraced, and shows two or three benches several feet apart. Only two explanations suggest themselves to account for its presence there. It may have been a hot sand avalanche, and, if so, it cannot have accumulated during the eruption of 1812, for when so great a mass of débris was projected into the lower part of the Rabaka Valley the accompanying hot blast must have been tremendous, and would have annihilated every living thing for miles around. But it does not look like a sand avalanche; in the stream sections it too often shows a well-marked bedding, and it contains many rounded blocks of lava, which certainly appeared to be water-worn boulders, such as are common in the valleys. Everything pointed rather to its having been a great mud lava, which had swept down the upper parts of the river's course with a high velocity, and had caught up and incorporated the gravel and boulders over which it passed. Then, when it reached the flat country at the lower end of the valley, it had been unable to flow further, and had come to rest, a great glacier of black mud and stones, which filled up the broad but shallow channel into which it had flowed. Such floods of mud followed the eruption of 1812, and may possibly occur in the near future as a consequence of the eruption of this year, and the discharge of such mud lakes as are forming in some of the higher streams would furnish most of the conditions necessary for the obstruction of the rivers in their lower parts by masses of stony mud exactly resembling the "lava bed."

The "May Dust" in Barbados, 1812.

Early in the morning of May 1st, 1812, sounds as of distant cannonading were heard in Barbados, and it was generally believed that a naval engagement was taking place somewhere off the coast. The garrison prepared to repel any attack. At Rozeau, in Dominica, similar noises were heard a little after midnight, and the regular forces were placed under arms and the militia called out. In Barbados dust began to fall about half past 1 o'clock, and there was intense darkness, but in Dominica there was no darkness and no fall of dust. For several days afterwards

there was great excitement, as ship after ship came in reporting that they had had the same experience, only varying in degree, and no explanation was obtained till word came from St. Vincent that the Soufrière had erupted.

SIR ROBERT SCHOMBURGK, in his history of Barbados, gives the diary of a gentleman residing in St. Peter's parish in the island, containing his observations on the unusual occurrence * :—

“At half-past 12 A.M. on May 1st, 1812, a heavy, dark cloud obscured the heavens completely, hanging so low as apparently to touch the ground, except in the south and north-east, where there was a fine, light blue tint, which closed in at half-past 1 A.M., when darkness visibly overspread this part of the island. At this period a sandy grit began to fall in small quantities. At 2 A.M. explosions heard to the southward and westward, resembling two frigates exchanging broadsides, to the amount of 18 or 20; went to the top of the house, but could perceive no flashes, though the sound seemed sufficiently near, light being perceptible at a much greater distance than sound can be heard, the sandy grit, converted into ashes, silently falling. From 2 to 6 A.M. low, murmuring, hollow distant thunder, but no lightning seen, except the vivid flashes which preceded two nearer peals. Between these periods smart squalls with rain and ashes mixed from the eastward, which seldom lasted above 40 seconds, the ashes bearing a greater proportion than the rain in this composition. At half-past 5 A.M. a small glimmering in the south and south-east resembling the appearance of daylight, but did not last 10 minutes before the atmosphere was completely obscured again, and the darkness more intense, if that was possible. At half-past 6 A.M. heavy fall of ashes, with light breezes and a hollow, low, undulating noise to the northward; expecting an earthquake, quitted the house and retired to a wattled negro hut. From 6 to 8 A.M. light breezes, with squalls of ashes and rain of the same description and duration as mentioned before. During these last two hours meteors resembling globes of fire, about the size of a 13-inch shell, appeared in the north-east and north-north-east, to the amount of 10 or 12, crossing each other in every direction, occasionally appearing and disappearing for the space of an hour and a half; so incessant a falling of ashes as to render it impossible to face the eastward. At 9 A.M. the sky to the northward assumed a purple torrid appearance, greatly resembling a vast town at a distance on fire, accompanied by a tremulous motion resembling the Aurora Borealis. The horrid glare of this sky made the surrounding darkness more awfully dreadful; the sky to the southward, in the direction towards Bridgetown, had occasionally the same colour, only the tinge much fainter, attended with no motion. The sky never approached in any direction by my calculation nearer than 7 miles; as I have no data to go on this is a mere matter of conjecture. From 9 A.M. to 12 at noon light breezes and constant and heavy fall of ashes. At 10 A.M. a large flight of birds passed over the hut, flying so very low that the fluttering of their wings was distinctly heard; the notes of these birds resembled the yelping of puppies. When daylight took place they proved to be marine birds, called mien-of-war and cobblers, so loaded with ashes they could scarcely raise themselves from the ground. At a quarter past 12 daylight appeared immediately over our heads; half-past 12 the form of the sun, obscured in clouds in the same place. At 1 P.M. daylight; returned to my own house. From 1 A.M. to half-past 12 P.M. the wind east to east-north-east; 1 A.M. light gentle breezes never varying above two points, but fluctuating between both, the wind dying away nearly to calm, but never perfectly a calm. This may be said to be the state of the weather during the whole 12 hours of total darkness, except when interrupted by the momentary squall of sand and ashes. The darkness was so impenetrable that, with the exception of the light that was visible in the south and south-east at 5 A.M., at no period could anything be discerned even within reach. From three admeasurements taken in the lowest places the fall of ashes was an inch and a half. When I left the house the thermometer was 70°, when I returned at 70°; as I left the instrument behind I know not what variation might have taken place in my absence. The other

* ‘The History of Barbados,’ by Sir ROBERT H. SCHOMBURGK, 1848, p. 72.

observations were made with my own eyes, and the watch in my hand. It will be observed the first two hours the sand was small in quantity and coarse in its nature, but the last 10 hours were ashes, reduced to an impalpable powder, and sublimated to the highest degree. That it is a calcined matter strongly impregnated with nitre and ferruginous particles does not admit of a doubt, if examined through a good microscope; and that it has come from the *eastward* may be supposed from its involving in its mass the men-of-war birds, which are generally found about 60 miles to the east end of the island, seldom approaching nearer. From 1 P.M. to 6 the fall of the ashes began to decrease; at 6 P.M. ceased altogether. At no period of the day did the light amount to more than a dull twilight, and at 5 P.M. the day closed altogether, and darkness succeeded until the morning."

According to the contemporary account already quoted (p. 464), it was shortly after 7 P.M. that the "lava" overflowed the north-west side of the crater, and about that time the noises from the mountain were very loud. These reports, apparently, were not heard at Barbados, or, if heard, were mistaken for thunder. About half-past one another overflow took place on the south side, towards the valleys of the Rabaka and Wallibu, and the detonations which attended this outburst were heard in Barbados at two in the morning, according to the diary. Ashes were falling in small quantity between 2 o'clock and 6 o'clock; there was much lightning, but it was not very dark. When the day broke there was a thick mist of falling dust, and at half-past six there was a considerable increase in the rain of ashes. This means, probably, that the material ejected by the great outburst at 2 A.M. had taken four hours and a half to reach Barbados, a distance of 99 miles. This year the main explosion was approximately at two in the afternoon, and the dust was falling freely in Barbados at half-past five, having taken only three and a half hours on its journey.

In 1812 ash began to fall "between 2 A.M. and 6 A.M.," fell freely from 6 A.M. to 1 P.M., and ceased altogether about 6 P.M., a duration of somewhat over 12 hours. In 1902 ash began to fall about 5 o'clock, and continued till daybreak next day, or approximately 12 hours also. In 1812 it fell mostly in the day; this time mostly through the night. The weather was dry this year, and the dust formed a fine, dry powder easily blown about by the wind; in 1812 there were frequent heavy showers, and much of the ash fell wet. Otherwise the two records are as similar as could well have been expected.

Another account of the rain of ashes in Barbados in May, 1812, has come down to us in the form of a private letter from a gentleman of St. George's parish to a correspondent in Great Britain. It is less precise and full of details than that just quoted, and was printed in TULLOCH'S 'Philosophical Magazine' of the year 1812 (p. 71).

In the morning at half-past six, when it should have been bright daylight and the sun above the horizon, he was astonished to find that it was still so dark that he could only compare it to moonlight on a night when the moon is at times clear and at times obscured by cloud. He had not heard the detonations about 2 o'clock in the morning, though they had led to a general opinion that an engagement between British and French ships of war had taken place somewhere in the neighbouring seas,

and preparations had been made among the troops to repel attack. Thick clouds covered the sky except near the southern horizon, where a bright light shone through a thin cloud-veil. The morning was still and calm, and the appearance of sky and air so unusual that he was apprehensive of an impending catastrophe, and hastened to the house of some friends. By 7 o'clock it was much darker, at half-past seven candles were required; by eight it was pitch dark—"so dark that we could not perceive our hands when held up before our faces at two feet distance." This continued till 12.25 P.M., when the light returned and rapidly increased till near objects could accurately be perceived. About 3 or 4 o'clock the light was fairly good, but the air hazy. Fine ashes fell all day till 8 o'clock at night.

"In order to ascertain the quantity which had fallen, Mr. H. last night took up that which lay upon a foot square, when it measured three pints somewhat pressed into the measure, and weighed $1\frac{3}{4}$ lb.

"This morning another square foot, where the surface was hard and level, gave, in $\frac{5}{8}$ inch and $\frac{1}{2}$ inch in depth, three pints loosely filled up in measure, and 1 lb. $7\frac{1}{2}$ ozs. in weight.

"Against the bottoms of windows, doors, and walls it was considerably deeper. But assuming the product of my experiment as the medium quantity which fell on a foot square throughout the island, and estimating from our best maps the quantity of land in the island at 106,470 acres, the total quantity of this extraneous substance which is now on its surface, independent of that which is on the trees, could not be less than 1,739,187,750 gallons wine measure, or 6,811,817,512 lb. avoirdupois."

According to this estimate, a total of about 3,000,000 tons fell on the surface of Barbados, which is nearly twice that estimated to have fallen this year. Although both accounts differ somewhat in respect of this point, they agree in making the fall of ashes greater than it was in May, 1902, but it is very doubtful if the old estimates are anything like as accurate as the modern ones.

Further Activity in 1814 and in 1880.

It is generally supposed that in 1812 the Soufrière had relapsed into repose, and that the lake which has occupied its crater since the last great eruption has been undisturbed by volcanic emissions till the beginning of this year. But there is good evidence that this is not altogether the case, and we are indebted to Dr. NICHOLLS, C.M.G., of Dominica, a gentleman who has done much for the advance of scientific knowledge regarding the Caribbean Islands, for bringing to our notice a most interesting correspondence which appeared in the 'Trinidad Chronicle' in 1880. Had his quick eye not detected its importance at the time, and had he not carefully preserved it, we should certainly have missed it altogether. Inquiries made at the Colonial Office showed that there was no mention of volcanic activity, on the dates mentioned, in the official papers and despatches.

It seems that in September, 1880, reports reached Trinidad that apprehension was being felt in St. Vincent regarding the state of the Soufrière, and that an eruption was feared. On September 28th the following letter was printed in the 'Trinidad Chronicle':—

" The Soufrière at St. Vincent.

" To the Editor,

" 'Trinidad Chronicle.'

" DEAR SIR,

" I observe that, in noticing what the fears of those who dwell in the vicinity of the Soufrière in St. Vincent lead them to regard as possible signs of a coming eruption, you state that the volcano has been quiescent since the great eruption in 1812. This is not quite correct. There was the small eruption of 1814, of which I have a manuscript account by an eye-witness. This has never been published. I think a few extracts from it may not be uninteresting to your readers :—

" ' On Sunday, January 9th, 1814, about 1 o'clock P.M., I observed a cloud of smoke issuing from the Soufrière. A part appeared to roll down the side of the mountain towards Wallibu ; and a large column shot upright to a great height. It continued to rise for upwards of half an hour, when it was detached from the mountain and proceeded in a compact body in a direction nearly opposite to that of the (lower stratum of) wind at the time. At about 5 o'clock it had reached the horizon, and before six had entirely disappeared. During its passage overhead the heat was excessive, but I had no thermometer to ascertain its degree.

" ' The eruption was preceded by loud noises . . . like the discharge of distant artillery. . . . It was preceded by a severe earthquake. Dr. , who was in a favourable position for observing the craters, not only remarked an intense light to issue from them, but saw rocks thrown to a great height, which seemed to fall back into them. All appearing quiet on Monday and Tuesday, the craters were visited on Wednesday. The eruption proved to have been from the old crater. Large rocks had been ejected to considerable distances, some having been found a quarter of a mile from the edge of the crater, which is strewn with them. There is little alteration in the appearance of the crater, but the water in it is boiling with great violence. The new crater remains unaltered. The rocks around are shattered in several places. One of these fractures is very large. . . . Many of the pieces which have been ejected . . . are entirely unaltered by fire . . . but many have been evidently acted upon and altered by fire, but none retained any heat. . . . Amongst others was a curious specimen of what I observed after the great eruption. It has the colour and somewhat the appearance of dry sponge, but is very friable, and seems principally composed of slender glass-like filaments, slightly connected, and enclosing numerous pieces of spar and other mineral substances.'

" I may mention that a correspondent of mine states, in addition to the particulars which you quote from your correspondent's letter, that all the beautiful vegetation, which for 60 years has adorned the sloping sides of the old crater, and has always been such a remarkable feature in the scene, has been destroyed, burnt up as if by fire, it is supposed by volumes of gas, which have rolled round the crater and overlapping its edge have rolled some way down the leeward side of the mountain and there, likewise, destroyed the vegetation. Mountain pigeons and some other birds are commonly seen on the heights—these seem to have forsaken the dangerous neighbourhood, and the melodious notes of the mysterious, and as popularly believed invisible, Soufrière bird are no longer heard.

" (Signed) H.

" September 28th, 1880."

The notes by an eye-witness of the eruption of 1814, contained in this letter, are so explicit and carefully worded that there can be little doubt that they describe a real occurrence, and not merely those deceitful shapes assumed by the drifting trade-wind cloud which frequently mislead superficial observers, and start the propagation of baseless rumours of eruptions. But it is quite improbable that any considerable disturbance took place, or that any great activity was manifested. We note

especially the absence of the premonitory symptoms which invariably at the Soufrière have preceded any resumption of volcanic action, and as the crater had then been quiescent for eighteen months (since June, 1812), it is unlikely that any violent eruption should have broken out without giving warning beforehand. It is a rule, to which we know of no exceptions, that the longer the period of quiescence at the Soufrière, the more marked are the disturbances and earthquakes which are observed before an eruption. The outburst of May 18th this year gave apparently no warning, though of this we cannot be quite certain, as it took place after dark; before all the others, earthquakes have been numerous, violent, and continued.

It is said that a steam column shot upwards from the crater and, reaching the higher currents of the air, floated away on the anti-trades. It spread out to form a cloud which, as the emission was brief, was soon separated from its parent stem and, having become detached, was borne away to the horizon. There is no description of any rain of ashes, but a few days afterwards large stones were found lying around the old crater, some of them several hundred yards from the rim. As by that time a covering of green vegetation had probably formed in some part of the upper slopes, which had been devastated two years before—though Mr. ANDERSON, in 1784, describes them as quite barren and stony—and as the observer was resident near and well acquainted with the volcano, there is not much chance of his having been in error on this point. Their distribution led to the belief that it was from the old crater the steam arose, which is quite likely, as undoubtedly that crater had taken the chief part in the eruption of 1812, and had discharged the great masses of sand which blocked the Wallibu and Rabaka Valleys, and were accompanied by the loud detonations that awakened alarm in Barbados.

This description of the steam cloud, “part of which rolled down the side of the mountain,” is curiously reminiscent of the behaviour of the great black cloud. But it is in no way probable that this phenomenon appeared on this occasion. It has never been known, except in the major eruptions, and had an avalanche of dust rolled down the hillside it would have left traces visible when the mountain was again visited three days afterwards, and too obvious to have been overlooked. Moreover, we are told that the water was still left in the crater, and little change was to be perceived in the aspect of the interior. Now the great black cloud never appears till after the crater lakes are emptied, and the ascending column of lava has forced its way to the surface.

In all probability this was merely an exacerbation of solfataric activity, such as has been experienced by more than one of the Soufrières of the Windward Islands, and after the emission of a great mass of vapour and sulphurous gases the short-lived eruption came to an end. Such an emission would carry with it many stones from the bottom of the crater lake, would blast any vegetation growing within the crater, produce minor changes in some part of the crater walls, and project into the air a column of mud and dense muddy vapours which would fall partly outside the lip and

flow for a short distance down the slopes, giving rise to clouds of steam as it coursed over the surface. The discharge of steam thereafter for some time would keep the crater lake in a state of furious ebullition.

Apparently again, in 1880, a considerably increased production of sulphuretted hydrogen had killed the growing bush within the crater. This is a not uncommon phenomenon around the Soufrières of Dominica and St. Lucia, where every increase or diminution in the amount of the poisonous gases emitted is attended by a widening or contraction of the surrounding region in which the vegetation has been killed, and the blackened, blasted stems of trees scattered through the green forest are the standing witnesses of epochs during which the noxious fumes were able to increase the area over which they had control, but which they were unable permanently to dominate.

THE SOUFRIÈRE AND MONTAGNE PELÉE.

Their Resemblances and their Differences.

It is not possible at the present juncture for us to give more than the main points of similarity and of difference in these two mountains, and the features of their activity. Our own visit to Martinique was short, and was brought to an unexpected termination by the eruption of July 9th. Smaller outbursts followed during the next two days, and neither was it safe to work on the mountain slopes, nor could porters be obtained in Carbet, which had been deserted on the night of July 9th, who would undertake the ascent. We had, indeed, planned an excursion to the southern lip of the crater on the day after the eruption (July 10th), but all the arrangements we had made were nullified by the sudden increase of volcanic activity on the previous evening, and without much delay it would have been impossible to make an examination of the upper slopes that face St. Pierre. We had, however, spent two days in the ruined city, and examined the desolation wrought in the historic catastrophe which overwhelmed it, and had made a more or less cursory inspection of the lower fringes of the volcano and the fields and bluffs around the town. But there were many points of great importance into which we had not had time to enter. To Professor LACROIX, of the Scientific Expedition sent by the Académie des Sciences of Paris, we are indebted for much valuable information, and for his kind offices with the officials and others in the island of Martinique. His preliminary account of the results of his first visit to the scenes of the eruption* has furnished us with additional details. In Fort de France we had the privilege of renewing our friendship with Professor JAGGAR, of Harvard, whom we had previously met in Barbados, and he and Mr. ROST, photographer to the United States Geological Survey, discussed with us most frankly the results of their observations, and the conclusions to which they had been led.

Much has already been written on the great eruption of Pelée, and the havoc it wrought in St. Pierre, but mostly from a popular point of view, and without reference

* 'Comptes Rendus,' vols. cxxxiv. and cxxxv.

to the causes which underlie the phenomena. Articles of more permanent value have appeared also in certain magazines and scientific periodicals, but perhaps we are right in saying that the geological history of that catastrophe has yet to be penned, and as more than one party of scientific men have addressed themselves to the task, there should be available, in no great lapse of time, the results of careful and thorough examination of all the effects of the eruption, with a judicious and critical digest of the evidence of what actually took place on the fatal morning of May 8th. We are well aware that much of what has been published is quite untrustworthy, and unsuited for scientific discussion, and we will rely mostly on the results of our own observation, and on information given us by Professor LACROIX and Professor JAGGAR, but we have also made more or less use of the articles which have appeared in the newspapers of Guadeloupe, Martinique, Barbados, Trinidad, and in the English journals, and of the reports by Professor R. T. HILL and Professor ISRAEL RUSSELL to the National Geographic Society of the United States, and their articles in the 'Century Magazine.' *

The resemblance between Pelée and the Soufrière in their geological relationships is so striking that it cannot fail to impress the most casual observer. Each stands at the northern end of its respective island, an isolated volcanic cone, bearing a summit crater or craters, and with its skirts descending to the sea on all sides except the south. Here in each case there is a broad, flat depression, more marked in St. Vincent than in Martinique, on the south side of which rises another volcano, or group of volcanoes, extinct, highly eroded, but still bearing in its internal structure, and less obviously in its external configuration, the proofs of its origin. The Piton de Carbet, which lies to the south of Pelée, has not been in eruption since the European colonisation of Martinique; but its crater, though partly destroyed, is not yet completely effaced by erosion, and like its analogue, the Morne Garu at St. Vincent, it seems to have been the last volcano in the island which has died out and become extinct.

In both islands the older volcanic piles lie in the south end, which consists mainly of lavas and agglomerates emitted from foci which have long since passed into repose. The lavas are andesitic, and alternate with vast sheets of coarse agglomerate, which testify to the frequency and violence with which explosive action took place when activity was at its maximum along the Caribbean chain. The epeirogenetic movements which have affected this border ridge between two oceans have left their marks in the raised beaches which are found along the shores of both islands, and the same conditions of accumulation and uplift, with intense and rapid erosion on steep slopes attacked by tropical climates and tropical rainfall, have in each case resulted in deep sculpturing by the agencies of subaerial denudation. The ravines of Pelée have

* R. T. HILL, "A Study of Pelée," 'Century Magazine,' September, 1902, vol. lxiv., p. 764. ISRAEL C. RUSSELL, "Phases of the West Indian Eruptions," 'Century Magazine,' September, 1902, vol. lxiv., p. 786.

all the depth and picturesqueness of those of the Soufrière, and where the volcanic blasts have withered up and swept away the vegetation, the same general resemblance to the rugged, naked cañons of the western American desert country is seen in both cases.* Vertical cliffs of lava alternate with sloping taluses of ash; every variation in the underground structure is reflected in the contours of the surface, and where the rich Antillean forest clothes the eroded surface, the beauty of the landscapes and the variety of colour lend a striking charm to the distant hills and the highly-cultivated shores.

In both mountains the conical volcanic form is well exhibited, though scored with deep radial ravines. Pelée is 4428 feet high, while the Somma rim of the Soufrière is 4050, but had the latter mountain not lost by some great explosion the upper part of its cone, it would probably have been somewhat the more lofty of the two. In each case the diameter of the cone at sea level is almost 8 miles, and the restless action of the waves has eaten back the land, and formed ranges of lofty cliffs which face the ocean.

At the summit of the Soufrière we have a concentric structure—a crater within a crater. The great convulsion during which the huge Somma crater† of the mountain was produced, antedates authentic history. The lower lip of this great “caldera” must have been the southern, near which the main crater of the present day stands, and this in turn bears on its north-east side another smaller parasitic vent (that of 1812). Pelée had at its apex a single small crater surrounded by a serrate range of cliffs some 200 feet high or less, and in the bowl-shaped depression lay a little lake, the Lac des Palmistes, 150 metres in circumference. But on Pelée were numerous lateral orifices, the parent sources of “soufrières” and hot springs, one of which, at Ajoupa Bouillon, has since attained notoriety as the focus of a minor eruption on September 3rd, 1902. Another lay near the gorge of the Rivière Sèche to the south of the summit; it emitted steam and sulphuretted hydrogen, and around it the rocks were decomposed by acids and often crusted over with sulphur. This was surrounded by high cliffs, and was very rarely visited, but from near the top of the mountain a view might sometimes be obtained of its interior. It lay apparently on one side of the cañon, and from the descriptions of it which we have received must have closely resembled many of the soufrières of Dominica.

This Soufrière has now assumed a new importance, for, according to some accounts, it was from it that rose the cloud of suffocating gases and red-hot dust which laid the fair city on the shore below in ruin and ashes. When we were at St. Pierre great towering balloon-shaped steam clouds would frequently ascend from the neigh-

* TEMPEST ANDERSON and JOHN S. FLETT, “Preliminary Report on the Recent Eruption of the Soufrière in St. Vincent,” ‘Proc. Roy. Soc.’ vol. lxx., p. 11, 1902.

† Mr. R. C. HILL suggests (‘National Geographic Magazine,’ vol. xiii., p. 233) that the upper part of the mountain was blown away in the eruption of 1718. Mr. Anderson’s description of the crater as it was in 1784 disproves this.

bourhood of the fissure in this gorge, and, as will be seen later, it had still not lost its virulence and destructive energy.

The Lac des Palmistes is now filled with hot stones and sand to within a few feet of the lip, and the present crater lies to the south-west of it, apparently on the site of the former Soufrière, which has been much enlarged, and a great fissure has been opened on its southern lip.

The Eruptions of Pelée.

The only previous eruption of Pelée of which there is historic record is that of 1851, which appears to have been brief and abortive. A few mud-flows in the rivers, a discharge of steam and a fine ashy dust, and the volcano relapsed into quiescence for another 50 years. The earlier stages of the eruption of this year were of a precisely similar character, and this was one of the causes which lulled the suspicions of the inhabitants of St. Pierre, and inspired them with a fatal feeling of security. The premonitory earthquakes, so frequent always at the Soufrière, where they have heralded the outburst of every eruption, and prepared the minds of the inhabitants for the approaching paroxysm, were apparently practically absent in Martinique, and although there are records of a few small shocks, we are not aware that they were numerous or disquieting. But at Pelée there was a long preliminary phase leading up to the crisis. Steam was seen to ascend from the crater about April 23rd, or a fortnight before the culmination of the eruption. At first the activity was so gentle that it awakened only curiosity, and several people made ascents and reported that both the upper and the lower Étangs were boiling and giving out much steam. But day by day the violence increased, and by May 2nd a good deal of apprehension was felt in St. Pierre, and some were meditating flight. That night there was great activity in the craters, loud noises were heard, and over the mountain top a bright glare was visible. The ashes were wafted by the wind over St. Pierre and Prêcheur, and next morning these were covered with a layer of light grey dust. The crops were blighted, the cattle starving for want of water and of food; the people from the country began to flock into town. The noises and the rain of cinders and of dust continued more or less intermittently from that time onward.

The rivers descending to the south-west from the summit had already become muddy on more than one occasion, but on the 5th a great flow of hot mud poured down the Rivière Blanche with great rapidity and overwhelmed the Usine Guérin, killing many people. At the same time there was a small sea wave which did no damage. This mud-flow was apparently the waters in the Étang Sec which had escaped through some fissure on the south side of the peak, driven out by the ascending column of lava and the constantly increasing pressure of the gases within the volcano. Next day the streams were still torrents of mud, and the harbour of St. Pierre was covered with floating trees and the wreckage of bridges and other

structures swept away by the floods. No better instance could be cited of the difference between Pelée and the Soufrière in the rapidity with which the crisis of the eruption came than this :—in Martinique the discharge of the crater lake took place four days before the climax arrived ; at St. Vincent the first overflows were seen only three hours before the avalanche of dust swept down the mountain slopes.

From May 2nd onwards, no doubt remained that the volcano had resumed activity, and as each day passed the outlook became more threatening, the discharges of steam more violent, the detonations louder, and the rains of ashes more frequent and heavier. On the 4th the cloud of dust was so thick over the leeward side of the mountain that the steamer "Topaze" could not call at Prêcheur. The ash was very fine and light grey, resembling cement or flour, and covered the trees and shrubs as if there had been a light fall of snow. Animals were dying of thirst and hunger ; birds, overcome by the fumes, or weighted down by the dust on their plumage, were lying dead by the sides of the paths ; the crops were withering, and the outlying districts already abandoned and deserted. On the 6th, and again on the 7th, loud noises rose from the crater, and the red glare was visible in the steam cloud. But as yet no earthquakes ; even though a rift could now be seen to have opened at the base of Morne Lacroix.

Wednesday, May 7th, the day of the great eruption of the Soufrière, was not marked by any special features in Martinique.* That day the mountain was, in fact, rather less violent than it had been on the 6th ; but on the following morning, at 7.50, the great convulsion came, and with it the end of all things for St. Pierre.

So deadly was the blast that swept the city, so awful in its completeness the destruction that it wrought, that few survived who saw the great black cloud descending from the mountain, but of those few there are some who have placed on record what they saw, and it is clear from their descriptions that in Martinique there was a repetition of what had happened in St. Vincent on the previous afternoon. The mountain burst open and a great cloud appeared near its summit. It arose with a loud, growling noise, and some say that in it they saw a bright red glare. Like an avalanche it poured upon the city, covering the distance in a few minutes, and enveloping all in total darkness. It passed almost as rapidly as it had come, and when the darkness lifted a little, it was seen from the ships lying in the harbour that the city was razed, and fierce fires had broken out in many places. The north end of the town was practically wiped out in an instant : nothing was left but blazing ruins, the inhabitants perished where they stood. But in the south end the devastation was not so complete, the walls were left standing in many of the houses, and the

* In the 'Report of the Commissioners of the Academy of Sciences' it is remarked that a tidal wave was observed on the 7th May both in Martinique and Guadeloupe, and that it did not correspond to any eruption (of Pelée) ; needless to say that was the day of the great eruption in St. Vincent. MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, "L'Eruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 390.

living occupants rushed into the streets, yelling with pain and terror, terribly injured, throwing themselves into the sea to mitigate the agony of their burns. It was no earthquake that levelled the town, neither was it lightning nor the weight of ashes. All who saw that calamity and have survived agree that a mighty blast came with the cloud and mowed down everything in its path. The origin of the conflagration is not quite so clear, and there may have been more than one cause. Lightning may have ignited some structures, and the fires in some of the houses may have played a part in setting the ruins ablaze. But the cloud was filled with hot ashes, and we have no doubt that, especially in the north end of the town, the temperature of the dust was sufficiently high to ignite combustible substances.

Captain FREEMAN, of the "Roddam," has described his fearful experiences in language so terse and vigorous that it is well suited to the occasion* :—

"At about 8.15 he was in the chart room; a good many of the sailors were leaning over the side of the vessel watching the distant mountain, which was emitting dense clouds of smoke and occasional flashes of light. Mr. CAMPBELL was talking to Mr. PLISSONNEAU on the deck. On a sudden he (the Captain) heard a tremendous noise, as though the entire land had parted asunder. Simultaneous with the noise there was a great rush of wind, which immediately agitated the sea, and tossed the shipping to and fro; he rushed out of the chart room, and looking over the town and across the hills he saw a sight he cannot describe. He remembers calling out to Mr. CAMPBELL, and saying: 'Look!'—then an avalanche of lava was upon them. It immediately caught the town afire as it passed over it, likewise the shipping. It struck his ship with the force of a mighty hammer, and the lava rained upon the deck. Everyone, as far as he could see, sought shelter at once, but the heat was so great, and the air so suffocating, that Mr. CAMPBELL and many of the crew, among whom was the chief mate, threw themselves in despair overboard. Some crawled from where they had hidden themselves on to the deck to obtain a breath of air, and were roasted upon the fiery hot ashes. He did not lose his head, his first thought was to try and save his ship and such of his crew as were still alive. He rang the bell for full speed astern, and the heroes below turned on the steam. He had time to slip his anchor, and he was off. As his steering gear was rather difficult to manage he once or twice nearly ran foul of the steamship 'Roraima,' which was on fire. He saw two still figures standing on the bridge with arms folded heroically awaiting their end. One of them waved a good-bye to him. There were a good many passengers on board, these were rushing up and down in anguish. When he was steaming out of port he looked down at the burning city. A pall had enveloped it, but through it he could plainly see the skeletons of burning houses, and the shadowy figures of men and women running hither and thither in their terror, and above the loud din of the falling cinders, the roar of the raging sea, he heard the agonised cries of 30,000 voices."

The cloud was red hot when it emerged, or, at any rate, a red glare was seen in the fissure from which it leaped. As it swept down it was black to those who saw it coming, and lightnings scintillated in its front. With it came the mighty wind which capsized the "Grappler," ruined the town, and laid the "Roraima" and the "Roddam" on their beam-ends till the water poured in through the lee ports. The dust cloud followed in an instant. The sea raged as in a storm. Its surface hissed with the hot dust, and must have been nearly boiling. Even late that night the engineer of the R.M.S. "Esk" found the temperature of the water of the bay five

* Blue Book 'On the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' p. 45.

degrees above the normal for the time of year. The wood deck of the "Roraima" was set on fire; the "Roddam" had iron decks, and was lying further to the south, in the quarantine station. Some of the crew of the "Tereso Lobico" also escaped. The rigging of the ships, where it was not cut away by the blast, was charred or set on fire by the heat of the dust.

It does not appear that the smell of burning sulphur was overpowering as in the black cloud at St. Vincent. There it was noticed by all; but in Martinique comparatively few of the survivors mentioned it. The hot dust entered nose, and mouth, and throat, and some stuffed their caps in their mouths to prevent getting burnt in the respiratory passages. They had great difficulty in breathing, and felt choked; some noticed a gentle return current, which brought fresh air and relief to the suffering. The burns on the bodies of those least severely injured were often beneath their clothes, which had not been ignited or destroyed. In the north end of the town the corpses were superficially charred; their clothing had often entirely disappeared. It is probable that in many cases death was instantaneous, though many of the stories which were printed in the papers as to the attitudes in which the bodies were found are not worthy of credit.

The Effects of the Eruptions on St. Pierre and its Vicinity.

We visited St. Pierre in the beginning of July, 1902, and are able to confirm from our own observations the majority of the facts already described by Mr. HILL, Professor RUSSELL, and the French Commissioners. Before we arrived much time had elapsed, and the eruptions of May 20th, 28th, and June 6th had added their quota to the sum of destruction. Moreover, the city had for 36 hours after the first eruption been the scene of a gigantic conflagration. The piles of coal on the wharves still smoked when we were there. No good evidence is yet available of its condition when the first blast had finished its deadly work, except, perhaps, the reports furnished by the relief party, who went from Barbados under Mr. NEWTON, the Colonial Secretary.* We have also some photographs taken on May 14th by Mr. POYER, of Barbados. These show that the eruption of the 20th had finished the destruction of the cathedral and demolished many houses, weakened as they must have been by the fire which consumed the city. In the north end of the city, across the Rivière Roxelane, all houses were levelled with the ground, except where they stood below the bluff which forms the river's bank and faces the sea. Everything was burnt up, all vegetation gone, but the copper telephone wires were not fused, and objects of metal were little affected by the direct heat of the blast. Those, of course, which were within the houses showed more alteration, but nothing that might not reasonably have been attributed to the action of the conflagration.

* Blue Book: 'Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' p. 47.

The blast had crossed the shallow valley of the Roxelane, had twisted the iron stanchions of the bridge over the stream, and planed off the upper parts of the houses which stood in the shelter of the steep northern bank. On the south side of the river the destruction was less, and it rapidly decreased as it was traced southwards, though still the desolation was so striking that it seemed as if no more utter ruin had ever

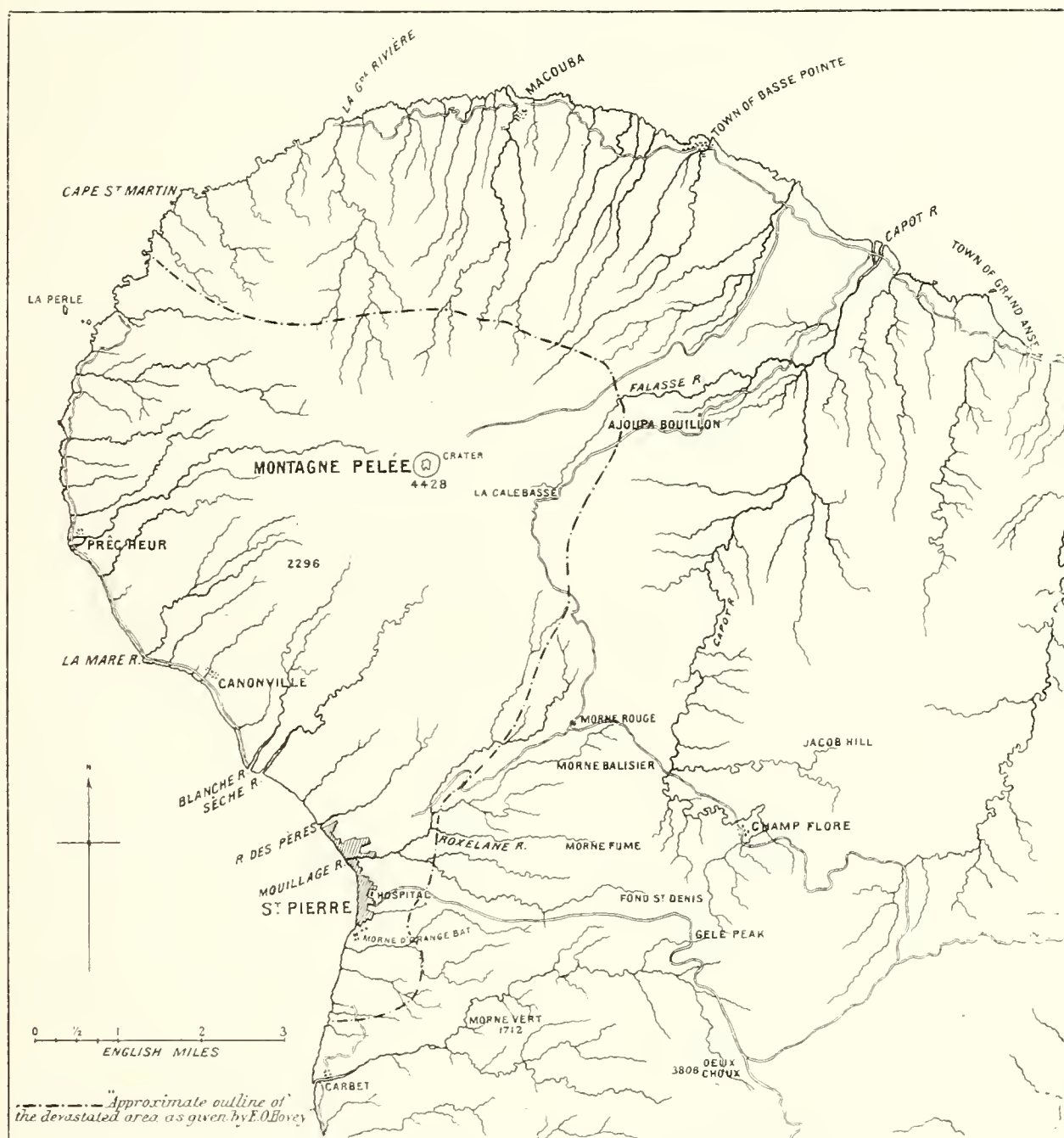


FIG. 2.—Sketch Map of the North End of Martinique.

overwhelmed a town. The streets were full of piles of stones, the remains of the walls which had been overturned. In many cases the walls running east and west were levelled, as they had faced the force of the blast, while those which ran north and south had offered their ends to it and successfully resisted its violence. It was curious to note how different in consequence was the aspect of the town when looked at from

the north of the Rivière Roxelane, and from the anchorage in the bay. From the one side it seemed entirely razed, hardly one stone left upon another; from the sea it was possible to make out where the streets had been by the long walls which had formed the frontages and were still standing. The houses were burnt out, and filled with blackened timber, ashes, and mud, much of which had apparently been washed down by the rains from the slopes behind the town. These were now nearly bare, though a little green vegetation was again showing itself, and evidently rivers of mud had flowed along the streets and come to rest on the flat ground on which the city had been built. It was in turn being washed out of the houses by every shower, and fresh heaps of skeletons were in this way daily being exposed to view.

The violence of this blast must have been terrific. The prostrate walls, the twisted ironwork of the verandahs, the ruined cathedral, the uprooted and dismembered trees, all spoke eloquently of this. The cannon in the fort near the south end of the town had been overthrown (they stood side on to the blast). The gigantic statue of the Virgin, which was planted on the edge of the bluff near the southern fort, was torn from its pedestal. This was not in itself wonderful, as the statue was fastened to the pedestal only by four or five iron bolts which had given way, but it was marvellous to see how it had been carried 40 feet away, and lay prone with its head pointing directly to the crater. Near it a low, stone wall, about 18 inches thick, had been broken from its foundations and forced southwards for several inches.

The conflagration, too, had left its mark. Nothing that was combustible had been preserved. Everything was more or less completely burnt, except some unsigned notes, which lay in a box in the strong room of the Banque Coloniale. We were told that robbers had rifled the houses of everything of value, but we saw very little—it was difficult even to obtain a curio to remind one of the visit. The iron safes of the business houses were there, most of them broken open; the books they contained were more or less charred. Coins found in the houses and shops were shown us, most of them blackened, and often sticking together as if they had been partially fused. The shops were empty, except those which had contained china, glass, or iron. The china was often superficially fused, the glass in some cases melted into lumps. The iron beams of one large building were curved and twisted like reeds.

From our point of view the condition of the trees was of special interest, as giving us data on which to base a comparison with what we had seen in St. Vincent.

To the north of the city they had vanished, and on the south side of the Roxelane they were overturned—even the largest charred and sand-blasted. But on the extreme south end, though many had fallen, some stood, and behind the town a banana was putting out its oar-shaped leaves of fresh green amid the desolation of ashes and death. The trees in the streets may have been partially protected by the surrounding houses. It is not wise to push the comparison too far, but in general the south end of St. Pierre was in much the same condition as St. Vincent on the

windward side, about a mile above Lot 14, so far as the condition of vegetation in the city was concerned. There were no houses, of course, in that region in St. Vincent.

In St. Pierre, as in St. Vincent, the blast was heavy and crept along the ground. At any rate, when it reached the lower slopes it flowed along, and was deflected by prominent ridges in its path. The bluff behind St. Pierre directed it along the coast, where it swept the shore to Carbet, killing or scorching all the cocoa-nut palms on the low beach, and singeing the face of the steep bank, but unable to mount the heights and devastate the country behind. This when we saw it was green; it had, perhaps, been covered with a thin sheet of ashes, but the crops, the trees, and even the buildings were little damaged, and nowhere was the line between the blasted, desolate, fire-swept area and that which had been injured, but not beyond the power of rapid recovery, so perfectly sharp and definite. In the length of one garden, near the statue of the Virgin on the cliff top, we could find every transition, from utter destruction at one end to very fair preservation at the other.* The edge of the dust cloud had been as well defined between St. Pierre and Carbet as in any part of St. Vincent.†

The depth of the layer of ashes was quite inconsiderable when compared with that on the south side of the Soufrière. On the cane fields at the north end of the town it had been originally perhaps a foot, but near the sea-shore, behind the rum distilleries, and near the mouth of the Rivière des Pères, perhaps 3 feet or more. In the town itself so largely had the deposit been re-arranged by water that it was not safe to deduce its original thickness, and everywhere the heavy rains had scoured away the loose, ashy sand, and only indirectly could we infer how much had once been there. The upper surface was wonderfully smooth, but Professor LACROIX and his colleagues‡ describe ridges "like sand dunes," which resemble the hog-backed mounds on the Wallibu Dry River. There did not appear to have been any great deposits in the valleys, which were mostly open and shallow, not deep, narrow ravines, but there was more in such situations than on the flat, and it was obvious that much of what had originally obstructed the stream courses had been cut out by running water and carried to the sea. In the Rivière Blanche we believe there is some depth of hot sand, which, as described by Professor HEILPRIN and Professor LACROIX, is slipping into the water and giving out gushes of steam, recalling on a smaller scale the explosions on the Wallibu. The sand was distinctly lighter in colour than that of the Soufrière, as may be inferred from the descriptions of its mineralogical and

* Mr. HOVEY states that "in many places the line of demarcation passed through single trees, leaving one side scorched and brown, while the other side remained as green as if no eruption had occurred." E. O. Hovey, "Martinique and St. Vincent: a Preliminary Report," 'Bull. Amer. Museum Nat. Hist.,' vol. xvi., 1902, p. 347.

† MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 382, 1902.

‡ MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 421, 1902.

chemical composition already published. It was astonishingly uniformly fine-grained; small lapilli were present, spongy, brownish-grey, and porphyritic, and were most numerous where the finer stuff had been removed by the rains, but large bombs are rare, except on the slopes around the crater, and we saw none near the city of St. Pierre. There were a few ejected blocks, so few that most of them already bore the marks of the hammers of the French and American geologists, who had been there before us. They were brittle, and flaked off on the surface like those already described as found on the Soufrière. All that we saw were old and somewhat decomposed porphyritic andesites.

The products of the magma of Pelée were distinctly more acid than those of the Soufrière, and more of the dust consisted of felspar, which gave it a pale grey colour. It is less rich in the ferro-magnesian silicates, especially augite, but contains a fair amount of hypersthene and some hornblende. No olivine has been reported. Mixed with the broken and entire crystals of these minerals are the remains of a base or ground-mass, which is often filled with fine microlites of hypersthene,* and this material of the later period of consolidation is, on the whole, more abundant than in the dust of the Soufrière. The felspars are also more rich in the albite molecule, and contain less of anorthite.

The area of devastation in Martinique is not only sharply defined, it is small, much smaller than that of St. Vincent, and occupies only a segment of the volcanic cone. Its base stretches along the coast from Carbet to Roche La Perle; its apex is a little to the north of the summit of Pelée; its boundaries are fairly straight, but are apt to be deflected by the irregularities of the surface and the obstruction offered by the projecting ridges to the outward flow of the black cloud near its margin. Within this region the desolation varies almost directly with the proximity to the crater. It is complete and total on the upper slopes and in the north end of St. Pierre, but decreases rapidly as traced southwards through the city and along the coast to Carbet. On the west side, Prêcheur is covered with ashes, but it was not swept by the awful blast. The wind constantly floats the fine ash projected from the crater with the rising steam clouds in this direction. The trees are powdered over, the vegetation blighted, the streets paved with dust, but much of this is due, not to the great eruption, but to the subsequent activity of the volcano.

Only about a third part of the mountain has been ravaged by the blast. The north and east sides have hardly suffered, or at any rate the deposit of ash on them was so thin that when we were in Martinique it had mostly been washed away, and the beautiful verdure which characterises that lovely island flourished to all appearance as if no eruption had ever taken place.

The area affected is rather more than one half that which has been blasted in St. Vincent, and the total mass of material ejected by Pelée, may be, perhaps,

* A. LACROIX, "Sur les Roches rejetées par l'Éruption actuelle de la Montagne Pelée," 'Comptes Rendus,' vol. cxxv., p. 453, 1902.

one-tenth of that which the Soufrière has furnished. The great avalanches of dust which fill the Rabaka and Wallibu valleys enormously surpass in magnitude any deposits of the same kind in Martinique, and the general sheet of deposit over the Soufrière is not only larger but probably also five times as deep as that which can be found on Pelée. The dust from the Soufrière fell in Barbados (1,700,000 tons), and all over the sea for 700 miles to the south-east of this.

The cloud which surged from the crater of Pelée had only one outlet. On north and east it was hemmed in by walls of rock; only to the south-west could it find a fissure of escape, and through this it rushed and swept down the slopes on the plain beneath and on the devoted city. It spread out somewhat, and the area it covered is fan-shaped, but all its violence was concentrated in that small space, and the havoc it wrought was fearful in consequence. Probably the force of the hot blast was as great in Martinique as in St. Vincent at equal distances from the point of origin. But in St. Vincent the total energy expended during the great explosion was vastly greater than in Martinique, for the avalanche poured over the whole lip of the great circular crater, though mostly over the notch on the south-west side, and radiated out in all directions from the centre. Still, that the blast in Pelée was so violent, while the amount of material ejected was so small, is one of the most interesting features of the eruptions, and calls for special discussion when we come to consider the mechanism of these discharges.

In St. Vincent no fissures are known to have opened on the hill-sides, no flows of hot mud lava have welled out of the mountain, and there is a complete absence of fumaroles or minor steam emissions, except from the boiling lake within the crater. But in Martinique early in the eruption the rivers began to flow full of muddy water, and on May 5th the Étang Sec broke through its barriers and, discharging into the Rivière Blanche, sent down a torrent of hot mud which buried the Usine Guérin. Fissure formation and hot mud-flows did not end here, they are one of the dominant features of the eruption of Pelée, and some interesting facts regarding them are contained in the Preliminary Report of the French Commissioners.* Along the course of the Rivière Sèche, and in the district between that and the Rivière Blanche, numerous fumaroles have appeared which indicate the presence of a series of fissures in that quarter. They are probably radial in direction, and may be crossed by another series (tangential) on which lie certain fumaroles near the sea-shore. On the north-west side of Montagne Pelée there is, at Ajoupa Bouillon, a fumarole which has poured out large quantities of muddy water, and this may lie on a prolongation of the radial fissures on the south-east side of the crater. They are not characterised by gaping cracks, but rise through the old or new ash deposits, and give evidence of their presence by emanations of steam and sulphuretted hydrogen. Some of them are at a temperature sufficiently high to melt pieces of lead inserted into the orifice.

* MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 381, 1902.

Their maximum activity does not coincide closely with the outburst of steam clouds from the crater.

When these fumaroles lie in the course of a stream or in the sea, the water flows into the open tube in periods of quiescence; when, on the other hand, the activity resumes, it is discharged as an overflow of hot mud, alternating with steam puffs and jets of muddy water. To this is due the sudden variations in the volume and the temperature of the streams. As in St. Vincent, the banks of hot ash overlooking the current are subject to frequent landslide, which give rise to ascending steam clouds when they meet the water; and this must not be confounded with the open fumaroles from which steam is also emitted. Professor LACROIX and his colleagues also remark that the large number of dead fishes cast ashore on some days in the Bay of St. Pierre may have been killed by the action of similar fumaroles beneath the sea, along a prolongation of the same fissures, and that the telegraph cable was broken apparently where it crossed this line, and when recovered its insulating material was found to have been melted.

On the north-western side of the mountain also, mud-flows have taken place on a large scale. One has covered the village of Basse Pointe, another flows down to Macouba. Even before the great eruption there were mud currents in some of the streams, and though they may be in part due to the washing action of rains on the ash covered soil, and consequently similar to the muddy rivers of St. Vincent, yet they cannot all be explained in this way. Some are due to the discharge of the crater lakes through fissures, others no doubt to the re-emergence of water which has been engulfed into fumaroles and open cracks, but for a fuller discussion and explanation of their origin and nature we must wait till the detailed reports of the French Commissioners are to hand.

The long preliminary stage of the eruption at Pelée, the scarcity of earthquakes, the development of fissures and of fumaroles, and the extensive flows of mud, together with the more acid nature and more uniform fineness of the ejecta, and the greater strength of the blast in proportion to the small amount of dust deposited, are the main differences which we find in making a comparison of the great eruptions of St. Vincent and Martinique. In all their principal features the two outbursts are parallel, and belong to a clearly-defined and highly-destructive type of volcanic action. These volcanoes are of the explosive class, and in their dust avalanches exhibit one of the most remarkable effects of the expansive power of the superheated steam in an igneous magma. Their fatal violence is owing rather to the physical processes at work and to the form their ejecta assume than to the magnitude of the eruptions themselves, for while that of the Soufrière was really a considerable eruption, and produced notable geological consequences, that of Pelée was comparatively small, and its geological effects are of no great importance, and most of them will disappear within a few months of the cessation of the explosions.

Another and a striking difference between the Soufrière and Pelée is in the manner of their behaviour during the time that has elapsed since their first outburst early in May this year. The Soufrière soon ceased to emit its column of vapour and of ashes, and between the successive eruptions periods of complete tranquillity (except for rumblings and slight earth tremors) have intervened. But Pelée, according to the accounts given us by many who have visited the mountain since the first great outbreak, has never quite discontinued to send out towering steam clouds at more or less regular intervals. When we were in St. Pierre in the early part of July this year, the great gaping rent on the south-west side of the mountain-top would every now and then discharge great puffs of steam which, rising in the air, would expand and become balloon-shaped, their surfaces covered with rounded, swelling convolutions, which rapidly multiplied and incessantly changed their form. They were very similar in appearance to the puffs which rose from the Wallibu in St. Vincent, and as they were sudden, and formed in a second or two, their upper parts soon separated from the stem, and floated off before the steady east-north-east trade wind. As they drifted across the face of the mountain the fine ash fell from them like a thick mist, which veiled the features of slope and scar and ravine. We often compared them to cauliflower-like flowers, or to bunches of grapes, and very similar effects may occasionally be noticed where a large locomotive sends one great blast of steam straight up from its funnel. When they leaped into the air, a low, dull rumble might often be heard. They ascended to heights of 4000 or 5000 feet above sea-level before their upward velocity was spent, and their graceful beauty of form, and the play of light and shade on their surfaces, as they ceaselessly expanded and their convolutions swelled and melted into the flattened drifting clouds, which the wind bore with it to leeward, were objects of continual interest and admiration to us. We did not see them carry up stones of any size, nor did they condense as they floated away, for the ash which fell was dry, and there was actually a lack of rain in Prêcheur, in which the falling dust had covered everything with an ashen pall.

One of these clouds would rise every 10 or 20 minutes, for hours at a time, then for an hour or more there would be none, and when the trade-wind cloud which always capped the mountain would lift and clear for a little, we could see with our binoculars the great V-shaped cleft which faces St. Pierre, and out of which welled the deadly blast that razed the city. A sloping scree of enormous angular blocks of rock lay in this gulch. At night a dull red glow is sometimes seen given out by these boulders, for they are intensely hot; and little landslides occasionally took place in them, the material being probably set in motion by the tremors which accompanied the rise of the steam jets.

This talus of great stones was formed, apparently, around the crater, where the ejecta which had been cast up, but had not sufficient velocity to surmount the summit and land on the windward side or in the apical lake (the Étang des Palmistes), would necessarily accumulate. Its formation was plainly due to the greater eruptions,

for the smaller jets of steam could add but little to the piles of rock which gathered there. The large size of the boulders showed that most of the finer dust had soared high in the air and travelled far before alighting, and little whiffs of steam rose sometimes from between these stones, and wafted away whatever finer stuff had temporarily landed among them.

These club-shaped steam columns which towered into the air have been seen by all visitors to St. Pierre, and when unusually large, have occasioned much trepidation, and given rise to many baseless rumours of eruptions. According to Professor JAGGAR, they had been visible every day he saw the mountain from that side, but varied a good deal in frequency and in size. They were quite harmless, even to those who were accidentally involved in their edges, and consisted of dry steam, dust, and a little sulphurous acid.

The Eruption of July 9th, 1902.

The night of July 9th was marked by one of the major eruptions of Pelée, and we had the exceptional good fortune to be in St. Pierre that day, and to have a magnificent view of the eruption, while we escaped entirely unscathed.

The morning had been exceedingly fine, and we spent it in St. Pierre examining the ruined city and the cane-fields on the banks of the Rivière des Pères. During the forenoon the mountain was in a state of almost complete quiescence. Hardly a single steam cloud rose from fissures in the Rivière Blanche, and at times the summit of the mountain could be made out rising above the great cleft near the top. To the west it formed a broken cliff, with extraordinary ruggedness, overlooking the old crater of the Lac des Palmistes. As mid-day passed the familiar steam jets appeared again, and in the clear air they were so beautiful, so large, so perfectly formed, that we perforce had to halt in our wanderings through the city and gaze on their varied transformations. Similar clouds of larger size we had seen previously, but never any so uniformly perfect in all their proportions. We had with us a small sail-boat, the "Minerva," of Grenada, of 10 tons register or less, which served as a base for our expedition, and in the afternoon we went aboard and cruised about the bay, sailing down along the coast to Prêcheur, taking photographs of the hill and of the steam clouds, which continued to rise at regular short intervals. Off Prêcheur we put about and stood back close-hauled across the bay. For a time the breeze fell away, but as we drew out from the land the fresh trade wind met us and bore us along to the south end of the anchorage. The perfect afternoon light showed up in deep relief the naked, scarred, and riven surface of the great volcano. As we were half-way across the bay of St. Pierre, about half-past five in the afternoon, the steam clouds, which hitherto had followed one another at short intervals sufficient to allow each to exhibit its perfect form and to pass through the various stages of ascent and expansion before another followed it, began to rise with greater frequency, so that one interfered with the isolated development of that which went before.

We ran down to Carbet, about $1\frac{1}{2}$ miles south of St. Pierre, where we came to anchor on a sandbank a little north of the village. One of us went ashore to make the final arrangements with the porters who were to come on board before dawn next day, and to ascend the hill with us. After purchasing some food he returned, and now the sun was setting behind a dense pall of ashes which hung over the leeward side of the mountain like a dark fog, and over the sea to the westward for a distance of several miles from the mountain. From the fissure in the volcano, clouds of pale slaty vapour rose constantly, and, spreading out, they floated away before the trade wind. We could see that puff followed puff; each could be distinguished, though they followed one another at intervals so brief that as each expanded it melted more or less completely into the streaming cloud mass which swept across the hill. The sun behind this cloud became a pale yellowish or greenish-white disc, easily observable with the naked eye long before it touched the rim of the horizon.

In the rapidly-falling twilight we sat on deck intently watching the activity of the volcano, and calculating the chances of an ascent next morning, when our attention was suddenly attracted to a cloud which was not exactly like any of the steam "cauliflowers" we had hitherto seen. It was globular, with a bulging, nodular surface; at first glance not unlike an ordinary steam jet, but darker in colour, being dark slate approaching black. But in its shape there was nothing very distinctive. Its behaviour, however, was unique. It did not rise in the air, but rested there, poised on the lip of the fissure, for quite a while as it seemed, and retained its shape so long that we could not suppose it to be a mere steam cloud. Evidently it had been emitted with sufficient violence to raise it over the lip of the crater, but it was too heavy to soar up in the air like a mass of vapour, and it lay rolling and spouting on the slopes of the hill. The wind had no power over it, fresh protuberances spouted out from its surface, but it did not drift to leeward any more than if it had been a gigantic boulder. For a little time we stood watching it, and slowly we realised that the cloud was not at rest but was rolling straight down the hill, gradually increasing in size as it came nearer and nearer. We consulted together; it seemed so strange and so unaccountable, but in a minute or two suspicion gave place to certainty. It seemed that the farther the cloud travelled the faster it came, and when we took our eyes off it for a second and then looked back it was nearer and still nearer than before. There was no room for doubt any longer. It was a "black cloud," a dust cloud, and was making directly for us. So with one accord we prepared to get out of its path. We helped the sailors to raise the anchor and, setting the head sails, we slipped away before the wind. By the time the mainsail was hoisted we had time to look back, but now there was a startling change. The cloud had cleared the slopes of the hill. It was immensely larger, but still rounded, globular, with boiling, pillowy surface, pitch black, and through it little streaks of lightning scintillated. It had now reached the north side of the bay, and along its base, where the black mass rested on the water, there was a line of sparkling lightnings that played incessantly.

Soon, however, it seemed to lose its velocity; its surface became less agitated, it formed a great black pall, with larger, less vigorous, more globular, bulging convolutions. Evidently its violence was spent, and it was not to strike us; it lay almost like a dead mass on the surface of the sea.

At first the wind was east and very gentle, a slight tide drew us southward and, as we slipped past Carbet, the church tower shone in the pale moonlight (the moon was in its first quarter and high in the heavens to the south-west), and there was still light to enable us to see the figures of men fleeing south along the cliffs, and we could hear their shouts of terror.

The black cloud rose from the fissure about 20 minutes to 8 o'clock. It took very little time for us to get the sails set, and for 20 or 30 minutes we sailed along with a gentle breeze from the east, every sail drawing, and the houses of Carbet lessening gradually as we sped south. Then the wind fell away, and it was practically a dead calm. In the deepening darkness we kept a close watch on the mountain, and soon the black cloud seemed to clear, for again we could see dimly the distant cloud-capped mass of Pelée, with a faint red glare above the fissure from which the cloud had come. This glare, however, had been seen once or twice during the previous month by various observers. But it slowly increased, and we could see bright, glowing masses describing parabolic paths through the air and then landing on the mountain slopes and rolling down the hill. These were clearly red-hot stones, and they must have been projected about a mile from the crater. The sailors had been often in St. Pierre during the previous three weeks, but they had never seen anything like this before, and it was apparent that the volcanic activity was unusually great.

Suddenly a great yellow or reddish glare lit up the whole cloud mass which veiled the summit. It was like the lights of a great city on the horizon, or the glare over large iron furnaces, as seen from a distance on a dark night reflected from an overhanging mist, but brighter and more yellow. Then from the mountain burst a prolonged angry growl, not a sharp detonation or a series of detonations, such as we had heard just before when the hot stones were launched from the crater, but a long, low, rumbling sound, like the sullen growl of an angry wild beast. It seems strange that this sound should have been heard as far as Barbados, for what struck us about it was not its loudness, but its snarling character.

Then in an instant a red-hot avalanche rose from the cleft in the hillside, and poured over the mountain slopes right down to the sea. It was dull red, and in it were brighter streaks, which we thought were large stones, as they seemed to give off tails of yellow sparks. They bowled along, apparently rebounding when they struck the surface of the ground, but never rising high in the air. The main mass of the avalanche was a darker red, and its surface was billowy like a cascade in a mountain brook. Its velocity was tremendous. The mist and steam on the mountain top did not allow us to see very clearly how the fiery avalanche arose, but we had a perfect view of its course over the lower flanks of the hill, and its glowing undulating surface

was clearly seen. Its similarity to an Alpine snow avalanche was complete in all respects, except the temperature of the respective masses. The red glow faded in a minute or two, and in its place we now saw, rushing forward over the sea, a great rounded, boiling cloud, black, and filled with lightnings. It came straight out of the avalanche, of which it was clearly only the lighter and cooler surface, and as it advanced it visibly swelled, getting larger and larger every minute. The moonlight shining on its face showed up the details of its surface. It was a fear-inspiring sight, coming straight over the water directly for us, where we lay with the sails flapping idly as the boat gently rolled on the waves of the sea.

The cloud was black, dense, solid, and opaque, absolutely impenetrable, like a mass of ink. It was globular as seen end on, very perfectly rounded, but covered with innumerable minor excrescences, rounded, and filled with terrific energy. They shot out, swelled, and multiplied till the whole surface seemed boiling; one had hardly time to form before another sprung up at its side; but they were directed mostly to the front and fewer at the margins, so that their effect was that the cloud drove onward without expanding laterally to any great extent. On the whole the resemblance to the rising towering cauliflowers of steam, which soared up from the fissure, was quite striking, only here the cloud lay on the water and sped on horizontally, and its ebony mass was a great contrast to the pale, pearly, ascending steam jets.

The display of lightning in the cloud was marvellous. In rapid flashes, so short that they often seemed mere points, and in larger, branching, crooked lines it continually flickered and scintillated through the whole vast mass. It was often greenish, perhaps, when seen through some slight depth of the dust cloud, at other times yellowish, and always rapid, short-lived—a mere succession of flashing points in the great black wall of cloud. Many of the flashes were horizontal, others shot obliquely from one lobe to another, while along the base, where the black cloud rested on the steel-grey sea, there was a line of sparkling lights, constantly changing, varying in amount but never disappearing. This feature was so pronounced and so apparent at the first glance, that we were at once reminded of the narratives given us by survivors in St. Vincent, in which it was stated that when the black cloud rolled down upon the sea it was filled with fire.

Nearer and nearer it came to where our little boat lay becalmed, right in the path of its murderous violence. We sat and gazed, mute with astonishment and wonder, overwhelmed by the magnificence of the spectacle, which we had heard so much about, and had never hoped to see. In our minds there was little room for terror, so absorbed were we in the terrible grandeur of the scene. But our sailors were in a frenzy of fear, they seized their oars and rowed for their lives, howling with dread every time they looked over their shoulders at the rushing cloud behind us. Their exertions did little good, as the boat was too heavy to row, and fear gave place to despair. But in a minute a slight puff of wind came from the south-east, very

gentle, but enough to ripple the water and fill the sails. We had drifted out from the shore, so we gave our boatmen instructions to keep the boat close-hauled, and draw in to the land, as the cloud was passing more to the westward. Then, when we looked at the cloud again; it was changed, it showed no more the boiling, spouting, furious vigour, but the various rounded lobes in its point swelled slowly and to greater size, while fresh ones did not shoot forward, and the mass had a more reposeful and less violent appearance. In the moonlight it was difficult to say how far away it was, but judging by our distance from the shore, we thought it was a mile off, or rather more.

It now lay before us nearly immobile, a gigantic wall, curiously reflecting the moonlight like a pall of black velvet. Its surface was strangely still after the turmoil it had exhibited before, and great black rounded folds hung vertically like those of an enormous curtain. This lasted a few minutes, and the folds became flatter and less convex, and a strange shimmering and change of colour began to steal through the great murky wall, like a transformation scene. It became brownish in places, and in others grey, or even white on the edges of the folds, and through the whole vast face of the cloud these changes gradually spread. Soon it was evident that the base was darker, and the paler summit was rising in the air and soaring obliquely upwards and forwards. The dust was sinking, and the pale steam set free from entanglement with the heavier solid particles, was following its own natural tendency to ascend, while still impelled forwards by the great onward impetus it had received.

The steam cloud crept southward, and was soon directly over our mast-head, travelling with a velocity of perhaps 20 miles an hour. It was grey and tongue-shaped, with a blunt, rounded apex, and at its sides and beneath it there were bars of fleecy white where the moonlight fell upon its thin edges and the lobes which hung beneath it. The mass of the cloud was slaty-grey, and as we looked up at its slightly rounded under surface we could see that still through it the expansive energy of the vapours was working everywhere, and many rounded convolutions were forming especially on its front, though no longer with the rapidity that they had formerly exhibited.

The lightnings were now reduced in number and frequency, but still, in branching tortuous lines, threaded the dark mass in every direction. A low rumbling noise was given out as the cloud worked its way across the clear, starry sky. In a little while it obscured the moon, and the night became very dark. It spread and spread, broadening and elongating till in a great, flattened, rounded mass it covered the whole sky, and in an hour or two only a narrow crescentic belt of stars could be seen away down on the southern horizon. The low rumbling noise continued, and the lightnings, though less numerous, were visible for a long time almost continuously flashing.

As the cloud reached the zenith a hail of pebbles fell in the sea and on our decks. We picked up the first that fell. It was about the size of a chestnut, and was cold to the touch, so we knew that we were safe. Then smaller pellets rattled on our decks,

like a rain of peas or small shot. A little afterward the fine grey ash came in little globules moist and adherent, noiselessly sinking through the air and sticking to everything on which they landed. They were not warm, and there was a slight but noticeable smell of sulphurous acid. After a few minutes the ash took the form of a dry powder, which got into our eyes and felt gritty between our fingers.

We had now a little wind, which rose gradually to a fresh breeze, and with many tacks we beat up towards Fort de France, with the lightning flickering still in the sheet of cloud overhead, and the fine ashes on everything that we touched or tasted. It covered the decks, and fell from the sails and the rigging overhead, but there was not much of it. A thin layer, perhaps one-sixteenth of an inch thick, was all we found when daylight broke, and we could ascertain its exact amount. That night dust fell on Fort de France and the whole south end of Martinique, but only in very small quantity.

As we beat up through the darkness to the harbour lights of Fort de France there were loud claps of thunder and bright flashes of lightning in the northern sky. They were quite distinct from the growl of the mountain, the short stabbing lightnings of the black cloud, and the low rumble we had noticed as it passed overhead. We watched them carefully, and thought they were only atmospheric, and the thunderstorm was certainly not all to the north of us, but partly also to the north-east, where there was no possibility it could have been mistaken for an eruption of Pelée. There is a good deal of evidence, however, connecting sudden thunderstorms with eruptions of Pelée and the Soufrière, and we cannot be sure that all the noises were thunder, or that the storm was not in part due to the atmospheric disturbances attendant on the explosion. The thunderstorm broke out about midnight, and by that time the lightnings in the ash-laden cloud overhead were practically over.

The avalanche of hot sand was discharged about 8.20 P.M. In a couple of minutes it had reached the sea, and was over. The second black cloud, which was all that remained of it when the heavier dust had subsided, travelled about 5 miles in six minutes, and very rapidly slowed down, coming to rest and rising from the sea in less than a quarter of an hour. The tongue-shaped steam and dust cloud was over our boat by 8.40. A few minutes after that the ash was falling on our decks.

The second black cloud did not differ in appearance from the first, except that it was larger, had a far greater velocity, and swept out at least twice as far across the sea. It was black from the first moment when we saw its boiling surface in the moonlight. Both travelled very rapidly over the lower part of the mountain, but slowed down after reaching the sea, and came to rest comparatively suddenly. The lightnings on the two clouds were similar in all respects.

No blast struck us—in fact, we were becalmed—and it seemed that when the black cloud ceased the blast was also over. Nor did the sea rage around us as some have described who were overtaken by the dust storm. When the cloud was passing overhead there was a slight rolling sea, but as the breeze freshened the boat steadied,

and there was no unusual disturbance. We watched carefully for a strong indraught, such as was described by more than one observer, but the wind that rose from the south-east was very gentle, and increased gradually to a full-sail breeze. There were no reef points in our sails, which were all set, and the boat carried them quite easily.

In the cloud there was a dull, low rumble, but we heard no detonations, and saw no sheet of flame, so that we both agreed that there had been no sudden ignition of quantities of explosive gases. The lights in the cloud, in our opinion, were lightning and nothing else.

No wave was noticed by us other than the slight roll already mentioned. Yet we learned in the morning that a sudden rise of the sea had been observed at Fort de France, and Professor LACROIX* states that on July 9th, as on May 8th, May 20th, May 26th, and June 6th, the dates of the greater eruptions of Pelée up till that time, the sea level oscillated in sympathy with the volcanic outbursts. It was not to be expected that any such disturbance would be very evident on board a boat on the open sea.

On the morning of July 11th we were strolling through the streets of Fort de France photographing the picturesque inhabitants of Martinique and the changing scenes of a tropical city, when a low rumbling sound fell on our ears. We paid little attention to it: it might have been the noise of a heavily-laden cart, or a military wagon passing along the roads, but in a minute or two we noticed that the people were gathering in clusters on the pavement, gazing up in the northern sky and exclaiming, "La montagne! La montagne!" We raised our eyes, and there against the background of blue sky a long, narrow, tongue-shaped cloud, fleecy white, with rolling, boiling, globular protuberances continually forming at its apex, was working its way upwards and southwards in the clear morning air.

That morning on Pelée a small outburst had taken place, following the larger eruption of July 9th after the lapse of a period of about 36 hours, just as in St. Vincent the small eruption of the morning of May 9th followed the great explosion of the afternoon of the 7th. For some days thereafter the mountain was in a restless, unquiet condition, and on the morning of Sunday, the 13th, there was a fall of very fine grey dust in Dominica after another eruption, which must have occurred about midnight on the previous night. About 6 A.M. a very fine powdery matter could be seen floating on the air and resting on the glossy leaves of the plants in the gardens. When gathered and examined microscopically it proved to be undoubtedly dust from Pelée. We made inquiries, and learned that an explosion had taken place in Martinique about six hours before. The path the dust had followed had been a very indirect one. It had been projected southward as usual, had risen into the anti-trade region and been floated away to the east-north-east, then sinking into the lower strata of the air had been borne northward to Dominica

* MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxv., p. 390, 1902.

by a gentle south-east breeze which was blowing that morning, so that in making the journey from St. Pierre to Rozeau, which are only 20 miles apart in a direct line, it must have travelled at least thrice as far through the air, and had taken about six hours to cover the distance.

We left Fort de France on the afternoon of July 11th, and on our way to Dominica we had a good opportunity of seeing the general effects of the eruptions of the 9th and of the 11th. Captain BARRETT, of the R.M.S. "Yare," most kindly took his ship right up into St. Pierre Bay, and round the shore as close as was possible. Evidently the avalanche had come down approximately along the line of the Rivière Sèche, for all over that part of the mountain there was a layer of fresh, pale-grey dust covering the black surface of the older ash. The effect was not unlike that of a fall of snow, but, of course, the country was treeless and perfectly bare. The grey dust was blowing over the surface in little clouds, stirred up by the wind, and the rivers were flowing down through their channels hot and steaming. We thought that there was more ash in the valleys than had been there on the afternoon of the 9th, when last we sailed along that shore, but as we had never landed there it was not possible for us to be certain. Evidently the total amount of matter ejected had not been large, only sufficient to add a thin additional layer to the older deposits, but not enough to dam up the valleys or make any essential difference in the surface features.

A COMPARATIVE STUDY OF THE PELÉAN TYPE OF ERUPTION.

The attention of mankind has this year been powerfully directed to the effects produced by eruptions of a type which hitherto has not been known to exist, and which differs from other more customary kinds of volcanic action in several important respects. As the catastrophe by which, in the month of May, 1902, the city of St. Pierre was erased from the map of the world will remain to all time a witness to the violence with which these eruptions are attended, we propose to adopt the term "A Peléan Eruption" to designate this group of phenomena. The earliest historic instance of such eruptions is that of St. Vincent, in 1812, which, as we have already shown, was clearly accompanied by the discharge of vast masses of hot sand and stones into the valleys of Wallibu, Larikai, and Rabaka. It is probable, but it is not certain, that the eruption of the same volcano in 1718 was also of this nature.

Eruptions of the Peléan type are distinguished by the occurrence of one or more discharges of incandescent sand, which rush down the slopes of the mountain in the form of a hot-sand avalanche, accompanied by a great black cloud of gases charged with hot dust, which sweeps over the country with a very high velocity, mowing down everything in its path. All living beings within the zone nearest the crater are killed; all plants reduced to charred and broken stumps. At greater distances men and animals are scorched by hot sand or mud; plants are burnt, eroded, and

stripped of leaves and branches ; but beyond the limits covered by the great black cloud no effects are produced, other than those consequent on the rain of ashes which precedes or follows the avalanche. Except for the dust avalanche, there is nothing unusual about these eruptions. They have, indeed, many points in common with those of volcanoes of the ordinary explosive type, to a sub-group of which they evidently belong.

It is possible for us at the present time only to compare the history of the different outbursts of Pelée and the Soufrière in order to ascertain what features they have in common which are distinct from those of other eruptions, and to form a general idea as to what are the stages of their action, and what variations are possible in the type they represent. Even for this purpose the data are not yet so full and exact as we hope they ultimately will be, when the various scientific men who are engaged in the study of these volcanoes at this moment have completed their labours, and given the world the results of their investigations. But sufficient is known already to enable us to formulate certain preliminary conclusions and working hypotheses.

We will avail ourselves, in the first place, of the results of our own observations in St. Vincent and Martinique, and of our experiences during the eruption of Pelée on July 9th, 1902. But the Reports of the Commissioners appointed by the French Academy of Sciences, and of the various parties of American scientific men, already quoted by us, have furnished many additional facts of importance, and not a few theoretical suggestions, which we shall be able in some cases to adopt as in accordance with the opinions we have been led to form by our own investigations. It is only by the collection of a large body of facts from authentic sources, well and carefully sifted, that this new branch of the science of volcanology will ultimately be established on a secure basis, and its results and conclusions entitled to the confidence of scientific men.

The Stages of the Eruptions.

We have in a Peléan eruption certain distinct stages to recognise and describe. They may be classified as follows :—

1. The premonitory symptoms which herald the volcanic outbreak, but are attended by no actual emissions from the volcano.
2. The preliminary stages, in which activity has been resumed and discharges are produced from the crater. Their violence increases more or less rapidly, and they lead up to—
3. The climax of the eruption, which is manifested by the appearance of the avalanche of incandescent sand and the passage of the great black cloud.
4. The concluding stages, during which the volcano sinks into more or less complete repose.

These stages must be admitted to be more or less artificial and arbitrary. In some of the eruptions they can all be recognised as distinct, in others there is a tendency

for one or more to become inconspicuous or disappear, while occasionally the preliminary stages of one eruption are continuous with the closing stages of another. We have adopted them as convenient rather than as necessary, and as affording merely an easy and simple method of classifying the observed facts.

1. The premonitory symptoms of the eruptions have been very noticeable at St. Vincent, and much less obvious in Martinique. They consist of numerous earthquakes, not so violent as to damage houses, but so frequent as to awaken apprehension. They have on more than one occasion been observed for a year before the outbreak. That was the case before the eruption of 1902, and also, according to HUMBOLDT, before that of 1812.* Before the eruption of 1718,† earthquakes were frequent during the previous month. They have invariably been most violent around the base of the volcanic cone, especially on the west and east sides, and have never been conspicuous in Chateaubelair and Georgetown. They are not known to have taken place before the eruption of May 18th, but in that case the area over which they are felt had been completely evacuated. When the throat of the crater has been recently cleared by a previous eruption they are never so violent as after a long period of quiescence, when the passages are occupied by masses of solid rock.

At Martinique the first indications of activity in April, 1902, were the increased action at the Soufrière, the emission of steam from the summit crater, the formation of lakes of boiling mud, and the fall of fine ashes on the surrounding country. Since the first great eruption that destroyed St. Pierre the subsequent outbursts have given little or no warning. Earthquakes have not been at all numerous in Martinique, either before or during these eruptions.

2. *The Preliminary Stages.*—In their duration, their violence, and the constancy of their occurrence, these vary as much as do the premonitory symptoms. They consist of the emission of steam in increasing volumes from the crater, with fine, ashy dust, small lapilli, and fragments of rock from the crater walls, the discharge of the crater lakes as torrents of water or of hot mud, with loud detonating noises, and in some cases numerous earthquakes accompanying the explosions. In Martinique the earthquakes were few and attracted little attention, but fissures were opened in the flanks of the volcano, and through these the crater lakes discharged, steam arose from many fumaroles, and the rivers were augmented by flows of boiling mud. In St. Vincent no fumaroles and no fissures are recorded, there was apparently no change in the temperature or volume of the springs, the crater lakes were driven over the lip of the depression, and the fall of ashes around the volcano was quite inconsiderable. In Martinique the preliminary stages of the great eruption were prolonged over a period of two weeks. Ash fell steadily on the leeward slopes of the mountain during that time. In St. Vincent, in 1812, the preliminary stages occupied

* HUMBOLDT'S 'Personal Narrative,' English translation by WILLIAMS, vol. ii., p. 226. See also anonymous narrative cited above, p. 463.

† DEFOE'S 'Narrative,' cited above, p. 456.

three or four days, and were on the whole similar to those witnessed on Pelée during the present year, except that fissures were not formed, and no flows of hot mud are spoken of in the accounts. This year the preliminary stages of the eruption of the Soufrière lasted only for a little more than 24 hours.

When one eruption follows another after a brief lapse of time the preliminary symptoms may be indistinguishable from the closing phases of the previous outbreak. This was the case on the Soufrière on May 9th, and on Pelée on July 9th there was nothing, except a slight increase in the amount of steam discharged, to give warning of the approaching outburst of the avalanche of incandescent dust. Equally sudden was the eruption of May 18th at the Soufrière, of which no symptoms were visible at nightfall—about 7 o'clock—yet at 8.30 an immense cloud of steam suddenly towered into the air, with vivid lightnings and loud detonations.

The only danger during these preliminary stages is the suddenness of the mud lavas that may flow down the valleys, burying the houses on their banks, as happened at the Usine Guérin near St. Pierre this year. Even the vegetation of the hill, though covered with fine dust, suffers very slightly, and a few showers of rain will remove all the ashes and restore the beauty of the foliage. In Martinique, and also in St. Vincent, it is possible that heavy falls of rain took place on the higher parts of that mountain during the earlier outbursts of steam, but these were local and did no real damage.

3. *The Climax and Descent of the Black Cloud.*—The preliminary stages gradually or suddenly pass into the culminating phases of the eruption. The great black cloud wells out of the crater and rushes down the slopes, obliterating all trace of vegetation, annihilating every living thing in its path, and leaving behind it a desert of ashes. Its appearance is usually terribly sudden, a few minutes previously the volcano may have been emitting only a little steam and fine dust, perhaps more than usual, but not enough to awaken any general alarm.

On the morning of May 8th the “Roddam” had just dropped her anchor; the sailors were leaning against the bulwarks, watching the great column of grey smoke which rose from the volcano, while the fine dust was gently falling on city and harbour. “Then came a sudden roar that shook the earth and the sea. The mountain uplifted, blew out, was rent in twain from top to bottom. From the vast chasm there belched up high into the sky a column of belching flame, and a great black pillar of cloud. That was all—just the one big roar of the shattering explosion, one flare, and then the cloud, shooting out from the rent, rushing down the mountain side on to the doomed city.” *

It will be seen that in Martinique little warning was given, and the residents in St. Pierre were uneasy, but not madly excited. Things looked no worse than they had done often during the previous five days. It was the feast of the Ascension, and many were hurrying to church. So sudden, so unexpected, was the great

* Captain FREEMAN, “The Doom of St. Pierre,” ‘Pearson’s Magazine,’ September, 1902, p. 316.

catastrophe, that death overtook most of the inhabitants before they knew what had happened.

It was otherwise in St. Vincent on the previous afternoon. For an hour or more the mountain had laboured and groaned, quivering in the throes of the eruption, great explosions followed one another at comparatively short intervals, the enormous steam cloud got larger and larger, the violence of its ascent more and more terrible, the lightnings flashed, and loud crashing noises burst on the ears of the observers. Yet after all the climax came suddenly. There was an "enormously increased activity over the whole area,"* and with a terrific noise the great black cloud surged from the crater, and in one torrent poured down upon the valleys and right out to sea.

Captain CALDER was of opinion that the cloud ascended in the air and then curled over and came down the mountain slopes exactly in the same way as some have described the fatal blast in Martinique:—"As the top of the stupendous cloud bent over toward our little village the weird fascination gave place to a feeling of impending doom. It was vividly apparent that in a very short space of time this dust-charged pall of sulphurous smoke must envelop the district for miles."†

The eruption of Pelée on May 24th was seen by Professor HILL. He describes it in the following words‡:—

"Stepping out of the door I saw before me a perfect tropical night. Not a cloud obscured the starlit firmament. Suddenly, to the north and above Pelée, there was a dim flare of light like the sheet-lightning of a summer storm. This was the reflection of the incandescent molten mass within. Following this a great spherical cloud, with hundreds of boiling and seething convolutions, slowly rose above the vent. It had hardly appeared before it was followed by a blinding flash of light, like a great gun flash, from the mouth of the crater, accompanied by long, deep-pitched detonations from the bosom of the mountain. Over the crater's rim followed a fountain shower of incandescent pumice, which looked like molten fire. Hardly had the cloud-ball reached the air when around and through it flashed a thousand lightning-like streaks, with here and there great balls of fire. While standing in mute amazement observing this phenomenon at the apparently safe distance of some 3 miles, I was horrified to see the cloud fall suddenly, flatten, and float out horizontally into the sky like an aerial river directly toward and above me—a ribbon of inky blackness, and coming slowly, yet so fast that it was easy for me to see that it was not to be escaped by running."

According to Professor HILL's statements it would appear that the black cloud floated out horizontally in the air. If so it cannot have been of the same nature as those we saw on July 9th or that of May 8th, for these flowed along the ground. But in this case also it came suddenly with a loud noise and a bright light from the crater. The "fountain shower of incandescent pumice" was the avalanche of red-hot ashes, which, as a rule, is visible only after dark.

* Mr. T. M. McDONALD's Diary, 'Sentry' Newspaper, Kingstown, St. Vincent, May, 1902.

† 'Century Magazine,' vol. lxiv., p. 636, 1902.

‡ 'Century Magazine,' September, 1902, vol. lxiv., p. 778.

On July 9th, when we were off Carbet, there were two black clouds. It was obvious before the first emerged that activity was increasing, but not in our opinion to a dangerous degree. The first cloud welled out quietly in the twilight, and was quite a small affair. It was black from the first—a surging, foaming, boiling mass. Slowly it gathered speed, and came rushing along. The second was much larger, and in the darkness we could see that with it there was a glowing avalanche of red-hot dust. It came down with far higher velocity, but, like the other, slowed down rapidly when it reached the sea. With the first we heard no noise, with the second there was a loud angry growl. Both were full of lightnings, and in the features of their surface so like that no essential difference could be distinguished between them.

It is in all cases difficult to learn what happens after the great black cloud has passed. Not many survive that experience, and the terrible shock they receive unfits them, as a rule, for making exact observations of what is going on around them. But there are a few both in St. Vincent and in Martinique who have passed through that ordeal of fire, and their recollections of what followed are interesting, and at any rate worth placing on record.

In the Carib Country of St. Vincent, as already described, the hot wave lasted a few minutes, perhaps at most two or three. The darkness was absolute at first, but cleared slowly till it was possible to make one's way about in the open air, but not within the homes. The mountain continued to thunder and roar; hot stones fell through the air; there was a strong smell of sulphurous acid, and the fine, dry dust irritated the eyes and throat. Apparently there was no rain, and no second visitation by the dread and deadly cloud, but great explosions of steam were taking place within the crater, and forcing into the air quantities of dust, stones, scoria, and bombs, which fell in a continuous hail on all the country round. The mountain was veiled in clouds of ashes and vapour, and no one on the island could see what was taking place there till the morning of the next day. But the noises, the rocking of the ground, the continuous fall of stones and ashes, exactly resemble the features of eruptions of the usual type in which the explosive violence of the steam within the magma acts mostly in an upward direction.

This is also true of the later eruptions of the Soufrière during May of this year. For hours the mountain roared, and stones and dust fell through the air, landing with great force. But there was no second outburst of the black cloud, and nothing which can be regarded as in any way unusual in an explosive volcano. Having delivered itself of the avalanche of dust, the volcano settles down to spout its gigantic column of vapour into the air, sending up with it great quantities of fine dust and hot stones, and this goes on till the subterranean forces are exhausted, and a period of quiescence, or, at least greatly reduced activity, ensues.

At St. Pierre hardly one was left to tell the tale of destruction, and those whose lives were spared were so busy in taking measures to relieve pain and escape from the fatal harbour that they had little time to attend to the scientific phenomena. But

from the narratives of Captain FREEMAN and Mr. ELLERY SCOTT, we can glean a few particulars as to what happened during the time they spent before the city. The great cloud passed in a few minutes, and the air cleared sufficiently to enable them to see the burning streets and the inhabitants fleeing hither and thither, stricken to death with wounds and burns. No second incursion of that destroying cloud swept over them. The air was thick with dust, hot stones were falling freely, and the mountain boomed in the distance. At times the darkness lifted a little, till they could make out more clearly the details of the awful scene on shore.* Working backward and forward to free his steering gear from the dust with which it was jammed, the captain of the "Roddam" had still light to see the land and to avoid collision with the wrecked and burning ships which strewed the bay. The "Roraima" was ablaze; most of the crew and passengers were dead; she lay a helpless, burning hulk, but those who were alive took steps to put out the flames, and were at least able to keep them in check till 3 o'clock in the afternoon, when the French cruiser "Suchet" came in and took them off. This was about seven hours after the great explosion, and it seems that at Pelée the period of comparative quiescence followed much sooner than at St. Vincent, where the noises and the rain of stones continued for 12 or 15 hours.

The later stages of the eruption of May 26th are not recorded by Professor HILL, and on July 9th we could not say exactly what followed the descent of the red-hot avalanche and the black cloud, for the mountain was concealed in a mass of dust and vapours, and we were gradually increasing our distance from it. But we believe that on that night the activity was short-lived, and soon came to an end.

4. *The Concluding Stages.*—The concluding phases of these eruptions are in most respects similar to the preliminary stages, except that the activity is now constantly decreasing. Steam rises from the crater in rolling, expanding clouds, densely charged with fine dust, which gives them a slaty-grey colour. The Soufrière relapsed into complete inactivity in seven or eight days, to break out again after a week's repose. Montagne Pelée has continued, between the greater outbursts, to send out, more or less frequently, the cauliflower steam clouds which well from the fissure in its side. In other respects the two mountains have behaved in much the same way when the period of waning violence arrived. Each day they have been less active than on the previous one, with occasional rare and temporary exceptions, on which the discharges increased for a few hours to again diminish more rapidly than before.

In St. Vincent the eruption of May 7th was followed by that of the 9th, without any intervening cessation; between that of the 9th and that of the 18th there was a considerable pause. After the eruption of the 8th at Pelée came that of the 20th, and then that of May 26th. On June 6th another took place, and thereafter for a

* Professor LACROIX and his colleagues state that the sky cleared in one hour after the passage of the black cloud. "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. exxxv., p. 425.

month there was no great outburst. It is true that several have been recorded both from St. Vincent and from Martinique on other dates, but in the former case they were false, and in the latter we have good reason for believing they were trivial, or did not occur at all. So excitable is the population after the terrible events of this spring, and so willing are the newspapers to publish sensational news without inquiring as to its veracity, that many eruptions have been reported which never happened. On Pelée there was an outbreak on July 9th, and two minor ones on the 11th and the 13th, and between these dates the smaller steam clouds were frequently emitted, though for hours at a time none might be seen, and as on the afternoon of the 11th, when we passed along the base of the mountain, there might be nothing to indicate its deadly virulence except the hot water in the streams, and the thin coating of fine, pale-grey ashes which had been scattered over its surface that very morning.

The Avalanche of Sand and the great Black Cloud.

A synthetic study of the features of these remarkable discharges, so far as they are known to us from our own observations and those of eye-witnesses on whom we can rely, involves the discussion of many matters which are more or less theoretical; and we have, in consequence, thought it best to separate this part of our Report from the previous chapters in which we describe what we and others have seen, referring only indirectly to the underlying causes.

All witnesses agree that the great black cloud consists of dust, stones, and gases at a very high temperature, and moving with a great rapidity. But as yet it is impossible to give a complete description of its properties, though sufficient evidence is available to enable us to consider its outstanding features and to justify an attempt to explain them.

The Beginning of the Blast.—We have ourselves observed one of the black clouds emerge from the crater and come rushing down the hill, and have already described what we saw on that occasion. The evening of July 9th was perfectly clear, and the mouth of the fissure, on the south-west side of Pelée, was clearly visible from where we were anchored, a little north of Carbet. The little black cloud ball rose from the crater and rested on the lip, tumbling and seething; it lay there for a little time, then began to travel down the hill, at first slowly, then faster and faster, till it rushed down the lower slopes with a velocity which must have approached 100 miles an hour.

Others have seen the black cloud rise from the fissure, but their descriptions do not entirely agree with ours. Several describe it as rising in the air a considerable distance and then curving downwards as if it overbalanced itself. It is probable that in the larger outbursts a considerable mass of dust is projected up into the air; and as this black cloud is too heavy to ascend or to be wafted away by the

wind, when its upward energy is spent it sinks to the ground, owing to its own weight, and then flows down the hill. But if we consider the great eruption of the Soufrière and the enormous amount of hot dust which was then shot into the valleys, it seems quite unlikely that more than a small fraction of this was elevated to any great height above the crater. The bulk of the material must have swept down in a river of hot sand, though over it a lighter cloud would form, consisting mostly of hot gases densely charged with dust.

All witnesses, however, agree that the outburst appears in the form of a cloud, which rises to a certain height, and then flows over the surface of the ground. None have seen it in the form of a fluid molten lava. Professor HILL describes the black cloud of May 26th as floating out horizontally in the air, but in this respect that eruption differs entirely from all the others.

The mixture of dust and gases is so heavy that it courses down the slopes like a torrent in a river, clinging to all the valley bottoms, ever availing itself of the steepest descents, and deflected by the projections and irregularities of the ground. That it does so we are convinced, not only from our own observations of its effects on the Soufrière and in St. Pierre, but also from what we saw on the night of July 9th. The black cloud poured down the hill like a torrent of inky water, except that it spread out, expanded, and its upper surface rose slowly in the air as it advanced. This explains also why the great mass of the ejecta at the Soufrière came down the south side of the hill, where the crater rim is lowest, while the north side was spared, as the Somma wall protected it.

That a mixture of gases and dust should behave in this way is certainly remarkable, but similar phenomena are to be observed in connection with even the minor steam jets emitted by Montagne Pelée. Professor LACROIX and his colleagues describe : *
 “Waves of dense vapour, heavy, dark coloured, often coppery, which roll over the external taluses of the crater, and down to the bottom of the fissures in the region of the Rivière Blanche. They are probably puffs of gas and steam very richly charged with ashes.” It is, moreover, not uncommon, when a steam cloud larger than usual has been ejected, to see its lower part dark, heavy, and laden with dust, rolling over the surface of the ground, while its summit of pure white steam steadily mounts into the air.

In considering this property of perfect fluidity, which the black cloud possesses, we must remember its origin. Within the crater it was a molten magma, in which a considerable number of small crystals floated in a liquid which contained enormous quantities of occluded steam. As it rose in the throat of the volcano the relief of pressure allowed the gases to expand, and to free themselves from the liquid in which they were held. Sooner or later the cohesion of the liquid was overcome, and from a spongy froth the mass changed to a cloud of particles, mostly solid, but, perhaps, in

* MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, “Sur l'Éruption de la Martinique,” ‘Comptes Rendus,’ vol. cxxxv., p. 380. See also vol. cxxxv., p. 426.

some part liquid, each surrounded on all sides by films of expanding gases, and thus the mixture of the ingredients from the first was perfect. Around each grain of dust there was a film of gases ready to expand enormously when the mass reached the upper air.

The amount of expansion of which these gases are capable is so great as to be almost incredible. The small black ball of cloud, which we saw emerge on July 9th, in a few minutes was a great black mass, which covered more than a square mile, and the white steam, which shoots upwards in the air when the cloud is dead, is still actively expanding at a great rate. This leads us to wonder whether, when the cloud emerges, it may not be partly at least composed of molten droplets, which, when they cool and pass into the solid condition as the cloud rolls on, give out gases which, till then, had been physically occluded or absorbed in the liquid. The mere pressure within the crater would almost seem insufficient to compress so great a volume of gas into so small a space, especially when we remember the very high temperature of the mass.

As this turbulent mixture of expanding gases and fine dust pours down the surface of the mountain, the small, solid grains are unable, at first, to rest on the ground, even when they may have sunk to the base of the cloud, and they are swept up again, and borne along till they reach some sheltered hollow, or the violence of the expansive forces lessens and the turmoil diminishes.

As noted by Professor LACROIX and his colleagues,* the cloud cools more rapidly when it passes over the sea than over the land, as witness the more complete destruction of the south end of St. Pierre than of the "Roraima" and other ships, which were lying off the shore. The explanation is simple: when the hot dust falls on the sea it can rise no more. The sea also, owing to its higher specific heat, rises more slowly in temperature than the land surface, and, though it may be raised to boiling point, it cannot exceed this. There is no such limit to the possible temperature of dry, bare earth. By the trapping of the dust and the cooling of the gases in contact with the water the cloud soon loses its heat, and with that its power of buoying up the solid matter it contains.

The temperature of the magma, when it rises within the crater, can be fixed within certain limits, though these are not very precise. We know that it was bright red hot, for all who have seen it at night have described it as incandescent. When the cloud struck the north end of St. Pierre the dust was hot enough to set fire to combustible articles, but did not fuse the copper wires of the telephone apparatus, or melt objects of brass or tin. And in the liquid within the crater were floating crystals of hypersthene and plagioclase felspar in great abundance, and in such perfection of crystalline form as to indicate that they had formed by crystallization out of a cooling molten mass. Professor JOLY gives the melting point of labradorite

* "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 424.

as 1229° C.,* but this, of course, is in the dry condition and under atmospheric pressure, not in a magma saturated with steam.

The velocity of the black cloud is different in different parts of its course. So far as our observations go it is comparatively low at first, and increases as it sweeps down the hill till it is probably at a maximum just before it reaches the plains beneath, or the sea. Then it again diminishes, and rapidly slows down, as is well seen in St. Vincent by its effects on the trees. We have compared its violence to that of a hurricane or tornado, but in many ways the comparison does not hold good. For equal velocities the destructive effects of the dust cloud must have been considerably greater than that of any wind, for the weight of the mass, or rather its momentum, is high, owing to the density of the mixture of gases and of dust. This is much heavier than air, and flows along the ground, but lighter than water, on which it always floats.

There are a few facts which seem to show that the mere weight of the cloud did much damage, as, for example, that the hatches of the "Roraima" were stove in.† We looked in St. Vincent for similar effects, but found none, even on Lot 14, which was nearest the crater. The roofs had in some cases collapsed, probably with the weight of ashes which gathered through the night. Many windows were broken, apparently by falling stones, as in Georgetown. There was no evidence that a great aerial shock, such as follows an explosion or the discharge of large guns, had broken the windows, though, as it was about mid-day, they were probably open, and may have escaped in consequence.

Professor LACROIX and his colleagues‡ consider that the blast which destroyed St. Pierre covered a distance of 8 kilometres in three minutes. No very good evidence is available on which to base exact estimates. When the blast passed over the north end of the town it was travelling with a velocity certainly over 100 miles an hour, but before it got to the Morne d'Orange, at the south end, it had considerably slowed.

The fundamental question remains to be discussed—What is the source of the energy which drives the cloud along? To this we believe there is only one answer—The motive power is supplied by the weight of the mass. It is in a condition comparable to that of a heavy and mobile fluid which has been elevated by the volcanic forces and poised on the edge of the crater, and proceeds to flow downward in obedience to the law of gravitation. This is, to our minds, the only conceivable explanation of what we saw on the evening of July 9th. The little black cloud rose from the crater exactly like a puff of steam, but lower and more globular. It did not

* The melting point of hypersthene has not been ascertained, but that of enstatite, a less ferriferous rhombic pyroxene, is given as 1295° C. The melting point of different kinds of monoclinic pyroxenes ranges between 1187° C. and 1300° C.

† 'Cosmopolitan,' July, 1902, p. 250.

‡ "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 425, 1902.

start its rush down the slopes at once, but rolled and tumbled, squirted and seethed, for quite a perceptible time. Then it began to move with greater and greater speed down the hillside. Faster and faster it came till it struck the sea, when its velocity began to diminish, at first slowly, then more and more rapidly. It was like a toboggan on a snow slide. It was not the blast of a gun, it was the rush of an avalanche. Gravity did the work and supplied the energy. There was no explosion; had there been any, we would certainly have seen it. The lightnings were visible only when the cloud came near.

Like an avalanche, it drives the air before it and parts it on each side. In the statements of Captain FREEMAN and Mr. SCOTT there is the clearest evidence that the blast struck the vessels and heeled them over before the hot dust began to fall* :—“On a sudden he (Captain FREEMAN) heard a tremendous noise, as though the entire land had parted asunder. Simultaneous with the noise there was a great rush of wind, which immediately agitated the sea, and tossed the shipping to and fro. He rushed out of the chart-room, and, looking over the town and across the hills, he saw a sight he cannot describe. He remembers calling out to Mr. Campbell and saying, ‘Look!’—and then an avalanche of lava was upon them.”

But these discharges are not mere avalanches; they are more, for they have properties unlike those of avalanches, and by means of which they approach more closely to blasts. In an ordinary avalanche the gases are more or less accidentally involved, and form only a small part of the whole mass. They are compressed by the weight of the moving solid mass, and when that pressure is relieved they expand again.

But here their presence is essential; they are an original part of the mass, and without them there would be no flow. They are expanding, surging of their own inherent energy, and lift the dust and sweep it along, while the dust in turn fetters them and compels them to keep to the surface of the ground.

The dust avalanches, or blasts emitted by these two volcanoes during the recent eruptions, though essentially similar, show minor points of difference. The first discharges have been in both cases those which contained most solid matter. In particular, the great avalanche which blocked the Wallibu and Rabaka Valleys in St. Vincent on May 7th must have consisted to so great an extent of red-hot sand that it may best be pictured as rivers of dust and stones flowing down from the crater. But before the main avalanche, and on each side of it, a great black cloud swept over the country; it consisted mostly of gases, though laden with hot dust, and resembled a hot blast far more than an avalanche.

On Montagne Pelée also the first eruption probably sent out more solid matter than any of the others, and was consequently more like an avalanche than those which succeeded it.

* Blue Book: “Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,” p. 45.

It will be understood also that, as a rule, the greater the mass of material ejected the farther the avalanche travels, and the greater is the black cloud which accompanies it. On July 9th the first black cloud was small, and soon came to rest; the second was much larger, and nearly overwhelmed us. Of course, we could not say what was the absolute velocity of either at any point in its course, but certainly the second rushed out twice as far as the first across the bay.

Probably the mass is never homogeneous throughout at any point after it has left the crater. The heavier solid ingredients must gradually sink to the bottom and flow over the ground, while the lighter gases will tend to rise to the surface and to dilate laterally. Thus from an early period in its history the cloud will have a heavy base, in which the solid particles preponderate, and an upper part lighter, and relatively richer in gases. The black cloud covers and envelopes a hot avalanche. The surface and sides of the mass are in contact with the air, mix with it, and are cooled. Hence those towards whom the cloud is coming may see a black advancing front, while when encircled by it they find it incandescent and red hot. That must have been the case with the unfortunate inhabitants of the north end of St. Pierre.

When we keep in mind the composite nature of these discharges, it becomes possible for us in some measure to understand the peculiar properties they possess at different points in their course and in different stages of their development. When they rise from the crater the gases are in a condition of indescribable turmoil, and buoy up the vast quantity of hot sand they bear with them. Each solid particle is cushioned on gases violently expanding, and the temperature of the whole mass is exceedingly high. Hence it seems quite probable that the black cloud rushes upwards, and may even shoot some distance into the air and fall back upon the slopes of the mountain when its first impulse is spent and its first heat has cooled. We may, at any rate, believe that the whole mass has then a mobility which it no longer possesses, when, by the expansion of the gases, by mixture with the air and by contact with the ground, its temperature has been lowered, and the solid matter is beginning to segregate from the gaseous, each taking its own course.

Some such explanation is required to account for the manner in which the upper part of the black cloud of May 7th surged over the Somma wall which overlooks the crater of the Soufrière, and poured down the north side of the mountain. The same cloud in the lower part of the valleys of the Wallibu and Rabaka Rivers was deflected by even comparatively slight ridges, and clung helplessly to the bottoms of the ravines—that is to say, its lower and heavier part flowed over the ground, for there is good evidence that above this there was an enormous mass of black, dust-laden vapours which filled the whole mountain valley between the Soufrière and Morne Garu, cutting down and sand-blasting the growing trees, though the mass of sand it left behind was comparatively small.

We have said that the blast which levelled the walls of the stone-built houses of St. Pierre was about as vigorous as that of the Soufrière at equal distances from the

crater, while the amount of solid matter discharged was very much smaller. Several possible explanations of this offer themselves. It may be that the velocity of the blast is greatest at that point where the slope of the hill gives place to the plain beneath, and the city stood exactly in that situation. It is more probable, however, that the avalanches of Pelée are of a more mobile and fluid character than those of the Soufrière, and hence acquire a greater velocity, and this must be a consequence of the slightly different proportions of gaseous and of solid matter which they contain. To judge by the crystalline state of the dust, as revealed by the microscope, there can have been no very great difference in the temperatures of the two magmas at the time when they were shivered into dust. That of Pelée contains rather more amorphous, glassy matter, and may have been slightly the hotter of the two, but not to any great degree. We may suppose that the percentage of gases in the dust cloud of Pelée was greater than in that of the Soufrière, the temperature at least equally high, the interfusion and admixture of the components equally perfect. The one may be compared to a heavy fluid, extremely mobile, the other to one still heavier, but more viscous. It would seem that in these circumstances, as the slopes down which they sped were about equally steep, the former would acquire a greater velocity after a certain specified period, or after covering a certain distance from the crater. It must also be kept in mind that the avalanche emitted by Montagne Pelée on May 8th was confined to a very limited space, and that the north end of St. Pierre stands, apparently, not far from the centre of its path.

This brings into prominence one property in which these discharges differ from ordinary avalanches. They have, in virtue of the expanding gases with which they are permeated, the power of moving down slopes much gentler than those on which an avalanche could start. The sides of Pelée and the Soufrière have an average inclination of 12 to 15 degrees, if we neglect the deep, steep-sided ravines with which they are seamed, and no avalanche, even of fine snow, would move on such gentle gradients. Had the hot sand been piled up there it would have rested peaceably; without the surging gases intermingled with the solid particles there would have been no motion. It is not correct to regard these cataracts of sand as mere avalanches; the idea of a blast is also essential, if we are to form a proper conception of the mechanism in operation.

Several other explanations have been advanced to account for the manner in which the deadly cloud of the Peléan eruption sweeps over the ground, but does not ascend in the air like the ordinary steam clouds with which all geologists are familiar in volcanic outbursts. One of these we may call the hypothesis of oblique discharges, the other the explosion hypothesis. It has also been hinted that electricity is responsible in some way or other for the devastation, but we are not aware that anyone has as yet formulated any workable hypothesis on this basis.

The great V-shaped fissure which looks down on St. Pierre from near the summit of Pelée is undoubtedly responsible, by its disposition and conformation, for the course

which the discharges take as they sweep down the mountain. The avalanche of dust rises in this cleft, it tumbles out on its lower or southern lip, where it is hemmed in by walls of rock on all sides but one. We can easily see that these are powerful factors in determining the path along which the blast will travel. But it has also been stated* that on the north side of the crater there are visible passages descending obliquely through the rocks of the mountain, and it is from these the avalanche of sand is launched in a direction which is nearly horizontal. That may be so, but even in that case it is not stated that these fissures are above the southern lip, or that they can emit blasts inclining downward at an angle of 12 to 20 degrees from the horizontal. And if from these fissures the dust-cloud is shot obliquely *upwards* so as to graze the southern rim of the fissure, it still remains to be explained why the cloud sinks down again and flows over the surface of the ground, refusing to rise in the air and float away as an ordinary cloud would certainly do. This, in fact, merely postpones the difficulties, and does not settle them. Moreover, if such an explanation were possible at Pelée, it would have no bearing whatever on the eruptions of the Soufrière, as there the crater is a vast bowl about half as deep as it is broad; the black cloud could never have been shot obliquely from the bottom of this, down the mountain side, but must have risen nearly vertically into the air.

The explanation proposed by Professor JAGGAR† is of a different kind: "These horizontal blasts are not hard to account for, and do not require a horizontal nozzle to project them. They are simply the effect of the down blast after the heavy gravel has begun to fall, acting against the upblast from the throat of the volcano, and both together deflected and thrown into terrific whirls or tornadoes by the prevailing wind, which on Mont Pelée is north-east." He would, in fact, account for them by the resistance which the sand and lapilli falling through the air offered to the ascent of subsequent discharges. Mr. E. O. HOVEY‡ has adopted this hypothesis without essential modification.

But, in our opinion, this is quite incompetent to explain the behaviour of the black clouds we saw on the night of the 9th of July. Before the first black cloud arose no very great amount of dust had been projected into the air, and the steam clouds were drifting steadily westwards before the trade wind towards Prêcheur. Before and after the appearance of the black cloud the steam ascended freely and apparently without hindrance. The black cloud took a different path, and once it had rolled a short way down to the mountain there was nothing above it to prevent it rising in the air; but it hugged the surface of the ground so closely that the conclusion was inevitable that it flowed down merely because it was too heavy to ascend.

* 'American Journal of Science,' Series IV., vol. xiv., p. 73, 1902.

† T. A. JAGGAR, "Field Notes of a Geologist in St. Vincent and Martinique," 'Popular Science Monthly,' vol. lxi., p. 366, 1902.

‡ E. O. HOVEY, "Martinique and St. Vincent: a Preliminary Report," 'Bull. Amer. Museum Nat. Hist.,' vol. xvi., p. 341.

The theory of explosions has also won a certain number of adherents, among whom are several distinguished American geologists. That there was an explosion—a steam explosion—is, of course, admitted by all. But some hold that there were also explosions of combustible gases attended by sheets of flame, and that these took place either during or shortly after the outburst from the crater. Our objection to these theories is that they have no sufficient basis of observed facts on which to stand. Neither what we know about the gases in the cloud, nor the recorded observations of eye-witnesses of the eruptions, make it in any way probable that such explosions took place except on a minor scale, or that they had an essential part in the propulsion of the discharges.

Many, of course, who saw the great black cloud emerge from Pelée or the Soufrière describe it as welling out of the hill with sheets of flame. But students of volcanic phenomena are too well acquainted with such descriptions to place any great reliance upon them. The popular mind is not careful to distinguish between a mere incandescent discharge, or the glow of the red-hot surface of the lava reflected from overhanging clouds, and true sheets of flame. The belief in burning mountains dies hard.

On two separate occasions we have witnessed the black cloud which rises from the fissure of Pelée, and on neither did we see any flame. Through the face of the cloud lights flickered and scintillated, but they were only lightnings. We have cross-examined many careful and accurate observers who have seen more than one eruption of the Soufrière, and not one of them described them as attended by flames. Neither were the noises they heard like the detonations of aerial explosions, they came from within the mountain. It is no doubt possible to obtain declarations from eye-witnesses, stating that great flames were visible, but they either break down altogether on careful cross-questioning, or are advanced by persons too uneducated, too excited, and often too inaccurate, to give evidence of any value in matters of this sort.

The Gases of the Cloud.—This raises the question—What were the gases in the cloud, and what were their properties? Unfortunately, this is a subject at present imperfectly known. We can only approach the question by indirect methods; no one has bottled a sample of the great black cloud. But what evidence is available is singularly consistent, and makes it highly improbable that great explosions due to chemical combinations between the gases of the cloud, or between these gases and those of the atmosphere, played any conspicuous part in the mechanism of the eruptions.

The gases of the cloud were the gases of the andesitic magma. The great black mass cleaved the air, driving it aside in virtue of its weight and the expansive forces within it. The atmosphere was passive, inert; it could not penetrate to the interior of the rushing inky torrent; only at the margins, where the cloud sent out curling wreaths of dusty vapour, was there any mingling between it and the air.

Even after the dust had settled down, and the steam mounted from the sea and forced a path obliquely upwards through the air, the white cloud cut its own way, maintaining its separate identity for a prolonged period.

Of these gases the most abundant was certainly steam. At first it was dry, invisible, superheated; afterwards condensation set in, and it formed a fleecy cloud of mist. All the symptoms of the injured are in harmony with the theory that steam and hot dust were the deadly agencies at work. The feeling of suffocation experienced by the survivors, and the exhaustion of the air so that it did not support respiration, are all explicable in this way. The falls of moist ash and of hot mud, which were more or less local, but were observed by many, are natural consequences of the abundance of superheated water vapour.

Next in abundance, in St. Vincent at any rate, was sulphurous acid. All the survivors agree in this; they describe it as the smell of burning matches. Most of the medical men who attended to the injured in St. Vincent considered that dry, hot steam and hot dust were the principal components of the cloud, and that sulphurous acid was very abundant, so abundant that it might have even caused some deaths, and have been responsible for some cases of bronchial catarrh and pharyngitis, though it was not possible to separate its action from that of the other constituents, which would have been, on the whole, similar. On account of its smell, the sulphur dioxide was very conspicuous, but it did not leave any very startling effects. We did not learn of any cases of bleaching or discoloration, or of the formation of crusts of sulphite or sulphate, on any objects of metal.

In St. Vincent sulphuretted hydrogen was also present to some extent. It was recognised by several competent observers. The silver ornaments on the arms of some of the coolie women turned black in a minute or two. Months afterwards the mud around the crater stank of sulphuretted hydrogen till on a hot day it gave one a headache. But the medical men did not attribute any deaths to it; symptoms of poisoning by its action were not observed; asphyxiation, burns, and shock were the chief causes of death.

It may be that in reality not sulphuretted hydrogen but metallic sulphides (such as calcium sulphide) were originally present, and that in presence of moisture they were decomposed at a certain temperature, and hydrogen sulphide produced. This is the most probable explanation of the abundance of that gas in the month of June in the wet mud near the summit of the hill.

As has been remarked by Professor LACROIX and his colleagues, there was comparatively little sulphur in the magma of Pelée, and analyses show that it is certainly more abundant in the ejecta of the Soufrière. But for weeks before the fatal May 8th there was a strong smell of sulphuretted hydrogen in the streets of St. Pierre, and the old name Soufrière, given to the Étang Sec, is an enduring witness to the presence of that gas. All the Caribbean volcanoes emit it; it has given a name to many a crater and fumarole in the various islands. It was certainly present in the

great black clouds in Martinique, and on the night of July 9th the wet ash which fell upon our decks smelt of sulphurous acid, though not strongly. Sulphuretted hydrogen is, of course, combustible, but would occasion no bright flame, and its presence cannot be blamed for the devastating forces which levelled the town.

We may also be certain that other gases were there, some of them highly inflammable, but it is impossible to believe that they were present in relatively large amount. In this line of inquiry the classic researches of FOUQUÉ on the gases of the andesitic magma of Santorin are the best guides. Hydrogen and various compounds of hydrogen and carbon were ascertained by him to be present in that case, and there is every probability that they were here also, but there are no data to enable us to judge what were their composition, proportions, and abundance. They may probably have occasioned trivial explosions, but could only have ignited when mixed with air—that is to say, on the outer fringes of the cloud. They were not its motive force, and could not have supplied the energy which launched it on its errand of destruction.

Hydrochloric acid has been mentioned by some as having been observed in connection with certain of the eruptions. No doubt it was there, but as it has left no visible effects, so far as we know, we cannot suppose it to have been plentiful. Nitrogen is another probable component, and so also is carbon dioxide, which accompanies practically all volcanic outbursts; but at the present moment little can be said about these, for, though we may admit that they were present, the evidence respecting them is altogether of a negative kind. There is nothing to show that they played any important part in the mechanism of the eruptions.

It has been suggested by Professor VERRILL* that the dissociation of steam within the volcano produced an explosive mixture of hydrogen and oxygen which combined with violence on emerging from the mouth of the crater, and that this explosion explains the disastrous force which razed the city of St. Pierre. That such dissociation takes place to a certain extent is made probable by the researches of FOUQUÉ,† but, owing to the enormous pressures, this could have affected only a small part of the water in the magma, which, moreover, was at a temperature not very much higher than that at which steam begins to be dissociated at atmospheric pressure.

The crystalline minerals which were floating in the magma before eruption show that it was at a temperature of less than 1200° C., and this renders it probable that the amount of dissociation produced would be quite inconsiderable. The magma of Santorin, to judge by the nature of the products, is considerably hotter than that of Pelée at the moment of effusion. It is also certain that when the lava broke forth, cooling would be continuous and not instantaneous, and the gases would

* 'American Journal of Science,' 4th Series, vol. xiv., p. 72 (July, 1902); also by Professor ISRAEL C. RUSSELL, 'Century Magazine,' vol. lxiv., p. 790, September, 1902.

† 'Santorin et ses Eruptions,' p. 232.

combine gradually and quietly, as the temperature fell, without occasioning any explosion whatever. Some, however, of the products of dissociation would escape combination, and these might, if not too much diluted with other ingredients, be ignited by lightning flashes at a later period, though we are not inclined to regard this process as having taken place on any but a very small scale.

Others, of whom Professor HEILPRIN* is one, hold that the cloud was highly charged with mephitic carbon gases. This theory is interesting, though we cannot endorse it:—

“What the exact constitution of this death-dealing cloud was will never perhaps be known, but its associations with the mud discharges, its heavy specific gravity, and the mephitic or oily odour of the products emitted by both the lower and upper craters, lend reasonable certainty to the belief that this glowing cloud was mainly composed of one of the heavier carbon gases brought under pressure to a condition of extreme incandescence, and whose liberation and contact with the oxygen of the atmosphere, assisted by electric discharges, wrought the explosion, or series of explosions, that developed the catastrophe.

“To the enquiry as to what was the source of this carbon gas—to my mind the main factor of the catastrophe—the geologist points to those vast bituminous deposits, like those of Venezuela and the island of Trinidad, which lie but little out of the line of the connected series of volcanoes, of which the Soufrière of St. Vincent, and Pelée of Martinique, are a part. He also points to the limestone deposits, with their enormous masses of locked-up carbon, forming the foundation on which these same volcanoes are implanted, which indicate a source of energy far greater than was required for the catastrophe of Pelée.”

When we were in St. Vincent we made most careful inquiries, both of the survivors and of the medical men who had attended the injured, as to the occurrence of poisoning by carbon dioxide, carbon monoxide, or poisonous hydrocarbons. We failed to find any evidence whatever of symptoms such as would have indicated that much of these gases was present, and the medical men were all convinced that they were not responsible for the fatal effects of the great black cloud. This seems to us all the more remarkable, as such gases must have been present in the cloud which, on May 7th, did such deadly havoc in the Carib Country, and this for a reason which has apparently escaped the American professor.

That afternoon, as the darkness was closing in around the north end of St. Vincent, the richly-wooded slopes were still covered with all their wealth of tropical forest up to the moment of the climax, when the great dust avalanche arose. In an instant all was changed. The hot sand which now fills the valleys is mingled with innumerable fragments of charred, broken trees, caught up, destroyed, and swept along in that burning flood. The great hot blast which radiated outwards mowed down the standing trees, scorched them, eroded them, consumed their leaves, twigs, and smaller branches. The amount of vegetable matter carbonised or semi-carbonised in that brief space was enormous. The charcoal in the valleys is washed out and floats on the sea, supplying the inhabitants with fuel for months to come. Never

* ‘Fortnightly Review,’ September, 1902, pp. 477 and 478.

was a fairy landscape changed to a blighted desert in a shorter space of time. What came of all the carbon compounds produced by the destructive distillation? They must have been incorporated in the great black cloud. This was in itself one of the most startling and marvellous features of that strange eruption.

What form the products took, what was their relative amount and the changes they passed through, we can only guess. There was little or no oxygen in the cloud, and the vegetable matters were practically distilled in presence of dry steam. Any oxygen which had not already combined with the hydrogen or hydrogen sulphide would unite with the carbon of the wood to form carbon monoxide or carbon dioxide, and the vast number of volatile organic compounds produced by the charring of wood in the absence of oxygen must all have been there in greater or less abundance. Many of these substances must have been capable of producing explosions when mixed with air or oxygen in suitable proportions, and we may believe that the black cloud at one stage of its history was entirely deprived of oxygen and filled with carbon gases. It is probable that the temperature fell too rapidly, and the process of mixture with the air was too slow to allow of any considerable part of these gases being burnt in the cloud. Before the oxygen could reach them they were too cold and too much diluted with steam to ignite. We may in this way explain why the ash which fell in Kingstown on the afternoon of May 7th had an odour of organic matters. Mr. POWELL described it to us as strongly resembling that of guano. The ash was moist, and the water mixed with it may have absorbed some of the more soluble organic products of the great black cloud.

Even when we were in St. Vincent in the month of June, after a few hours of dry weather and sunshine, wreaths of blue smoke would be seen curling up into the air from the banks of hot sand where pieces of carbonised wood lay partly exposed to the atmosphere. The air then smelt strongly of burning wood, and specimens of tar have since been sent us, which were brought down by the streams when the ashes were washed away by the rains in the Rabaka Valley. This tar is a product of the distillation of the wood as it lies embedded in the hot ashes.

No other eruption was attended by these circumstances; no second black cloud involved in its mass the forest growth which for 90 years had clothed the surface of the mountain. It had vanished like a fall of snow, and in its place was now a waste of ashes. Carbonic acid and carbon gases there may be in the later outbursts, but these are from the magma, original and not accidental ingredients. We may, in fact, be certain that they are there, but we may be equally certain that they are not the principal gaseous components, and that neither does their density determine the behaviour of the heavy cloud, nor does their potential energy furnish the motive force which impels it.

The gases emitted by volcanoes have been the subject of much study, though they are still less known than any of the other products. In the interests of science it is desirable that some day we may be able to obtain samples of the gaseous emanations

of Pelée and the Soufrière. It is to be hoped that the investigations of the French Commissioners will greatly advance our knowledge in this respect, for the gases of the Caribbean volcanoes may be supposed to present important points of difference from those which accompany less violent and more continuous activity. We regret that the results of our own inquiries are so purely negative in character, but St. Vincent does not offer the conditions necessary for the investigation.

The Physical Condition of the Magma.—Accounts have already appeared of the composition and structure of the dust which fell on Barbados on May 7th,* and of the earlier falls of ashes in St. Pierre and Fort de France.† It has not yet been possible for us to find time to make a comparative study of all the specimens which we have collected during our visit to the West Indies, or have been sent us by correspondents residing there. But there are certain general conclusions regarding the composition and structure of the materials emitted which we may draw from the facts at present before us, which throw some light on the causes which determined the magma to assume the form of a dust cloud.

All the samples, and notably the Barbados dust, contain a very large proportion of crystalline fragments of volcanic minerals, often broken, but frequently showing perfect crystalline form. Volcanic glass is present also, in fine threads and broken splinters and in thin pellicles, coating the surface of the crystals. The same material occasionally forms little rounded pellets, in which small microliths may be seen embedded in the glassy matrix. That the Barbados dust should be so largely of a crystalline nature has awakened general astonishment, especially when it is compared with the fine air-borne dust of Krakatoa and Cotopaxi. The glassy fragments are of lower specific gravity than the minerals, and while the latter frequently are sub-cubical in form, the former are highly irregular, and have a very large surface in comparison with their volume and weight. In consequence, the splinters of glass should fall through the air more slowly, and should be carried by the wind to greater distances before subsiding, than the heavier and more compact minerals. The ash which fell on ships some 200 or 300 miles to the east of Barbados

* JOHN S. FLETT and W. POLLARD, 'Quarterly Journal Geological Society,' vol. lviii., p. 368. J. S. DILLER, "Volcanic Rocks of Martinique and St. Vincent," 'National Geographic Magazine,' vol. xiii., p. 285. W. F. HILDEBRAND, "Chemical Discussion of Analyses of Volcanic Ejecta from Martinique and St. Vincent," 'National Geographic Magazine,' vol. xiii., p. 296. JOHN D. FALCONER, 'Nature,' vol. lxvi., p. 132. Professor CARMODY, Blue Book on "Volcanic Eruptions in St. Vincent and Martinique in May, 1902," p. 83. LONGFIELD SMITH, Blue Book on "Volcanic Eruptions in St. Vincent and Martinique in May, 1902," p. 79. T. C. PORTER, 'Nature,' vol. lxvi., p. 131.

† MICHEL LEVY, "Sur la Composition des Cendres Projetées, le 3 Mai, 1902, par la Montagne Pelée," 'Comptes Rendus,' 134, p. 1123. MM. A. LACROIX, ROLLET DE L'ISLE & GIRAUD, "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 377. A. LACROIX, "Sur les Roches rejetées par l'Éruption actuelle de la Montagne Pelée," 'Comptes Rendus,' vol. cxxxv., p. 451. T. G. BONNEY, 'Quarterly Journal Geological Society,' vol. lviii., p. 86, 1902.

contained a higher percentage of glassy matter than that which was gathered on the island.* From their freshness, their idiomorphism, and other characters, the crystals were clearly formed in a fluid magma, and the glass may be taken to represent the still liquid material at the moment of eruption. If so, it seems clear that such a rock could never have formed a pumice. It is too highly crystalline, and contains too little glass. When the steam separated out in little bubbles, which expanded and expanded as the retaining pressures diminished, a time arrived when the fluid part of the magma had passed into a spongy froth, in which innumerable crystals occupied the walls between the vesicles, and on further expansion the mass could no longer hold together, but passed into a mist of droplets, most of which held a crystal in their centre, while some consisted mostly of the glassy material.

Such a magma, with its high percentage of solid inextensible crystals and its small proportion of fluid rock, when the steam within it began to expand, very soon passed beyond its limits of cohesion, and was rent into separate particles, each of which was in most cases a crystal surrounded by a film of glass. The crystals themselves were not only incapable of extension, they were also anhydrous, and acted as passive ingredients; only the liquid magma contained imprisoned water, and the immense amount of steam developed in the explosions is all the more remarkable when we reflect on the relatively small proportion of the substances which held it in solution.

We may infer from its highly crystalline condition that the magma was also at a comparatively low temperature. The minerals have the characteristics found in those of volcanic rocks, and were not formed at very great depths or under such pressures as determine the production of rocks with plutonic structures. The molten mass was lying in the conduits of the volcano, cooling gradually there, and the separation of the first crop of crystals proceeded no doubt for a considerable period and under exceptionally favourable circumstances. Had not the subterranean pressures increased and driven the mass upwards, and the temperatures at greater depths been so high as to render the ascending forces irrepressible, the upper part of the lava column might in a short time have completely solidified.

In all probability crystallization was more advanced near the surface than in the deeper parts, and the lower portion of the magma was more completely fluid than that which was ejected as the great black cloud. In this regard it is important to remember that the dust avalanche is always the first product of the crowning stage of the eruption. As soon as the obstruction in the orifices of the volcano has been overcome the cloud wells forth. It is the upper part of the ascending column of molten rock. We may almost say that the great black cloud is the froth that is blown off the surface of the subterranean reservoirs. When it is once over everything, so far as we know, goes on in the usual manner. Bubbles of steam arise, burst, and

* J. S. DILLER, 'National Geographic Magazine,' vol. xiii., p. 293.

project showers of hot bombs and dust into the air. The later stages of the Peléan eruptions do not essentially differ from those of the eruptions of other volcanoes.

There is, in fact, a considerable amount of evidence to show that that part of the magma which gave rise to the great black cloud was not in a state of complete fusion, but that even the matrix separating the individual crystals was semi-solid and partially crystallized.* The glassy fragments in the dust of Pelée contain many microliths. According to the descriptions of Mr. J. S. DILLER,† “at least half the mass (of the dust which fell on the ‘Roddam,’ in St. Pierre on May 8th) is dark microlithic, more or less felty, but not vesicular ground-mass, often enclosing or clinging to crystals, and appears identical with the ground-mass of the lavas of Mount Pelée antedating the last eruption.”

Professor LACROIX,* describing the ashes which fell in St. Pierre on May 2nd and 3rd, 1902, says:—“The examination of the glassy matter which plays an important part in these ashes is not without interest; it is compact, with few cavities, or at most enclosing a few minute vesicles; it contains very few microliths of felspar, but a pretty large number of opaque globulites, and occasionally some crystallites of hypersthene. It is not in reality a pumice composed of vesicular, filamentous glass, like that which characterised the great explosion of Krakatoa, and that which in pre-historic time blew out the bay of Santorin.”

From these descriptions it is clear that a second generation of crystals had begun to form in the magma within the mountain, or at least in the upper part of it before it was forced to the surface and blown into dust by the expansion of the gases it contained. Mr. DILLER, indeed, advances the view that these glassy fragments belong to the older rocks of the mountain:—“It appears certain that the greater portion of the material which fell on the deck of the ‘Roddam’ was derived from the pulverisation of solid rock about the volcanic vent of Mont Pelée, and only a small portion from the molten magma which was the seat of the eruption.” He arrives at the same conclusion as a result of his examination of the dust which fell in Kingstown, St. Vincent, on May 7th, 1902:—“The larger particles are of dusty glass, rarely clear, and colourless, and full of bubbles. Others contained a multitude of minute crystals. Those filled with these microlites are of pulverised older rock, while the dirty vesicular glass ones, like the ground-mass of the pumice, represent the molten magma of the eruption.”‡

This theory, that much of the dust was due to the comminution of the older rocks of the hill, was suggested to us by Professor JAGGAR, whom we met in Barbados

* Professor ISRAEL C. RUSSELL also is of opinion that the magma had partly consolidated. *Century Magazine*, vol. lxiv., p. 795, September, 1902.

† “Volcanic Rocks of Martinique and St. Vincent,” *National Geographic Magazine*, vol. xiii., p. 290.

‡ *National Geographic Magazine*, vol. xiii., p. 290.

before we arrived in St. Vincent, and we directed a considerable amount of attention to the evidence in favour of it and against it while engaged in our work on the Soufrière. It cannot be denied that, especially in St. Vincent, an enormous amount of the pre-existing rock around the crater has been blown into the air, and this is sufficiently attested by the changes in the conformation of the crater-walls and the increased depth of its floor. Angular fragments of the old ashes and lavas, often several feet in diameter, are very numerous, and are easily distinguished by their form, appearance, and structure from the bombs which originated from this eruption. No doubt, also, fine dust of this nature mingled with the cloud, but we saw no evidence that any considerable part of the mountain had been shattered to minute, almost impalpable, powder, and disseminated through the air by the explosions. It seemed rather that the older igneous masses had yielded fragments usually of some considerable size, and that a breccia, and not a dust cloud, would have been produced in this way.

The presence of glassy fragments, some of which are full of crystallites, while others are practically free from them, may be explained in more than one way. It may be that the partly devitrified material was that which formed the surface and the sides of the column of molten lava, the more vitreous substance its centre and deeper parts. But we should also be prepared to admit that the magma was not entirely homogeneous, and that parts may have been more liquid, or less crystalline, than others. This would have given rise to a banded structure, such as is so common in the vitreous lavas, had the mass poured out in a coulée and rapidly solidified to form an andesitic lava flow.

It is also possible that the gases may not have been quite equally abundant throughout, and to these two causes, conjointly or separately, we may ascribe the presence of a considerable number of small vesicular lapilli, and even occasionally of rounded pieces of pumice mixed with the dust. There was very little pumice in St. Vincent, but many scoriaceous lapilli with embedded crystals, while at Pelée pumice was more abundant, though still not in sufficient quantity to form a considerable proportion of the ejecta.

In this connection we may remark that in Martinique the eruption of July 9th* ejected more vesicular pumiceous glass than any of the previous outbursts, and we have received information from St. Vincent that pumice is more abundant in the ashes of September 3rd. It looks as if the magma in these volcanoes was undergoing some change in its physical condition, and that the highly crystalline and comparatively cold condition of the material at first ejected was not quite so pronounced in the later stages. What effect this may have on the future phases of the volcanic activity we can only guess at present, but it is a problem of much interest, and well worthy of careful attention.

* MM. LACROIX, ROLLET DE L'ISLE & GIRAUD, "Sur les Roches rejetées par l'Éruption actuelle de la Montagne Pelée," 'Comptes Rendus,' vol. cxxv., p. 452.

We have seen that the magma of these Antillean volcanoes, or at least the uppermost part of it which gave rise to the great black cloud, was to a large extent crystallized, contained comparatively little fluid matter, and that it was accordingly near the temperature of consolidation, or may even in part have solidified already. These may all have been powerful factors in determining it to assume the form of a dust cloud as soon as the retaining pressures were relieved, and the vast quantities of steam and other gases it contained were free to expand.

But the chemical aspects of the problem must not be neglected; they may be of even greater importance than the physical. Both magmas are andesitic, but the analyses and microscopic investigations which have already been published make it certain that they are not exactly the same, so that it does not appear likely that the dust-avalanche type of eruption is a consequence merely of the presence of a magma with a certain narrowly-restricted chemical composition.

On the other hand, it is a significant fact that many of the most terrible instances of disastrous and sudden volcanic explosions have been furnished by volcanoes emitting hypersthene andesite. Krakatoa and Bandaisan are two examples of quite recent date. This group of rocks is, of course, a very diverse one, and includes many important variations; but as contrasted with the basalts, which in so many cases give rise to floods of liquid lava quietly welling out of craters or fissures, they seem predisposed to violent and disruptive activity. The rarity of dykes in St. Vincent, and the abundance and thickness of the sheets of coarse agglomerate, point to the same conclusion.

What influence the nature of the gases present in such quantities may exert is a problem equally interesting and obscure. Till we know more of what these gases are, and in what proportions they exist in the magma, it is premature to enter into a discussion of the part they play in the origination of the great black cloud. Any conditions whatever, whether chemical or otherwise, which diminish the cohesion of the magma in which the gases were absorbed will, *primâ facie*, facilitate the conversion of the mass from a spongy froth into a cloud of liquid droplets.

The Cause of the Deaths.

Owing to the exigencies of the case, it was impossible to perform autopsies on the bodies of the dead in St. Vincent, and however much we may regret this from a scientific point of view, we must recognise that the first call on the energies of the medical men was to attend to the wounded, and give them what assistance they could. At first the number of doctors in the island was too small to enable them to cope with the sudden emergency, but help was rapidly provided from the adjacent British islands, and the final results were brilliant. It was several days before the state of the volcano warranted the exploration of the devastated country on an extensive scale, and when the bodies were finally all discovered and interred, they

were very often in a condition which left little hope of obtaining any important information by means of *post-mortem* examinations. Fortunately, the evidence as to the lethal agencies at work is fairly clear and conclusive, and the opinions formed by the doctors who had care of the survivors are entirely in accordance with the geological facts regarding the nature of the catastrophe. From Dr. C. W. BRANCH, Dr. DUNBAR HUGHES, and Dr. AUSTIN, of St. Vincent, and from Major WILLS, R.A.M.C., and Dr. HUTSON, of Barbados, we obtained most of the information on which we have based our conclusions. We had also an opportunity of examining many survivors who had passed the afternoon and night of May 7th in the Carib Country. Some of these had completely recovered, others were in the hospital in Kingstown; many were very unwilling to retail the horrors of that afternoon when their friends, relatives, and families had perished at their side.

Without doubt steam laden with hot dust was the principal cause of the fatalities. When the hot wave struck the houses all the occupants felt a sudden pain in their mouths and throats. This was principally due to the fine hot dust, which was intensely irritant. Many stuck their caps into their mouths, and this relieved the burning feeling, thus proving that it was due to the fine particles floating in the air. It produced intense pain in the eyes, and among the wounded not a few had their faces nearly unburnt about the eyelids and the temples, while their brows and the backs of their hands were severely scorched. When struck with the hot blast they had covered their eyes with their hands in order to protect them. The mucous membranes of the nose and mouth were scorched, and subsequently in some cases desquamated,* and the hot dust gathered on the beards of the men and singed the skin beneath. Those parts of the body which were covered with the clothes, as a rule, escaped injury, or were only slightly burnt. As the black labourers wear thin cotton garments, this must have been because the hot dust was unable to pass through the cloth, and the gases were not at so high a temperature as to do much injury. It is in every way probable that when the black cloud swept over the lower grounds in St. Vincent the dust it contained was hotter than the gases, for these latter were actively expanding, and were cooled in consequence, while the solid matters were merely passive, and being also bad conductors, would only slowly part with their heat to the surrounding medium; but, owing to the small size of the particles, such differences in temperature may have been inconsiderable.

After a minute or so the feeling of pain in nose, mouth, and throat, and on the exposed parts of the body, was oppressive, and many who survived complained that they also felt a burning sensation in their breasts and abdomens—the result probably of their having inhaled the hot dust into their bronchi or swallowed it, and thus scorched their throats, or perhaps also their stomachs. It is clear that it acted as an intense irritant on all parts with which it came in contact. Very rapidly, also,

* *see* also Professor ISRAEL C. RUSSELL, 'Century Magazine,' vol. lxiv., p. 798, September, 1902.

a sense of suffocation supervened. The sufferers gasped and cried for breath, but soon their cries were stilled by the approach of asphyxiation. They felt as if someone was powerfully compressing their throats, and at the same time their thirst was excessive. They complained also of the choking smell of burning sulphur, and in St. Vincent it is clear that in the blast there was much sulphurous acid, though in St. Pierre it was not conspicuous, at least on May 8th, when so many perished. Some of the doctors were inclined to ascribe many of the fatal consequences of the cloud to its presence; others merely regarded it as a subordinate factor, and this seems most probable, in view of what happened in Martinique.

The duration of the fatal wave of hot gases and dust was certainly brief, probably not more than three minutes, but it seems clear that death was not instantaneous in St. Vincent, or at any rate on the estates in the lower part of the Carib Country, as it probably was in the north end of St. Pierre, for all the survivors gave a distinct and consistent account of the gradual though rapid onset of the symptoms. At the same time it must be remarked that apparently after a minute or two the conditions had a lethal effect on the great majority of those subjected to them. The cries were succeeded by silence and inarticulate groans, and death followed almost at once. We were told that in some of the houses where the dead were heaped upon the floor, as in Sutherland's shop in Overland Village, where 87 perished in a little room, the bodies lay regularly piled on one another—the whole mass, living and dead, had fallen at once. It is not possible to separate the effects of the hot gases and the dust. They acted together, and probably neither alone would have produced all the effects. The dust was irritant, and cauterised the epidermal surfaces; the steam, sulphurous acid and other gases in the cloud, especially as they were not mixed with oxygen, produced the suffocation and finished the deadly work.

Only those survived who had shut themselves up in cellars and rooms with tightly-closed windows. We have already given some instances which prove how important it was to avoid direct contact with the cloud. In the cellar at Orange Hill 40 survived; but 30 who were in the passage leading into it died. In Turema all who had escaped had taken refuge in tightly shut-up rooms. At Rabaka many were saved in the same way. All human beings and animals which were in the open air perished. In some cases one or two occupants of a house were spared, while all the others died. This was not the case at Lot 14, or on the leeward side—that is to say, in those parts where the blast was hottest; but in the Carib Country there were not a few of these miraculous escapes. Similarly, on the "Roddam" and "Roraima," at St. Pierre, some were little injured, though apparently as much exposed as others who died. In all probability accidental circumstances, which can now no longer be brought to light, were the determining causes. We cannot say, for example, why in some cases one survived in a room where all the others, to the number of 10 or more, died almost at once.

In Overland Village, and a few other places, the first ash brought in the blast was wet, and stuck to the walls of the houses. This was the case also in Fancy and in St. Pierre. This wet mud occasioned severe burns, as it adhered to the naked skin, and many of the sufferers had extensive burns, from which the skin was peeling. But in most cases the dust was dry, and clung principally to those parts which were moist, like the lips, or were covered with short hairs, like the backs of the forearms. In the hospital in St. Vincent one patient had a cake of dust, one-eighth of an inch thick, adherent to the scalp, which was slowly healing beneath it.

Probably not less than nine-tenths of the fatalities were produced by the causes above enumerated, but there were others which certainly were in operation, though we cannot now establish exactly to what extent any of them swelled the list.

Many of the injured lay among the dead bodies on the floor, covered with a thin film of ashes, groaning, with parched throats, unable to raise themselves and search for water, waiting for death. Most of these died within an hour or two, others dragged on for a couple of days; one case was taken to hospital, and died three days after the eruption. The direct cause of death was shock and exhaustion.

Others were removed and died under treatment, partly from the shock of their burns and the terrible experiences through which they had passed, partly from the other secondary effects of their injuries. Fortunately, they were comparatively few; only 70 deaths from burns and other causes occurred in the hospitals.*

Some also were killed by lightning—how many it is impossible to say. One woman was seen to fall dead during a bright flash of lightning in the yard at Orange Hill. Probably others met a similar death, but this could have affected practically only those who were fleeing from one house to another after the blast had passed.

Many of the roofs of the huts, and even of the more substantially-built stores, collapsed through the night under the weight of ashes, and beneath them the searchers found, in some cases, heaps of dead bodies. Had any within these huts been spared by the fatal blast, they must have been suffocated by the roofs falling with their weight of hot sand.

Similarly, many houses were ignited by incandescent stones or by lightning, and any wounded they contained must have been burned to death, but it is not certainly known that there were any such fatalities.

Undoubtedly some were killed by falling stones. Dr. HUTSON, of Barbados, told us he saw three cases of fractured skulls, and Captain CALDER narrates that when 8 miles from the volcano he was struck and almost rendered unconscious by a piece of rock. There are no statistics, however, to show whether such cases were frequent. No one in Chateaubelair, so far as we know, was injured in this way; one little girl in Georgetown was wounded by a stone in the afternoon of May 7th.

* Sir R. LLEWELYN, Blue Book: 'Correspondence relating to the Volcanic Eruptions in St. Vincent and Martinique in May, 1902,' p. 65.

Probably, if careful inquiries had been made at the time, many similar cases would have been brought to light.*

At least one party suffered death by drowning. They were in a boat, coming from Campobello southwards along the east coast, and were never seen again after the cloud had passed.

We inquired carefully of the doctors whether they had reason to believe that carbonic dioxide, carbon monoxide, or poisonous hydrocarbon gases could be considered to have produced any of the fatal effects, and whether sulphuretted hydrogen was present in such quantities as to have contributed in any way, but they all were of opinion that steam, hot dust, and sulphurous acid were the only important lethal components of the cloud, and though other gases may have been there they were in small quantities, and left no visible consequences.

The Air Waves.

All the greater eruptions of Pelée and the Soufrière during the months of May, June, and July, 1902, have probably been accompanied by both air waves and sea waves. The observations and records in our hands are far from complete or satisfactory, and we do not intend to attempt a final discussion of the phenomena in this paper, but we have given the evidence relating to the eruptions of the Soufrière, and our statements are corroborated by those of the French Commissioners.† The number and form of these waves, their dimensions, velocity of dispersion and range can be fully investigated only when all the data are to hand, and in particular when the various barographic records obtained at meteorological stations around the Caribbean Sea are available. The best records are those of the recording barometers used in the stations of the American Weather Bureau scattered through the islands, and those of the French Government in Martinique, and no doubt they will be fully discussed in all their bearings in the reports of the French and American scientists.

The study of these waves is of importance not only from the hydrographic and the meteorological point of view, they also give very valuable information to the volcanologist. They may help us to fix the exact time of the outburst of the black clouds. It is probable that they will also indicate the relative magnitude of the various eruptions, and in some measure enable us to compare those of the Soufrière with those of Montagne Pelée.

The origin of the air waves is not far to seek. They are due to the sudden and localised increase of pressure occasioned by the outburst of large quantities of steam and other gases into the air. The waves must travel outwards radially, in all directions, from the centre. Their amplitude depends principally on the magnitude

* Professor ISRAEL C. RUSSELL met with certain cases of injuries of this nature, 'Century Magazine,' vol. lxiv., p. 798, September, 1902.

† "Sur l'Eruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 389.

of the explosion. According to the investigations of Lord RAYLEIGH, quoted in the Krakatoa report to the Royal Society,* it diminishes in proportion to the square of the distance from the focus, and Lieutenant-General STRACHEY, R.E., calculates that the initial velocity of the air waves generated by that eruption was about 713 miles an hour. The great explosion at the Soufrière should, in consequence, have affected the barometric column in Barbados in less than 10 minutes. So far as we know at present, the eruption in St. Vincent on May 18th did not leave any record on the barograms, either in Barbados or in Martinique.

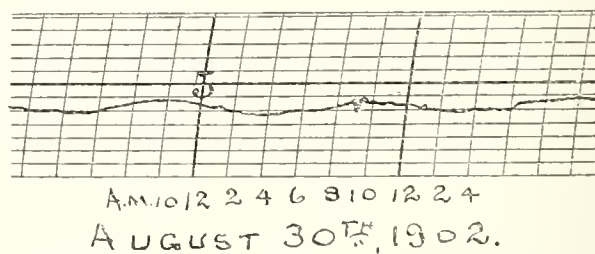
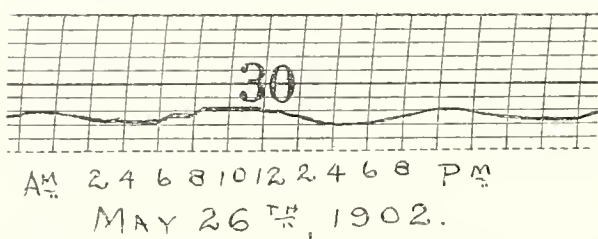
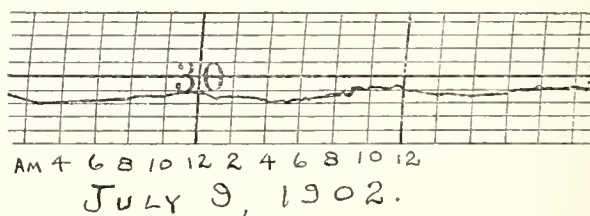
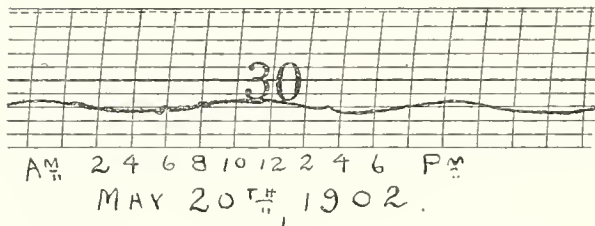
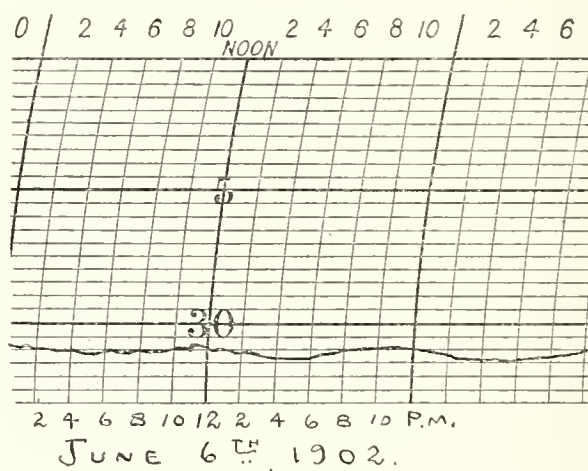
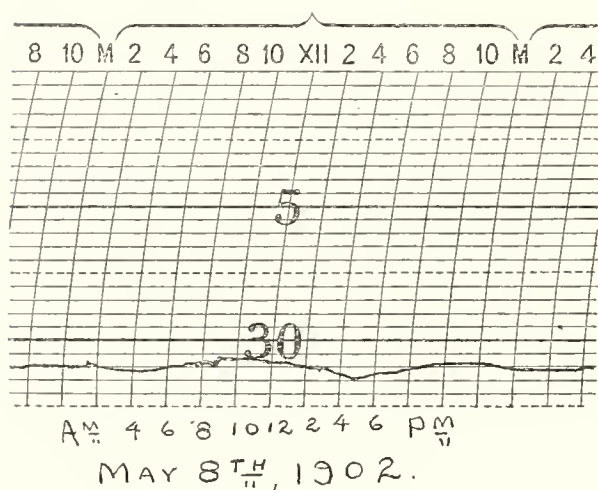


FIG. 3.—Barographic tracings, showing the air waves generated by the eruptions of Montagne Pelée. Taken in the station of the American Weather Bureau, Rozeau, Dominica.

The form of the wave tracing is precisely that given by sudden and violent explosions. We have obtained from Mr. PORTER, of Rozeau, in Dominica, photographic copies of the barograms yielded by his recording barometer at that station (through the kindness of Mr. HOBBS, of the American Weather Bureau, St. Kitts). (Fig. 3.) The characteristics of the waves have already been emphasised by Professor LACROIX

* "The Eruption of Krakatoa," 'Royal Society's Report,' p. 64, 1888.

and his colleagues.* It is the presence of a considerable, short, sharp depression, in advance of an elevation, which is more continued, and may last for a quarter of an hour or more. With these we reproduce also, by the kindness of the Meteorological Council, the trace left by the explosion of 12 tons of gunpowder on board the ship "Lottie Sleigh," lying in Liverpool Harbour, on the recording barograph at Liverpool Observatory, on January 15th, 1864. (Fig. 4.) The similarity between these waves is very striking: in each we have the initial, short, rapid fall, followed by an almost equally sudden rise, which lasts for a longer period, and is then succeeded in the Liverpool record by a second depression, not so great as the first.

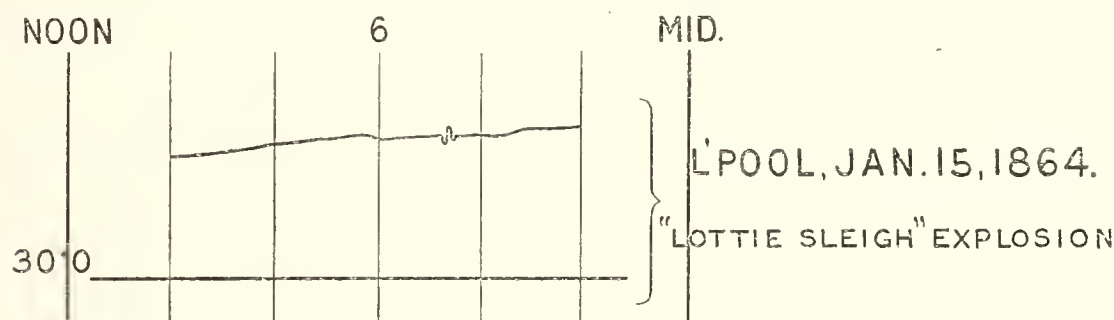


FIG. 4.—Barographic tracing of the King's barograph at the Liverpool Observatory, showing the effect of the explosion of twelve tons of gunpowder on board the "Lottie Sleigh," three miles distant.

(Reproduced from the 'Proceedings of the Royal Society,' vol. xxxvi., Pl. I.)

In the air waves which were produced by the eruption of Krakatoa these features were by no means so marked, and, in fact, Lieutenant-General STRACHEY was led to the conclusion, partly from his studies of the actual barographic tracings, and partly from the theoretical researches of Lord RAYLEIGH, that "the rise of the barometer indicating a sudden increase of pressure was the first and direct result of the explosion, and that the succeeding fall of the barometer . . . required some considerable time for its development."

The Sea Waves.

We have already detailed the observations regarding sea waves occasioned by the great eruption of St. Vincent on May 7th in Barbados, Bequia, St. Lucia, Martinique, and Guadeloupe. (See p. 406 and p. 410.) In Barbados and in Bequia its amplitude was $2\frac{1}{2}$ feet, in Martinique about 1 foot. Assuming that it originated in St. Vincent, and was synchronous with the great explosion and discharge of the avalanche of dust about 2 o'clock in the afternoon, we find that it took 70 minutes to cross the channel between that island and Barbados, where the first crest arrived at 3.10 P.M., and had a velocity of about 90 miles an hour.

According to the preliminary report of the French Commissioners, sea waves have been observed in Fort de France on the occasion of all the more important eruptions

* 'Comptes Rendus,' vol. cxxxv., p. 390.

of Montagne Pelée. On May 8th the first phenomenon observed at Fort de France was a recession of the sea ; it was followed by a considerable rise, and at intervals of several minutes minor oscillations succeeded the principal one to the number of five or six. In St. Pierre and in Carbet these waves have invaded the land and done considerable damage, especially on the morning of the great eruption (8th May), when the wave at Carbet was estimated to have an amplitude of nearly 7 feet (2 metres). This wave was not observed at Guadeloupe, though that of the previous afternoon, attendant on the eruption of the Soufrière, was clearly visible there.*

The most remarkable feature of these sea waves is that they have never been observed in St. Vincent, even on the afternoon of May 7th, when many people along the leeward coast of the island were embarking in boats or landing on the shore at the moment of the climax of the eruption. They cannot certainly have taken place on so large a scale as at St. Pierre, as in Chateaubelair many huts stand on the low beach but little above high-water mark, and none of them was destroyed or damaged. Yet the wave originated by this outburst was felt in Guadeloupe and in Barbados.

To some extent these waves may have been caused by the sudden increase in atmospheric pressure which started the air wave, but this cannot have been a very important factor in their production, as there is no reason to believe that the rise of the barometer was in any case more than a quarter of an inch.

They may also be partly a result of the concussion or shock which was occasioned by the great explosion, and may in this way partake of the nature of "earthquake waves."

Our knowledge of the extent and magnitude of the changes which have taken place on the sea bottom around the Windward Islands is at present unsatisfactory. It may be regarded as highly probable that the loose material lying on the steep submarine slopes on both sides of the islands has been set in motion, and has slipped downwards over considerable areas. The subsidence at Wallibu can hardly be the only case in which this has occurred. Interesting evidence on this point is given in a letter by Mr. C. B. CRUICKSHANK to Professor JOHN MILNE, F.R.S., from which we take the following :—

"I regret exceedingly that the amount of information which I can give you as to changes in ocean depths off Martinique and St. Vincent is practically nil.

"During the time I was out in the West Indies after the eruptions we did not attempt to tackle any repairs off Martinique. The greater part of the time was occupied with cables off St. Vincent, and the first cable we tackled was one which was laid in 1898. As no soundings were taken in this line before the laying of the cable, and this was the first break that had occurred, it is difficult to say whether or not the depths were changed.

"As to the other cable, the one between St. Lucia and St. Vincent, we found no great alteration in the soundings, and what little difference we did find may have been due to errors in taking the soundings, for we had very strong currents to contend with, making accurate sounding difficult.

* 'Comptes Rendus,' vol. cxxxv., p. 390.

"It might be interesting, however, for you to know in what state we found the former of these two cables—viz., the one between St. Lucia and Grenada. This cable passes off the island of St. Vincent, running in an almost north by east and south by west direction, and passing off the Soufrière at its nearest point to the centre of the crater at a distance of about 16 miles. The cable was hooked about 24 miles in a direct line from the crater (towards St. Lucia), and a remnant picked up towards Grenada, after buoying St. Lucia side, about a mile in length. The first part of this came up in a good condition, but the balance came up in variable condition, in some places almost all the sheathing wires being cut through, as if by the sharp edge of a heavy rock coming down on it, and in lots of places twisted and knotted in a very similar manner to that shown in Fig. 8 of your paper on 'Suboceanic Changes.'

"For a considerable distance from this break the cable seems to have been either buried or carried away out of position; at all events, we failed to hook it or feel any signs of it till we were down almost due west of Grand Bonhomme. In grappling, the grapnels were found to have been ploughing deep in soft grey mud, lots of which came up on the teeth, with occasional particles of a bright red colour."

We may reasonably expect that if there has been extensive and sudden displacement of the sediments on the sea floor, as is indicated apparently by the condition of the cables, they would have been attended by sea waves of greater or less importance.

It is interesting in this connection to note that on the 5th May, when the crater lake of the Étang Sec burst, and a deluge of mud poured down the Rivière Blanche and buried the Usine Guérin, there was a distinct sea wave which did some damage at the mouth of the stream.* As no black cloud was emitted in this case, this wave cannot have been caused by its action, but, as stated by the French Commissioners, a line of fissures has formed almost exactly along this radius of the mountain, and submarine movement may have taken place and given rise to a sea wave.

But these causes are not sufficient in themselves to account for all the phenomena, and in particular for the great disparity in magnitude between the waves observed in St. Pierre and Chateaubelair. Some additional and local factor must be in operation, and this is, in all probability, the direct action of the great black cloud on the surface of the sea. The avalanches of dust and gases which are emitted by Pelée roll right down the slopes of the mountain upon the bay beneath, driving back its waters by their weight and momentum. So great a mass of matter sweeping downwards with so high a velocity must certainly have a considerable effect in disturbing the hydrostatic equilibrium of the sea surface, and must start a local wave which travels outwards in all directions.†

In St. Vincent the avalanche of dust is discharged over the southern lip of the crater, and rolls down into the broad valley between the Soufrière and Morne Garu. Its onward course is obstructed by the latter mountain, and its current is split into two parts, one taking the direction of the Rabaka Valley to windward, the other

* LACROIX, ROLLET DE L'ISLE, & GIRAUD, "Sur l'Éruption de la Martinique," 'Comptes Rendus,' vol. cxxxv., p. 390.

† We find that this suggestion has previously been made by Professor ISRAEL C. RUSSELL, 'Century Magazine,' vol. lxiv., p. 800, September, 1902.

passing to leeward by Wallibu. Before they reach the coast most of their energy is spent, and their velocity so greatly diminished that the black cloud, when it flowed out over Richmond Village, was travelling at the rate of only 30 miles an hour, and did not produce any great disturbance in the level of the sea surface, while that which passed over the mouth of the Rivière des Pères at St. Pierre had a velocity of at least 100 miles an hour. Under the circumstances it is not difficult to believe that the one may have had very much greater effect upon the level of the waters than the other.

Magnetic Disturbances.

We understand from the reports of Professor LACROIX and his colleagues, and of Professor HILL,* that at several observatories magnetic disturbances have been noted corresponding in time to certain of the eruptions of this spring. As we are not in possession of any special information regarding these observations, it is not possible for us to discuss them in this Report.

THE GENERAL SEQUENCE OF VOLCANIC PHENOMENA IN THE ANTILLES AND CENTRAL AMERICA IN THE EARLY PART OF 1902.

The disturbances in St. Vincent which culminated in the eruptions of Pelée and the Soufrière can be traced back for more than a year. They began at least as early as February, 1901, at which time the Caribs were alarmed by the numerous earthquakes around the Soufrière. After a temporary quiescence they resumed in March and April, 1902. Montagne Pelée began to emit steam about April 23rd, though, according to Professor JAGGAR, who has made a special study of the premonitory symptoms, the water in the crater lake was noticed to be warm in January.† On April 28th the violence of the earthquakes in St. Vincent was startling.

On April 18th—that is to say, just before actual volcanic action was observed at Martinique—a powerful earthquake shook Guatemala, and destroyed the town of Ruez-Altenango.

On May 5th the crater lake of Pelée burst, and the Usine Guérin was destroyed. On the 7th the great eruption occurred in St. Vincent. On the 8th the city of St. Pierre was destroyed. It was not till the 15th that the Soufrière of St. Vincent passed into a state of temporary quiescence.

The second important eruption of the Soufrière was on the 18th. (There had been a minor outburst on the 9th.) That of Montagne Pelée followed on the 20th.

In Martinique further outbursts took place on May 26th and June 6th. There was no corresponding activity in St. Vincent.

* 'Comptes Rendus,' vol. cxxxv., p. 391 ; 'National Geographic Magazine,' vol. xiii., p. 255, July, 1902.

† 'Popular Science Monthly,' 1902, p. 363.

On July 9th, at 8.20 p.m., Montagne Pelée broke again into activity. That afternoon there were several earthquakes in St. Vincent, so severe as to cause great apprehension.

In the end of August there was renewed activity at Pelée, and on the 15th, 28th, and 30th of that month and September 3rd there were eruptions. On September 1st and September 3rd the Soufrière was active, the latter especially being one of the greater eruptions of this year.

These facts are sufficient to show that a distinct connection exists between the outbursts of these two mountains during the present year. The eruptions of the Soufrière have sometimes followed, sometimes preceded, those in Martinique, but, as a rule, the greater eruptions of the one volcano have been accompanied by eruptions at the other within a period of one or two days. The outbursts of Pelée have been more numerous than those of the Soufrière. In particular, those of the end of May, of June, and of July had no corresponding eruptions in the sister island, but though more frequent they have been of less magnitude, and have produced less widespread effects. Pelée has also been more constantly in action of a subordinate kind during the intervals between the major eruptions.

Some general cause underlying the volcanic activity is required to explain the synchronism in the superficial phenomena. It is to be found in the existence of internal pressures and stresses in that part of the earth's crust, of which the Caribbean fold is one of the dominant ridges. The volcanic chain of the Windward Islands occupies the summit of one of the great earth folds of this region.

Great earth movements have taken place around the Caribbean Sea in Tertiary times, and are still in progress. It is in consequence one of the great earthquake centres of the globe,* and the connection which subsists between volcanic activity in the volcanoes of the Lesser Antilles and earthquakes in the surrounding region has been emphasised by HUMBOLDT and by many subsequent writers.

Although there is apparently no record of great earthquakes having attended the eruption of the Soufrière in 1718, that of April 24th, 1812, was preceded by a most violent earthquake in Venezuela on March 26th of that year. The city of Caracas was levelled with the ground, and it is estimated that 10,000 of its inhabitants perished.† The disturbances had begun about three months previously (in December, 1811), and were not confined to Venezuela, but affected also a wide area of Central and North America. In the valleys of the Mississippi and the Ohio there were numerous earthquakes from December 16th, 1811, onwards. In St. Vincent over 200 shocks were counted during the twelve months before the eruption.‡

* See Professor J. MILNE'S Seismic Map of the Globe, 'Geographical Journal,' January, 1903.

† HUMBOLDT, 'Personal Narrative of Travels to the Equinoctial Regions of the New Continent,' English translation by Mrs. WILLIAMS, vol. iv., chap. 1.

‡ HUMBOLDT, *op. cit.*, vol. ii., p. 231.

In 1880 there were symptoms of activity at the Soufrière of St. Vincent, and an outburst occurred in Dominica on January 4th.* Many earthquakes were felt about that time along the chain of the Greater Antilles. On January 22nd there were several shocks in Havana; others were felt in Cienfuegos, San Diego, and Santiago, and the town of San Cristobal was almost destroyed.† From December 21st, 1879, to January 10th, 1880, there was seismic disturbance in San Salvador, and on February 5th several earthquakes were reported from Mexico.‡

The eruption of Montagne Pelée in 1851 was perhaps connected with the great earthquake which shook Chili on April 2nd of that year.§

HUMBOLDT has given many interesting facts to confirm his hypothesis that earthquakes and volcanic activity in this region go hand in hand. In 1692 Port Royal, in Jamaica, was destroyed, and the volcano of St. Kitts was in eruption. In 1766 there was a violent earthquake in Cumana, Venezuela, and many shocks were felt in Jamaica, Trinidad, and the Lesser Antilles. Qualibou, in St. Lucia, burst into activity. In 1796 there was a tremendous earthquake in Quito. On December 14th, 1797, the town of Cumana (Venezuela) was razed by a terrible shock. On September 27th, 1796, an eruption took place in Guadeloupe. In 1800, 1801, and 1802 many earthquakes were felt in Maracaibo, Porto Cabello, and Caracas, and in February, 1802, volcanic activity broke out in Guadeloupe.||

The connection between earthquakes and volcanic activity is further emphasised by the fact that in Guatemala a violent earthquake took place on April 18th, 1902, just before Montagne Pelée began to emit steam. This earthquake was recorded on the seismographs in the Isle of Wight.¶ It was most severe in western Guatemala, but affected also Salvador and Honduras. Quezaltenango was destroyed, and about 500 lives were lost, and extensive damage was done to the coffee and sugar plantations in all the surrounding district.**

Further earthquakes, attended by eruptions, followed in October, and the exceptionally long duration of the disturbances in Martinique and St. Vincent during this year—to which there is no parallel in the history of the Antilles—is to be ascribed to the continuance of crustal adjustments affecting the whole borders of the Caribbean Sea.

* H. ALFORD NICHOLLS, 'Nature,' vol. xxi., 1880, p. 372.

† 'Nature,' vol. xxi., 1880, pp. 306 and 357.

‡ 'Nature,' vol. xxi., 1880, p. 452.

§ J. MILNE, "Seismological Observations and Earth Physics," 'Geographical Journal,' January, 1903, vol. xxi., p. 14.

|| HUMBOLDT, 'Personal Narrative of Travels to the Equinoctial Regions of the New Continent,' vol. iv., chap. i. English Translation by Mrs. WILLIAMS. See also Professor MILNE, as above, 'Geographical Journal,' vol. xxi., pp. 12-15, January, 1903.

¶ Professor J. MILNE, 'Nature,' vol. lxvi., p. 57.

** EDWIN ROCKSTROH, 'Nature,' vol. lxvi., p. 150; KARL SAPPER, Petermann's 'Mitteilungen,' vol. xlviii., p. 193.

The minor earthquakes are local, and more probably due to the changes beneath the volcano before an outburst takes place. It is significant in this respect that on July 9th, though there was no discharge from the crater of the Soufrière, that mountain showed its sympathy with the eruption in Martinique by means of strong earthquake shocks.

No earthquakes of great violence have been experienced this year in Martinique, St. Vincent, and St. Lucia, though small shocks were frequent. In Trinidad the Milne seismograph in the Botanic Gardens has shown none but small disturbances.*

In Montserrat, since the year 1896, earthquakes have been very frequent; in fact, it is said that they have been of almost daily occurrence, and in some cases as many as 100 have been counted in one day.†

It is apparently a matter of indifference which of the volcanoes of the chain is in operation; one appears to be able to function in place of the others. The previous eruptions of the Soufrière and Montagne Pelée did not coincide in point of time, and that both have this year been in violent action further proves the unusual magnitude of the changes in progress in that part of the earth's crust.

Nowhere along the Caribbean islands have changes of level been proved to have taken place upon the coasts. Sensational paragraphs have been published regarding alterations in the soundings, but they are probably unreliable. The frequent ruptures of the submarine cables, however, are sufficient to prove that there has been disturbance on the sea bottom.

Concomitant Activity in the Adjacent Islands.

In the other islands of the Lesser Antilles there has been remarkably little disturbance. In St. Lucia, which lies between Martinique and St. Vincent, and which contains a volcano—Qualibou—which is said to have been erupted in 1776, nothing unusual has been noted, except a slightly increased activity at the Soufrière in the south end of the island.

It is possible also, as will be seen from the following report by Major HODDER, R.E., that some submarine activity has taken place off Castries. He writes:—

“On Friday, the 9th instant, about 12 noon, I observed two large white patches on the sea, bearing about 294° from the Garrison Office at Morne Fortuné. They were at a distance (estimated by various persons) of 8,000 to 10,000 yards. These patches remained in the same position in sight till about 1.30 P.M., when they disappeared. At first I considered they were floating pumice-stone, but soon came to a different conclusion when I saw they did not shift their position to any extent; besides, they gave the appearance of bubbling. The patches were irregular in shape, but approximating to oval. The large patch was perhaps 150 yards long and 100 in diameter; the smaller one, say 100 long and 60 in diameter. They were distant from each other by about half a mile.

* Botanical Department, Trinidad, ‘Bulletin of Miscellaneous Information,’ July, 1902, p. 450.

† EMIL DECKERT, “Die Westindische Vulkankatastrophe und ihre Schauplätze,” ‘Zeitsch. d. Gesell. für Erdkunde. Berlin,’ 1902, p. 420.

"Staff-Sergeant CROWHURST, R.E., states he saw these patches at 8.30 A.M. on the same day, and that they never shifted their position until they disappeared at 1.30 P.M. All this leads me to conclude that a volcanic vent exists in the sea at this point.

"On the following day I think I detected a slight white patch of a similar sort in exactly the same place, but am not certain of this."

Grenada.—It having been reported that the lagoon, a sheet of water connected with the "carenage" or harbour of Grenada, had shown signs of volcanic activity, Dr. ANDERSON, at the request of Sir R. LLEWELYN, the Governor, visited the locality.

The lagoon is a nearly circular sheet of water about a quarter of a mile in diameter and about 25 feet deep in the centre, to the south-east of the carenage, and connected to it by a shallow, narrow channel about an eighth of a mile wide and only a few feet deep. As shown in Plate XVII., fig. 1, it is almost surrounded by hills, arranged in a manner that certainly at first sight suggests a similarity to an old volcanic crater.

They are all composed of beds of volcanic material, chiefly if not entirely scoria and ashes consolidated into tuffs and agglomerates. Closer examination, however, soon showed that the bedding does not follow the slopes of the surface, or even dip with any apparent reference to the centre of the lagoon. Many of the beds can also be traced into the *massif* of other surrounding hills, which obviously owe their present shape to denudation, and not to the mode of deposition of the beds. We therefore conclude that the lagoon and surrounding hills owe their configuration to the same general causes, and are not a crater.

Père LABAT, who visited Grenada in 1705, gives a map in his book, 'Voyage aux Isles d'Amerique,' in which the then town is shown on an isthmus between the carenage and the lagoon. The lagoon at that time appears to have contained only fresh water, and a brook through its isthmus was its only connection with the carenage.

About the middle of the eighteenth century the town was removed to the opposite side of the carenage, and the isthmus became submerged, and remains as a reef. As this was the exact position of the disturbance in May, 1902, referred to below, it seemed desirable to ascertain the particulars of this submergence. The island was at the time in the occupation of the French, and, through the good offices of the Colonial and Foreign Offices, the French Government have caused a search to be made in their archives, but without throwing any light on the subject.

It having been reported to us that some sort of a volcanic eruption took place in the carenage of Grenada about 1867, we asked the authorities at the Colonial Office that search might be made in the records of that office for any entry bearing on the subject. Such search was made, and we were very courteously allowed to inspect the original documents, which were chiefly cuttings from the newspapers of the period. As they are correctly summarised in the following extract from 'The Grenada Handbook for 1897,' it is unnecessary further to particularise them:—

"1867.—On November 18th Grenada gave proof of her volcanic origin and tendencies. Between 5 and 5.20 o'clock P.M. the sea, having been previously very calm, a sudden subsidence of the waters in the St. George's Harbour took place, the sea falling about 5 feet, and exposing the reef in front of the lagoon and the adjacent shores. In a few seconds, with a slight rumbling noise, the water over the 'Green Hole' (a spot between the old watering pipe for ships known as the 'Spout' and the opposite point on the north) began to boil and emit sulphurous vapour. This part of the harbour, it should be remarked, was, in the days of the French settlement, an excellent anchorage and careening ground, but there was, at the time of this eruption, as there is now, hardly 3 feet of water on it, showing that there has since been an upheaval of the land in that vicinity. Immediately after this ebullition of the 'Green Hole' the sea in the harbour rose rapidly to about 4 feet over its normal high-water mark, and rushed violently up to the head of the carenage. The phenomenon was three or four times repeated, and great damage was done to buildings and boats, but no lives were lost. The mason work of the 'Spout,' which projects into the sea, was completely demolished, presenting the appearance of having been twisted round in a whirlpool, and the 'Green Hole,' which was very deep, was completely filled up.

"The wave was experienced along the western seaboard as far north as the town of Gouyave, where it appears to have attained its maximum dimensions, rising so high at Dougalston Estate as to cover the bridge at the mouth of the river and inundate the adjoining cane fields.

"At 9 P.M. there was a shock of earthquake, and another at 1 A.M. on the following morning, the motion being perpendicular in both cases.

"It should be noted, in connection with this eruption, that at 2.40 P.M. on the same day, at the island of St. Thomas, there was a severe earthquake, followed by an enormous tidal wave, fully 50 feet high, and that there was concurrently a volcanic outbreak at the neighbouring island of Little Saba, which emitted smoke and lava."

Though both the writers of the original articles and of the handbook appear to have thought that the disturbance referred to was due to some volcanic action in or about the Green Hole, it is clear to anyone carefully reading the records or the above summary that the actual fact was a tidal wave originating in the neighbourhood of Little Saba and St. Thomas, and affecting in a less degree the shores of the island of Grenada.

The harbour of St. George was really less severely affected than other parts. The Green Hole is probably a submerged cave extending some distance underground, and the water in it rushed out when the level of the water in the carenage was lowered by the tidal wave. It is almost inconceivable that a volcanic outburst should have taken place from the Green Hole at the exact moment when the water in the carenage was lowered, and yet have ceased so suddenly as to have allowed the hole to be filled up by the action of the waves which followed immediately afterwards.

If the temperature of the water had been taken by a thermometer, as was done by Major BAYLY on a subsequent occasion, we have no doubt the cause would have been found the same in both cases.

The Disturbances of June 6th, 1902.—Inquiries on the spot from Major BAYLY, Chief of Police, whose office overlooked the carenage opposite the entrance of the lagoon, and Mr. RICHARD HEALD, Superintendent of the Prison, which is situated on the top of a hill on the opposite side of the lagoon, and whose house commands

a full view of it (both of whom were eye-witnesses), elicited the fact that on Friday, June 6th, disturbances were noticed more than once in the course of the forenoon in the water about the entrance of the lagoon, the appearances being described as a rippling or bubbling.

Major BAYLY had, in consequence, gone out to the spot that morning, when, by using a thermometer, he ascertained that there was no rise of temperature in the water. He was good enough to convey Dr. ANDERSON in a boat to the exact spot where the rippling had been observed, when it was found to coincide with the shallow bar across the mouth of the lagoon, over which the water had a depth of only $4\frac{1}{2}$ feet, which fact was verified by sounding.

Further inquiries from the chief boatman to the Customs elicited the fact that on the morning in question, which was also that of one of the eruptions of Montagne Pelée, the level of the water in the carenage rose and fell repeatedly, at intervals of about 12 or 15 minutes, to a height of about a foot.

We conclude that there is no reasonable doubt that the currents set up over the bar by this small tidal wave were the cause of the disturbance in question, and that it had no connection with any local volcanic manifestations.

Dominica.—After our visit to Martinique we proceeded to Dominica, where it was reported that anxiety had been felt regarding the condition of the Boiling Lake and the Grand Soufrière. In December, 1901, a young man who was on a tour through the islands and a Dominican boy were killed at the lake. Since then it had been visited several times, and was found to be more active than usual. Mr. C. F. BRANCH, at the kind request of the Administrator, undertook to conduct Dr. FLETT there, as the guides showed some unwillingness to face the journey. Under his energetic and able guidance this was successfully accomplished.

The road from Rozeau is by Laudat, a settlement in the mountains above the town at an elevation of 1585 feet. We left before dawn, and rode along a good though steep and winding path. As the sun rose over the hill-tops and broke through the mists which hovered over the ridges the scene was one of superlative beauty. The river foaming in its deep ravine, the craggy, richly-wooded slopes, the waterfalls over which the mountain rivulets threw themselves into the valley, and the dark, dense, tropical forest which clothed the island, were such as Dominica can show better than any other of the Antilles. Every here and there a puff of steam would rise from among the trees, the sign of a hot spring or Soufrière, of which there are many in Dominica, so many that the atmosphere in Rozeau is often redolent of sulphuretted hydrogen.

After leaving Laudat we struck a path through the woods, and crossed several streams before we emerged on the top of the narrow ridge, in which the central mountain range of the island culminates. Its altitude is 2930 feet. From this we descended into a valley on the eastern side. At the head of this lies the Grand

Soufrière, a cirque bounded on three sides by walls so steep as to be almost precipitous, but open to the north, where a stream, hot and turbid with precipitated sulphur, flows out to make its way to the sea. In Mr. BRANCH's opinion this Soufrière was more active than usual. Half a dozen orifices spouted steam and boiling water into the air with a loud hissing noise, like that of a locomotive. In the bottom of the pit no trees grew, and the naked walls that overlook it showed that the poisonous gases had entirely prevented the growth of vegetation, and had, in addition, attacked the exposed rock surfaces of coarse agglomeratic tuff, which were crumbling away and bleached by the acids in the air. Formerly there were times when these springs were quiescent, or only few of them emitted steam. They were now in vigorous activity, all puffing and casting up little columns of mingled steam and water; the smell was overpowering, and the heat of the steam, with the tropical sun beating down on the bare, rocky walls, made it a place in which one did not desire to loiter.

We then followed the stream down the valley, along a path so seldom used that it was almost obliterated by the dense growth of calumet and razor grasses, through which we waded up to our necks. The whole of this valley contained hot springs charged with sulphurous gases: lateral streams entered the main one—some cool, clear, and potable; others hot, dirty, and laden with sulphur. After walking rather more than half a mile, we turned to the left up a side valley, and, crossing a small ridge, we came to the famous boiling lake.

It is a cup-shaped depression, a bowl nearly circular in outline, perhaps 100 yards across, with high bare walls of weathered tuff surrounding it. At the east and west sides the bowl has two deep notches in its margin. Through these, two small streams enter on the west—one pure and cool, the other sulphurous; and on the east the effluent stream emerges. It is, in fact, an enlargement of part of the channel produced by the action of a powerful soufrière, which has decomposed the rocks around its orifice, and produced a funnel-shaped cavity through which the stream flows, and out of which it washes all the finer mud due to the churning of the water by the uprising steam. The north and south walls are perhaps 50 feet high, and show the effects of the acids generated by the oxidation of the hydrogen sulphide in their crumbling decayed surfaces.

The pool was full of milky, greenish water, boiling furiously towards its centre, where it was seething like a gigantic caldron. The smell was oppressive, especially when the wind blew towards us and carried with it the steam and gases. We found that the only danger was that of being poisoned by the sulphuretted hydrogen, and this could only take place in the bottom of the depression, where there was least chance of the gases being mixed and diluted with air. The deaths which took place here this spring were, in the opinion of Dr. NICHOLLS, occasioned by the visitors having gone down to the edge of the water. They had fallen unconscious, and the guides had been afraid to go to their rescue.

The configuration of the whole valley showed that it had been eroded by running water. It contained no actual structural craters, only "soufrières," rising apparently along some line of fissure which runs from the cirque on the south along the western side of the stream channel. It has been noted that this year the activity is exceptionally great, but no special violence has been seen to accompany the eruptions of Pelée or the Soufrière of St. Vincent.*

According to the observations of Dr. NICHOLLS,† there was in January, 1880, a great outburst from the Grand Soufrière, which projected much steam and fine broken rock into the air. This was carried before the wind, and fell in Rozeau. The trees around the Soufrière were blackened and blasted, and the largest ones were projected to some distance. Since then the bush has never quite recovered its former luxuriance of growth.

It is reported also that in Guadeloupe there have been emissions of steam and fine ashes from the volcano; but as so much that has appeared in the newspapers is unworthy of credit, and as we have had no opportunity of verifying the statements on the spot, we cannot say of what nature the disturbances have been. This matter will no doubt be fully treated of by the French Commissioners, who have included that island in the range of their inquiries.

In conclusion, we desire to express our indebtedness to the many residents in the British West Indies and in Martinique, who gave us invaluable assistance in pursuing our investigations. The Scientific Commission was sent out, at the suggestion of the Colonial Office, by the Royal Society of London, who defrayed Dr. FLETT's expenses out of the Government grant for scientific investigations.

SIR ROBERT BAXTER LLEWELYN, K.C.M.G., Governor and Commander-in-Chief of the Windward Islands, and SIR FREDERIC MITCHELL HODGSON, K.C.M.G., Governor of Barbados, gave us every facility for pursuing our work, and MR. EDWARD JOHN CAMERON, Administrator of St. Vincent, COLONEL DALRYMPLE HAY, Administrator of St. Lucia, and MR. H. HESKETH BELL, Administrator of Dominica, rendered us great service by taking charge of our arrangements in their respective islands.

In St. Vincent much information and assistance was given us by MR. F. W. GRIFFITHS, MR. J. H. PRESTON, Lieutenant ROBINSON, R.E., Surgeon-Major WILLS, R.A.M.C., MR. T. M. McDONALD, MR. HENRY POWELL, MR. JAMES E. RICHARDS, the Rev. MR. DARRELL, Dr. C. W. BRANCH, Dr. DUNBAR HUGHES, Dr. AUSTIN, MR. PORTER, MR. KNOWLES, MR. ROBERTSON, MRS. KELLY, MR. ISAACS and the officials in his

* Descriptions of the Grand Soufrière and the Boiling Lake will be found in F. A. OBER, 'Camps in the Caribbees,' 1880; and W. GIFFORD PALGRAVE, 'Ulysses: or, Scenes and Studies in Many Lands,' 1887.

† H. ALFORD NICHOLLS, 'Nature,' vol. xxi. (1880), p. 372

district at Georgetown, Captain CALDER and the members of the police force throughout the island, the Rev. Mr. BELL, the Rev. Mr. LESLIE, and many others whom space will not allow us to mention.

The planters and merchants with whom we came in contact in every case did everything they could to help us, and the numerous replies which we received to our printed schedule of inquiries have very largely been embodied in the text of this report, and acknowledgment made of the sources from which the information was derived. We desire to thank all those who have in this and other ways enabled us to collect the data on which our report is founded.

From St. Lucia, Major HODDER, R.E., has sent us many interesting details of his observations during the whole period of the eruptions.

Dr. MORRIS, head of the Imperial Department of Agriculture for the West Indies, has supplied us with much important scientific information; and the officials of his department, both in Barbados and throughout the islands, have forwarded to us valuable material, and have been of the greatest assistance to us in many ways.

We wish to acknowledge also the information given us by Bishop SWABY, of Barbados, and by the Rev. N. B. WATSON.

In Dominica, Dr. H. ALFORD NICHOLLS, C.M.G., placed at our service his intimate knowledge of the island and the history of its eruptions, and Mr. G. F. BRANCH, at the kind request of the Administrator, made the necessary arrangements for visiting the Boiling Lake. To Mr. PORTER, of the West India and Panama Telegraph Company, we are indebted for the barographic tracings reproduced in the text (Fig. 3). Dr. TEMPEST ANDERSON was hospitably entertained by Mr. DUNCAN NAISH, Picard, and Mr. J. SOWTRAY.

Our best thanks are due to M. L'HUERRE, the Governor of Martinique, who received us with great courtesy, and to Professor A. LACROIX for the kindness with which he explained to us the results of his investigations in the island.

In Grenada Dr. TEMPEST ANDERSON was most hospitably entertained by Sir R. LLEWELYN, and to that gentleman and his family, and Mr. PRESTON, his private secretary, his warmest thanks are due. He desires also to thank Major BAYLY and Mr. RICHARD HEALD for assistance on the spot, and Mr. FRANK ROWNTREE, and Mr. J. BOWES MORRELL, of York, for historical information, also Mr. C. P. LUCAS, C.B., and Mr. WALTER SCOTT, of the Colonial Office, for searching out and verifying the details in the library of that office.

APPENDIX I.

REPLIES to our printed Schedule, asking for information, specimens of volcanic ejecta, or written communications containing facts relative to the eruptions, have been sent us by the following :—

BARBADOS.

Dr. D. Morris, C.M.G.
Bishop Swaby.
Mr. F. J. Newton, C.M.G.
Mr. John R. Bovell.
Rev. N. B. Watson.
Mr. W. G. Freeman.
Mr. Radclyffe Hall.

Dr. Longfield Smith.
Mr. John W. Kirkham.
Mr. J. J. O'Donnell.
Captain Owen.
Mr. Lewis B. Brown.
Mr. H. Maxwell-Lefroy.
Mr. Skeete.

ST. VINCENT.

Mr. E. J. Cameron.
Mr. F. W. Griffiths.
Mr. J. T. Preston.
Mr. Isaacs.
Mr. G. Gentle.
Mr. George Durrant.
Rev. Mr. Darrell.
Mr. Henry Powell.
Mr. T. M. McDonald.
Mr. Robertson.
Mrs. Kelly.
Mr. Charles Knowles.
Mr. Proudfoot.
Dr. C. W. Branch.
Dr. Dunbar Hughes.
Dr. W. Bruce Austin.
Mr. Porter.

Mr. J. C. Wilson.
Mr. Rowland Winn.
Mr. P. Foster Huggins.
Mr. Effingham Dun, Owia.
Mr. E. M. Browne, Belair.
Mr. A. L. Darrell, Kingstown.
Rev. Mr. Bell, Georgetown.
Rev. Mr. Huckerby, Chateaubelair.
Rev. Mr. Leslie, Georgetown.
Captain Calder, Kingstown.
Mr. J. W. Clarke, Georgetown.
Mr. H. A. Allen, Chateaubelair.
Mr. Morgan, Chateaubelair.
Mr. Jos. W. Cubbin, Fancy.
Sergeant Ballantine, Chateaubelair.
Mr. Cyril Inmiss, Georgetown.

ST. LUCIA.

Mr. Okell, Castries.
Major Hodder, R.E.
Captain Ford, Harbour-master.

Lieutenant A. C. Robinson, R.E.
Mr. Gerald Devaux, Cul de Sac.

BEQUIA.—The Rev. Mr. Duffus.

MONTserrat.—Mr. F. H. Watkins, Commissioner.

ST. KITTS.—Mr. Hermann E. Hobbs. Dr. W. J. Branch.

DEMERARA.—Professor J. B. Harrison.

JAMAICA.

Mr. T. Laurence Roxburgh.
Mr. William Fawcett.
Mr. H. H. Cousins.
Mr. John D'Aeth.

Mr. Maxwell Hall.
Mr. S. T. Scharschmidt.
Mr. J. F. Brennan.

TRINIDAD.

Mr. S. W. Knaggs.
Mr. J. H. Hart, F.L.S.
Mr. W. M. Gordon.
Mr. E. R. Smart.
Mr. J. E. Lickfold.
Mr. H. C. Warner.

Mr. H. C. Huggins.
Mr. W. C. Nock.
Mr. J. F. A. Redhead.
Mr. T. J. Potter, F.L.S.
Mr. J. Haynes.

DOMINICA.

Dr. H. Alford Nicholls, C.M.G.
Mr. G. F. Branch.
Mr. Porter.
Mr. Bellot, Soufrière.

Mr. Duncan Naish, Picard.
Mr. J. Sowtray.
Mr. Wm. Jackson.
Mr. C. S. Kitching.

ANTIGUA.

Mr. J. S. Udal, Chief Justice.

Mr. Francis Watts, F.C.S.

APPENDIX II.

Notes taken by Mr. T. M. McDonald on the Recent Eruption of the Soufrière.

(Reprinted from the 2nd edition of 'The Sentry' newspaper, Kingstown, St. Vincent, May 16th, 1902.)

LANDED at beach of Richmond Vale Estate about 6 P.M. on the 6th instant, and up to this time was sceptical as to any eruption having taken place, as during our approach by sea from Wallilabou nothing unusual in appearance had been noticed, and the summit of the Soufrière was enveloped in the usual white clouds. Within a minute or two of landing, however, someone exclaimed, "Soufrière bursting now," and on looking saw an enormous vertical column of white vapour being ejected—practically noiselessly—and was quite convinced that an eruption had been and was now taking place. People were coming in from the direction of the mountain in an agitated condition.

Went up to Richmond Vale House, from which place the summit of the Soufrière is plainly seen, and invited Mr. MATHES, a German gentleman on a visit to Chateaubelair, to come and stay the night and observe.

The following notes, taken by Mr. MATHES, are inserted here:—

"*Tuesday, 6th, 2.40 P.M.*—First appearance of white steam in consequence of a noise like a gunshot. 4 P.M. the people arrived at Chateaubelair who had fled from Richmond, and at 4.30 P.M. people from Morne Ronde came excitedly into Chateaubelair. At 4.35 P.M. the reflection of fire on steam clouds was seen quite distinctly. 5.15 P.M., very thick smoke rising from foot of the Soufrière on the right side New Crater. 5.20 P.M., reflection of fire in the Old Crater, and now for the first time to be seen—also issue of the smoke from the New Crater at top of the mountain. 5.40 P.M., both smoke and steam clouds disappeared, and summit of mountain clear and clean. At 6.05 P.M. there was a new eruption with very thick smoke."

Mr. McDONALD now continues his notes from Richmond Vale House:—

At about 6.30 P.M. (6th) a greater discharge of vapour took place, with flame along the whole rim of the crater, forming a red, sparkling line between base of column of vapour and rim of crater, accompanied by a loud noise. At intervals of about two hours during the night similar discharges to the preceding took place, and at midnight flames were seen from Chateaubelair round the rim of the crater.

No further observation was noted at Richmond Vale House till shortly after 6 A.M. on the 7th, when a discharge took place with the usual column of thick vapour, but beneath this was a much shorter column of almost dense black, and of a heavier nature, as it quickly subsided back into the crater. This was the first appearance noted of what was probably solid matter being erupted, the white vapour being no doubt vapour of water only. At about 7.4 A.M. an enormous high column of white vapour was ejected, and it may be here mentioned that these tall columns rose in a very short space of time—say, about a minute—to heights of about 30,000 feet and over, by comparison seven or eight times the height of the mountain (nearly 4,000 feet). Outbursts took place now at shorter intervals, and at about 10.30 A.M. the eruption became continuous, enormous volumes of vapour reaching to a very great height.

11.10 A.M.—At this time there was thunder and lightning, showers of black and heavy material could now be seen thrown outwards and falling downwards from the column of whitish vapour, associated with loud noises and more violent outbursts. From the commencement the Old Crater seemed to be the scene of activity, but at times it seemed as though some of the discharges proceeded from what is known as the New Crater, a little north-eastwards from Chateaubelair. The area of the escape of vapour seemed now to be extending in a direction corresponding with Morne Ronde (westward).

11.15 A.M.—Thunder and lightning still continuing, and associated each time with a more violent outburst from crater.

11.35 A.M.—Discharge still violent, and Old Crater apparently the great centre of activity, enormous volumes ascending in curling and whirling waves, those beneath forcing those above higher and still higher; the colour of the vapour now assuming a darker shade—white changing to light grey, and low, rumbling noises audible.

11.40 A.M.—The edge of the old crater was quite distinct, but was belching out over the whole area. Flash and peal were continuous. The contour of the whole mountain was unaltered, and vegetation still remained fresh and green, with one enormous pillar of vapour overhead.

12.25 P.M.—Small vents seemed to be forming on slopes near old road facing Richmond Vale, and jets of vapour emitted from them; then a more violent outburst, which seemed to be extending the crater westward, with dense black upheavals and rumblings.

12.35 P.M.—It seemed as if slope to left of old road up Soufrière had formed into fissure, as vapour was issuing from small vents, and at 12.40 P.M. these fissures were unmistakable, and discharges from crater were extended to windward.

12.50 P.M.—Enormous outburst through vent or front of mountain, as far as could be ascertained, the mountain being largely enveloped in vapour, &c.

1 P.M.—There was tremendous roaring, stones being thrown out to windward thousands of feet high.

1.15 P.M.—Activity seemed shifting to windward and Wallibu River Valley direction, the eruption continuing unabated in violence.

At 1.25 P.M. there was still further extension of activity in the direction of Wallibu River and Morne Garou to right of old road.

1.30 P.M.—Violent action to right, with heavy falls of streams of fine matter and black stones.

1.32 P.M.—Violent to left also, with showers of blackish material. A minute afterwards volumes of vapour covered the whole area.

At 1.50 P.M. there was a black outburst to right, and showers of large and small stones shot eastward and downwards with tails of fine, black matter following. These stones issuing from interior of enormous column of vapour, thousands of feet above the mountain. Some large stones were also seen falling from thousands of feet up on face of column to westward, and some were also seen falling from windward side.

1.55 P.M.—Rumbling. Large black outburst with showers of stones all to windward, and enormously increased activity over the whole area. A terrific huge purplish and reddish curtain advancing up to and over Richmond Estate. At this stage left Richmond Vale House and hurried into and pushed off boat a few minutes after 2 P.M. Saw vapour, as we rowed hard across Chateaubelair Bay, coming down to sea level past Richmond Point. Sea peppered all round with stones, one of which, about a cubic inch, fell inside the boat, in which were 11 persons. The huge curtain referred to was advancing after the racing boat, which never seemed likely to get out of range of it or the falling stones, which latter varied from the size of one's fist downward. All in the boat felt that their end was near, and someone cried out, "We are all done for—head for shore"! This was done, and the boat beached between Petit Bordel and Rosebank. Got on to public road, where streams of people were hurrying along, all anxious to get to some place of safety. The lightning and thunder at this time was terrific, and there were noises inland. Everything seemed to point to a general break-up both on land and on sea. Fortunately, the writer found a stray horse at Rosebank, which he mounted without a saddle and rode

slowly along after the rest of the party. On reaching Troumaca Hill the bulk of the party refused to face the descent into the ravine, fearing darkness seen advancing from eastward.

Small stones were coming down all the time in a continuous shower, and Troumaca stream was thick from ashes. At Cumberland a saddle was obtained, and the journey to Wallilabou continued in bodily comfort. Reached the last-named place about 6 P.M., and found everything covered with dust from the eruption nearly one-eighth of an inch thick, and small stones had also reached there. Horses were being despatched to Cumberland to assist Captain CALDER and Dr. HUGHES, who with all others had made a general escape from Chateaubelair about 2 P.M.

8th.—The next morning (8th) the writer returned to Chateaubelair with the two above mentioned, also Mr. MATHES, Mr. ALLEN, and Mr. GENTLE. Slaty-coloured vapours were still being discharged from the old crater, and on the wind blowing from north showers of dust descended, and darkness set in, producing general alarm. At 2.20 P.M. the discharge of slaty-coloured vapours was continuous, but the new new crater, or some point to the right, appeared more active, the volumes being denser and blacker.

9th, 6.50 A.M.—Continuous rumbling noise for about half an hour, and then an increased discharge from crater, and steam and darker vapours appeared in large quantities, evidently from surface of sea, and, as seen from Chateaubelair Police Station, over Richmond Point, it was concluded to be a discharge of lava. In afternoon went in police boat, with Captain CALDER and Dr. HUGHES, along coast towards Wallibu to observe, but could not proceed further than opposite Richmond Estate, nor could a further view be obtained than the spur at which the flat land of "Fraser" terminates. The impression received was that there were about three lava streams issuing from the same number of ravines in sides of the mountain—one at the back of above-mentioned spur, the next at north side of spur on which ran the Soufrière road, and the third flowed through Wallibu Estate. The general level of all the flat land as far as "Fraser's" was much raised—by 40 or 50 feet, more or less—and terminated in abrupt, almost vertical, bluffs at the sea, the fronts of which frequently broke away and fell into the sea. The whole of the Richmond Village was buried deep with lava or ashes (30 feet, more or less), the highest being nearest to north of Wallibu River. Occasionally discharges of vapour would take place from the furthest first-mentioned ravine, and each was accompanied by a flash and peal of lightning and thunder. Slaty-coloured vapours were discharged from the crater continuously the whole day.

10th.—For some time after daylight crater was almost free from discharges.

9.23 A.M.—At this time there was a lofty grey outburst, and these continued with lessened force throughout the day.

11th.—Discharges continuous and of the same slaty colour.

At about 11 A.M. left in police boat with Captain CALDER and others to make further observations. Opposite Wallibu the sea had encroached along a length of shore beginning at Wallibu River to a point beyond Wallibu Works. The hill tops and crests of ridges had a comparatively thin covering of ashes, but the "flats" near the sea and the main river had considerable depths of volcanic matter. Owing to the enveloping vapours, a complete view of the mountain could not be obtained, and it was impossible to know exactly what changes had taken place.

On the **12th** and **13th** the volcano, although very much quieter, still gave signs of agitation at irregular intervals by sluggish discharges of slaty vapours, accompanied by low rumbling noise.

14th.—Dense cloud still over crater, but less lofty. Few small pebbles fell at Richmond Vale. Towards evening the summit was very clear, but distinct discharges of white steam.

15th.—At 9.30 A.M. there was a slight escape of steam, otherwise the mountain remained clear all day.

APPENDIX III.

ON account of the discrepancy (*see* p. 396) between the narratives given by Dr. HOVEY and in this Report, in respect of what happened in the cellar at Orange Hill on the afternoon of May 7th, we thought it advisable to write to Mr. F. W. GRIFFITHS, Kingstown, to ascertain, if possible, what were the actual facts. We are much indebted to him for making further inquiries, and sending us the following statement:—

Letter from Mr. F. W. Griffiths, Government Office, St. Vincent, December 29th, 1902.

“The constable from Georgetown, who was of the first to visit Orange Hill cellar after the terrible catastrophe, states that he counted 37 dead bodies, and that this was verified afterwards. He further states that 18 people were in the cellar alive when he arrived, but cannot say how many were saved, as they had nearly all come out, and their ideas were so confused, in consequence of the terrible experiences they had passed through, that there were few, if any, able to give coherent accounts of what occurred. He is certain of one thing, however, and that is that the people were killed in consequence of the door being open. Only those at the door were killed.

“A woman to whom I have just spoken was in the cellar. She states that there was a large number of people in there when the outburst occurred—over 150. About 30 were killed; over 100 were saved. Those killed were all standing either just inside or outside the door, which was open. Everyone near the doorway were killed She added that if the door had not been open they would have been suffocated, as the windows were all closed, or, at all events, the place was shut in all round.”

DESCRIPTION OF PLATES.

PLATE 21, FIG. 1.

Georgetown and the Carib Country, St. Vincent, as seen from the Windward Road, north of Black Point, one mile south of Georgetown.

On the roadside the old tuffs and agglomerates are exposed. Georgetown stands on a sloping plain, which extends for several miles on the east side of Morne Garn and the Soufrière. In the background a spur of the Soufrière is seen, running down towards Overland Village, and behind Georgetown the land rises in a series of steep, rounded bluffs. It will be seen that on the cliffs and the beach south of Georgetown the trees were not destroyed by the eruption.

PLATE 21, FIG. 2.

Chateaubelair from the North.

The view is taken from the ridge which separates the village from Richmond Valley. Sharp spurs are seen running down to the coast, where they form rocky headlands. Traces of old terraces are furnished by the flattening of the profiles just above the cliffs. This country was covered with ashes, but the vegetation rapidly recovered, and in the month of June, 1902, flourished as vigorously as before. The fields behind Chateaubelair are mostly planted with arrowroot.

PLATE 22, FIG. 1.

Section Exposed on the Sea Cliffs on the Leeward Coast, near Cumberland.

To the left are well-bedded tuffs in horizontal layers. These have been cut into by a stream, and a valley has been formed which, at a later stage, has been filled up with water-worn volcanic conglomerate, the bedding of which is discordant with that of the ash which forms the walls of the valley.

PLATE 22, FIG. 2.

Section Exposed in the Roadside near Colonarie, 2½ miles south of Georgetown.

The rocks are weathered to a dark brown earth, which contains stones of various sizes, and in places exhibits a marked bedding. In this case an upper series of inclined strata rests with an apparent unconformity on a lower series which is nearly horizontal.

PLATE 23, FIG. 1.

Explosions of Steam on the Wallibu River after Rains, as seen from Sea View Cottage, near Chateaubelair, at a distance of Two Miles.

The whole valley is filled with steam: one explosion has just taken place, and the cloud is ascending in the air, expanding as it rises; another is floating to leeward. Their height was usually about 2000 feet.

PLATE 23, FIG. 2.

The Mouth of the Wallibu River, St. Vincent.

A gush of boiling mud, several inches deep, is rushing down the stream, and its surface is steaming vigorously. The valley is eroded out of soft well-bedded tuffs. On the left side, several feet of new ashes lie on the old soil. The vegetation is ruined by the eruption. A fan is forming at the mouth of the river, owing to the amount of mud which is being deposited.

PLATE 24, FIG. 1.

Section of the New Ash Deposits on the Sea Shore, a little south of the Mouth of the Wallibu River, St. Vincent.

The ash has gathered here on a low flat beach, which is covered to a depth of 20 to 40 feet. It is black on the surface when wet, but is quite hot in the interior, and is grey in a few spots where it has dried after the rains. Deep rills have been cut in it by the showers. In the background explosions of steam are rising from the river. On the shore lies a charred tree trunk. Richmond Peak is seen on the right-hand.

PLATE 24, FIG. 2.

Scene in Chateaubelair, St. Vincent, showing the Contrast between the Country which has been Spared and that which is Devastated.

PLATE 25, FIG. 1.

Landing below Wallibu Plantation, St. Vincent.

Before the eruptions there was a broad, flat beach at this place, and on it stood a village of labourers' houses beside the public road. That beach has disappeared, and the bluffs which formerly rose behind it now present a vertical face to the sea. Large masses frequently fall from these cliffs, and a narrow new beach has formed at their base. Soft stratified tuffs are exposed in the cliffs, and over them lies a deposit of new ashes from 5 to 10 feet deep. The remains of the chimney of Wallibu Plantation are seen above the boats on the right. The valley of the Wallibu Dry River lies to the left, and in it the new ash forms round-backed mounds. In the background is the Soufrière, with the Somma wall on the left of the crater, the rim of which is seen in the centre of the picture.

PLATE 25, FIG. 2.

Wallibu Plantation Works, Half Buried in Ashes.

The roof has collapsed, but the woodwork is unburnt. The level ground is covered with several feet of ashes, channelled by the rains. The steeper slopes behind have been washed nearly bare. The trees are blasted, but not overturned. In the background the Soufrière rises to the north of the valley of the Wallibu Dry River.

PLATE 26.

The Plantation Grounds of Wallibu, St. Vincent.

The fields are covered with a layer of ash, which is several feet deep on the flat grounds, but less on the slopes behind. The surface is channelled with rain rills, the arrangement of which varies with the slope. The trees are blasted and stripped of their branches on the side looking up the valley, so that the effects of the blast and the direction in which it was travelling are clearly visible.

PLATE 27, FIG. 1.

The Burnt-out Houses of Wallibu, St. Vincent.

These stood on the ridge behind the plantation works (*see* Plate 25, Fig. 2). They are surrounded by ashes, and the trees are overturned, but their branches have not been consumed or torn off the stems. Richmond Peak rises in the background.

PLATE 27, FIG. 2.

The Fields of Wallibu Plantation.

This view is taken further up the valley than Fig. 1 above, and shows an increase in the amount of devastation. In the background is a spur of Richmond Peak, on which the forest is blasted but not entirely overthrown. On the naked side of the valley a lava flow is exposed, capped by thick masses of tuff. At the foot of this slope the Wallibu is flowing through terraced accumulations of new ash.

PLATE 28, FIG. 1.

The Fields of Wallibu Plantation.

This plate shows the remarkable variety of the sculpturing of the new ash by the rain torrents. The trees are blasted, and in some cases overthrown. The direction in which they have fallen indicates the course of the blast.

PLATE 28, FIG. 2.

The Upper Part of the Wallibu Valley, St. Vincent.

The valley is filled with a thick deposit of new ashes, still very hot in the interior. It is somewhat eroded and terraced, but the rolling character of the original surface is still recognisable. In the background is a spur of Richmond Peak, with a thick lava flow dipping down stream. The trees on this slope are blasted, but still erect. In the right foreground stands a small mud crater in the field of ashes.

PLATE 29, FIG. 1.

The Upper Part of the Valley of the Wallibu River, St. Vincent.

The banks of the old valley are seen on the right of the Plate and in the background. New hot ashes now fill the gorge, from which they are being eroded by the stream, and terraces have formed on each side of the river and its main tributaries. After showers, the surface of the ash is cold and dark coloured, and rain may collect in little pools, but the deeper parts of the mass are still intensely hot, and, as the material dries, landslips are constantly taking place in the banks which overlook the terraces.

PLATE 29, FIG. 2.

The River Wallibu, St. Vincent, Eroding the Thick Deposits of Hot Ash.

In the centre the stream, which is nearly dry, is flowing through a deep gorge which it has cut in the ashes. These are dark on the surface when wet, but grey when dry, and quite incoherent, so that landslides are frequent, and the hot, dry sand may be seen tumbling down the cliffs which overlook the channel. These cliffs are 40 to 60 feet high. On the right the rolling character of the original

surface can still be traced, and on both sides the old valley walls are seen covered with burnt forest (in the middle distance). In the background a spur of Richmond Peak rises in bare precipices, which consist of lavas and tuffs in alternating series, dipping outwards from the centre of the mountain.

PLATE 30, FIG. 1.

Small Secondary Crater-pit in the Hot Sand Deposits a little north-east of Wallibu Plantation, St. Vincent.

On the left a channel has been cut in the new ashes by a small rivulet, which drains the slopes above. In the foreground a bowl-shaped depression in which a little water has gathered, and, behind this, two others less perfect and dry. These are produced by the explosions of steam which take place when the water of the stream comes in contact with the hot material in the deeper parts of the layer of recent volcanic ash.

PLATE 30, FIG. 2.

One of the Minor Ravines on the South-west Slopes of the Soufrière, St. Vincent.

A deep, narrow gorge has been cut out of the soft tuffs which form this part of the mountain. It has been filled nearly to the top with new volcanic ash, which has been in large measure washed away by the stream, but a considerable thickness still remains. The upper surface shows well-marked terraces. The ridges above have received only a thin covering, most of which has since been removed by the rains.

PLATE 31, FIG. 1.

The ash-covered cane-fields of the Carib Country, with, in the distance, the Rabaka Dry River pouring down in flood after a heavy shower, and sending up great clouds of steam as the water comes in contact with the hot sand which fills the old channel. At this level the bushes have suffered only slightly. The layer of new ash on the level fields is eroded by the rains.

PLATE 31, FIG. 2.

The Upper Part of the Carib Country, above Lot 14, and the Spurs and Ravines at the Base of the Soufrière on the Windward Side.

The undulating surface of the ground is covered with 4 or 5 feet of sand, in which the rain has worked a feather pattern of rills. The trees are erect, but reduced to mere trunks, without leaves or branches. On the mountain behind, the ash on the knife-edges shines in the sun, and the forest has been completely overturned or destroyed.

PLATE 32.

The Upper Part of the Valley of the Rabaka Dry River obstructed by the Avalanche of Sand.

Before the eruption this valley was over 200 feet deep. It is now nearly completely filled up. The new ash has a hummocky, irregular surface, and the stream flowing through it has cut a shallow channel, which in some places is flanked by inconspicuous terraces. Between the showers the flow of water ceases, though the ash, where it is wet, is freely steaming. In the background rise the lower spurs of the Soufrière and Morne Garu.

PLATE 33, FIG. 1.

The Valley of the Rabaka Dry River, St. Vincent.

The view is taken from the north bank, looking south across the surface of the sand avalanche towards Georgetown. In the foreground the dead trees, eroded by the sand blast and stripped of their leaves and branches, are seen on the back of a ridge which separated the main valley from one of its tributaries. Beyond this ridge lie two great semicircular crater-bowls, out of which explosions of steam have been emitted. Around each of them there is a low cone, and the fields of ash are covered with stones thrown out of the craters.

PLATE 33, FIG. 2.

Nearer View of one of the Crater Bowls shown in the previous Figure.

The dark and cold material forming the cone of ejection is sharply defined from the lighter-coloured and hot ash of the avalanche beneath. At both sides of the picture transverse sections of the low cone surrounding the crater-bowl can be seen. The ash fields around are strewn with stones. In the background the irregular surface of the deposit is marked in places by flat-topped terraces. The river is cutting a new gorge in the foreground, and on the side of this the explosion funnels are situated.

PLATE 34, FIG. 1.

Lake of Water occupying one of the Lateral Valleys opening out into the Main Channel of the Rabaka Dry River, St. Vincent.

The rolling surface of the avalanche of sand is seen in the background. On each side of the picture the cane-fields of Lot 14 are covered with several feet of ashes, but, as is apparent from the state of the bushes which project above the surface, the depth of the layer is inconsiderable, when compared with that of the great mass which blocks the valley.

PLATE 34, FIG. 2.

Lake of Mud in One of the Lateral Valleys which Open Out into the Valley of the Rabaka Dry River, St. Vincent.

A deep, narrow gorge has been cut into the soft bedded tuffs which form the southern part of the Soufrière. This gorge is obstructed at its mouth by the sand avalanche in the main valley, and a lake of mud has collected. The rains have stripped the new ash from the slopes, except near their base, but in places the irregular surface of the masses which occupied the valley bottom can still be seen projecting above the surface of the mud. The forest which clothed the mountain is broken down and destroyed. Most of it has vanished, only one stump remains erect. In the background the knife edges of the spurs are shining as the light is reflected from the layer of fine ash which still rests upon them.

PLATE 35.

The Windward Slopes of the Soufrière, St. Vincent.

The mountain is scored with deep ravines, between which lie sharp-backed spurs. On the knife edges the fine sand still remains, forming narrow strips which reflect the sunlight. The flanks of the ridges are almost cleared of the new ash, which can be seen only in scattered patches. The surface of the older

tuffs and agglomerates is now exposed. In the bottom of the valley the course of the stream is marked by a layer of shining mud. Hardly a tree remains standing, and the prostrate trunks lie parallel, with their roots towards the crater and their apices pointing down the slopes.

PLATE 36, FIG. 1.

View taken in the Streets of St. Pierre, Martinique, looking South towards the Cathedral.

PLATE 36, FIG. 2.

View taken in the Streets of St. Pierre.

The trees are broken and eroded by the sand blast. The streets are filled with scoria and sand. A stone, which has fallen from the roof of a building on the right, has been caught on the spikes of the iron railing, one of which has entered a hole made for the reception of an iron bar.

PLATE 37, FIG. 1.

Grenada.—The Lagoon and Carenage looking South from Government House.

The lagoon is to the left, the carenage or harbour to the right. The narrow channel between the two is the seat of the disturbance of June 6th, 1902. The Green Hole is situated in the carenage more in the foreground. The vegetation shows the luxuriance of growth usual in the tropics.

PLATE 37, FIG. 2.

The Crater Lake of the Soufrière as it was before the Eruptions of May, 1902. From a Panoramic Photograph by Mr. J. C. Wilson, Kingstown, St. Vincent.

The mists hovering over the mountain-top are reflected from the surface of the water, except in the centre of the lake, where the reflection of clear sky appears. The photograph was taken from the southern lip of the depression.

PLATE 38.

Montagne Pelée, Martinique, as it appeared on the Evening of July 9th, 1902, emitting Great Clouds of Steam and Dust, which Floated away to Leeward.

On the right of the picture the trade-wind cloud is seen covering the summit, and paler in colour than that discharged from the crater. The first black cloud rolled down the slopes a little to the right of the centre of the picture.

PLATE 39.

Map of the North End of St. Vincent. Reproduced from the British Admiralty Chart, by permission.



Fig. 1.—Carib Country, with Georgetown.



Fig. 2.—Chateaubelair.

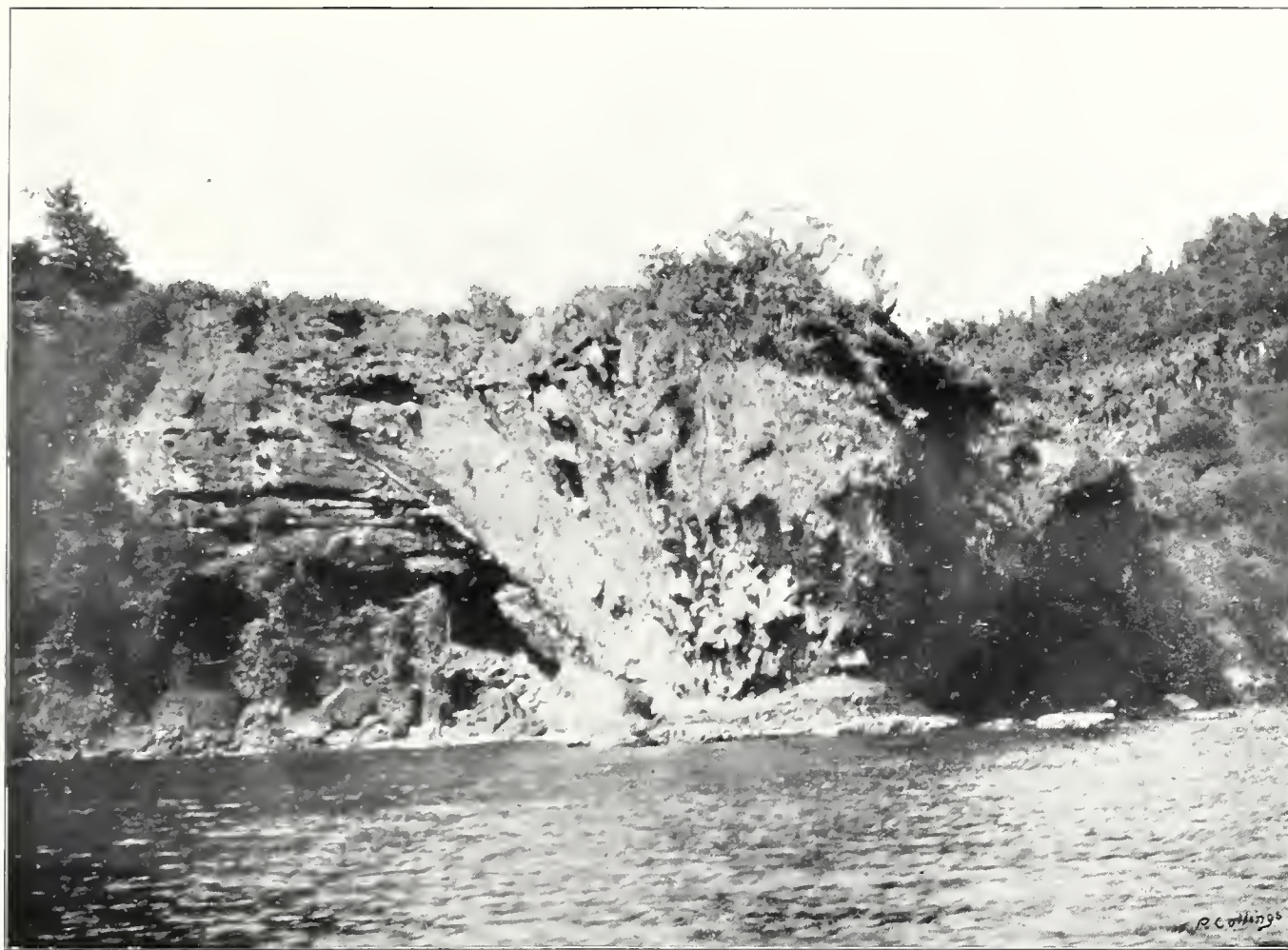


Fig. 1.—Cliff section near Cumberland.



Fig. 2.—Roadside section, Colonarie.



Fig. 1.—Steam explosions on Wallibu River.



Fig. 2.—Mouth of Wallibu River.



Fig. 1.—Shore section near Wallibu River.



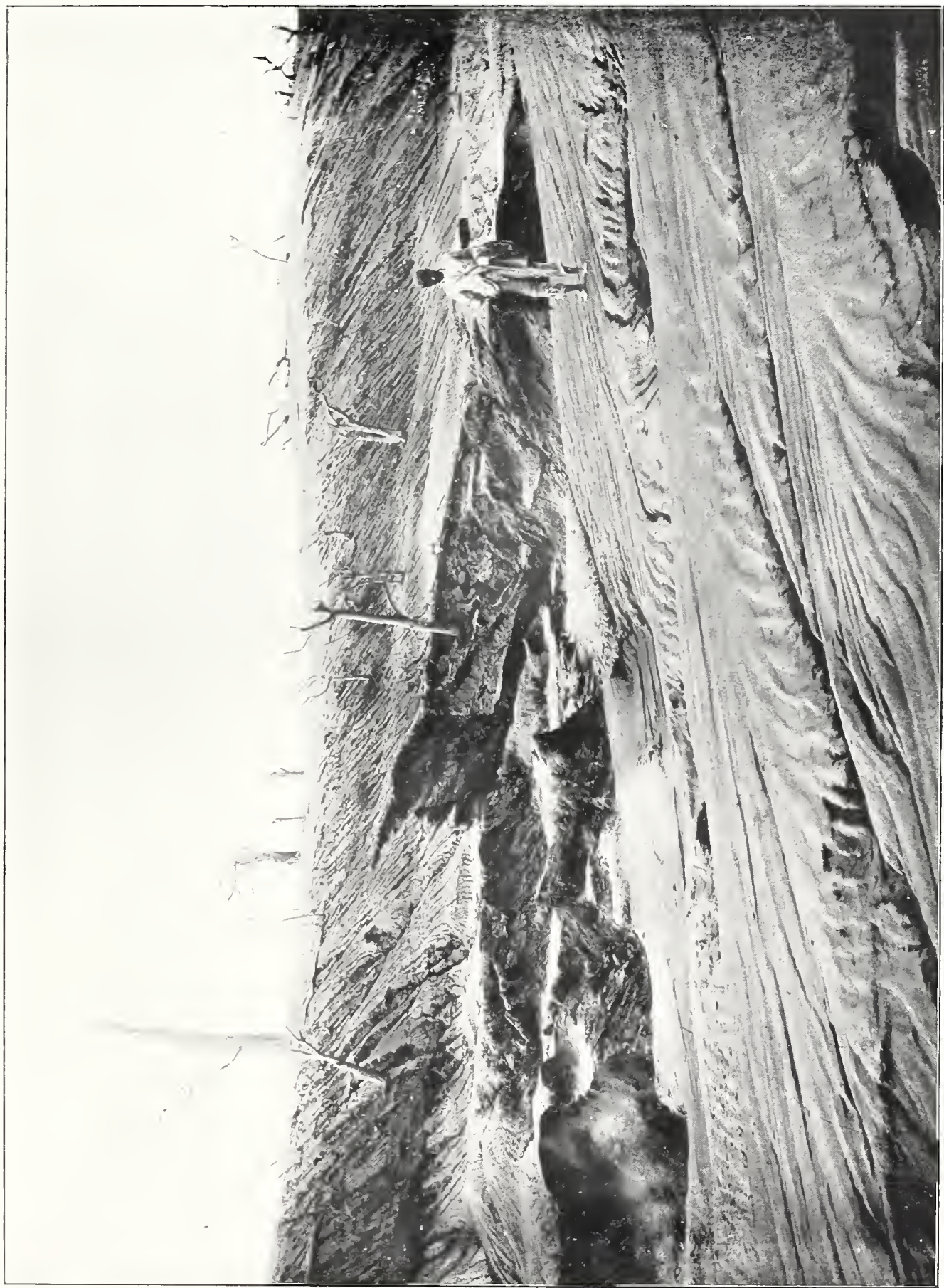
Fig. 2.—Coast scene, Chateaubelair.



Fig. 11.—Landing, Wallibu Plantation.



Fig. 2.—Wallibu Plantation ruins.



Wallibu Plantation grounds.



Fig. 1.—Wallibu, burnt-out houses.



Fig. 2.—Wallibu Plantation fields.

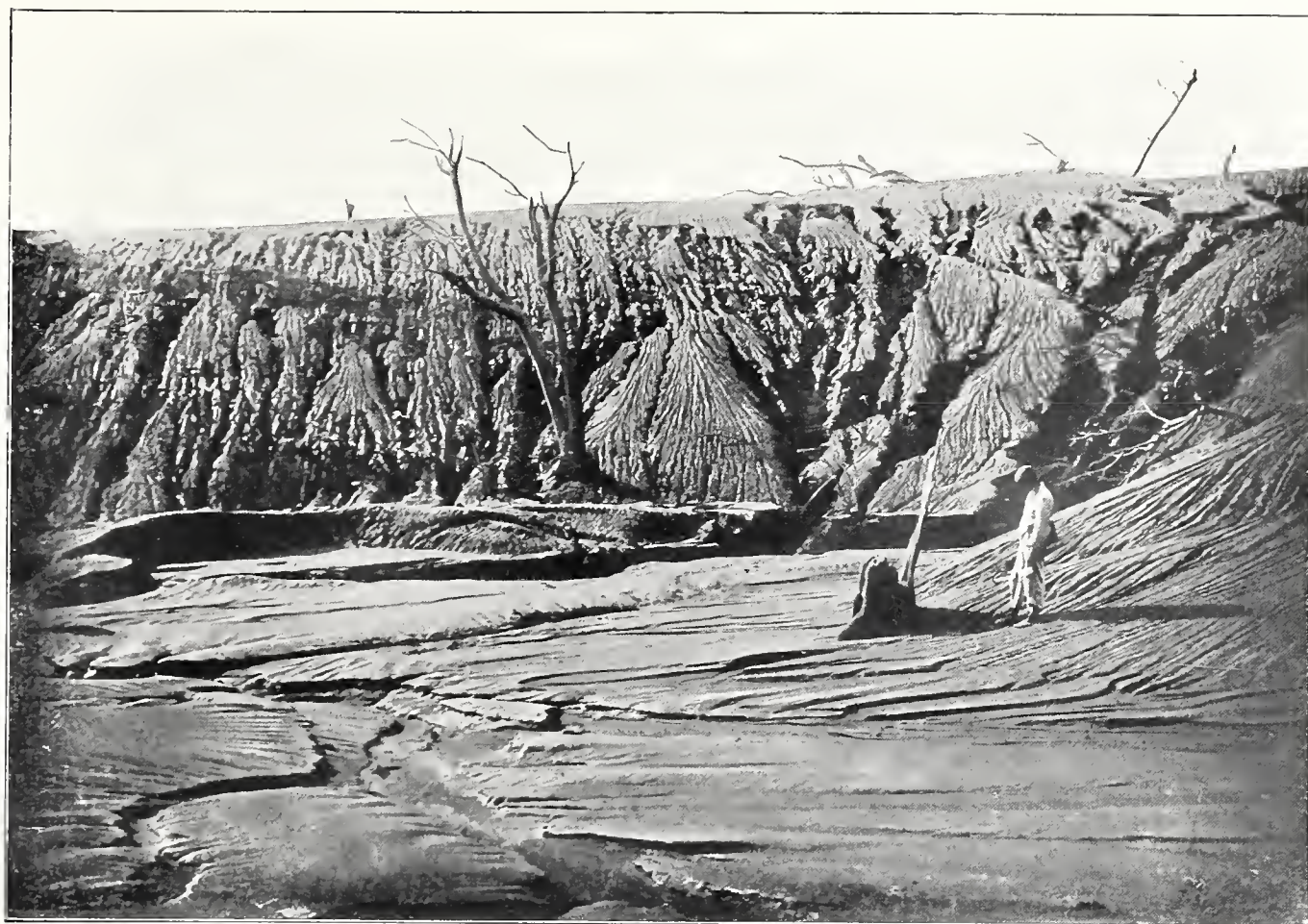


Fig. 1.—Wallibu Plantation fields.



Fig. 2.—Upper Wallibu Valley.



Fig. 1.—Upper Wallibu Valley.



Fig. 2.—Wallibu River, eroded bed.



Fig. 1.—Secondary Crater pit in the ash near Wallibu.



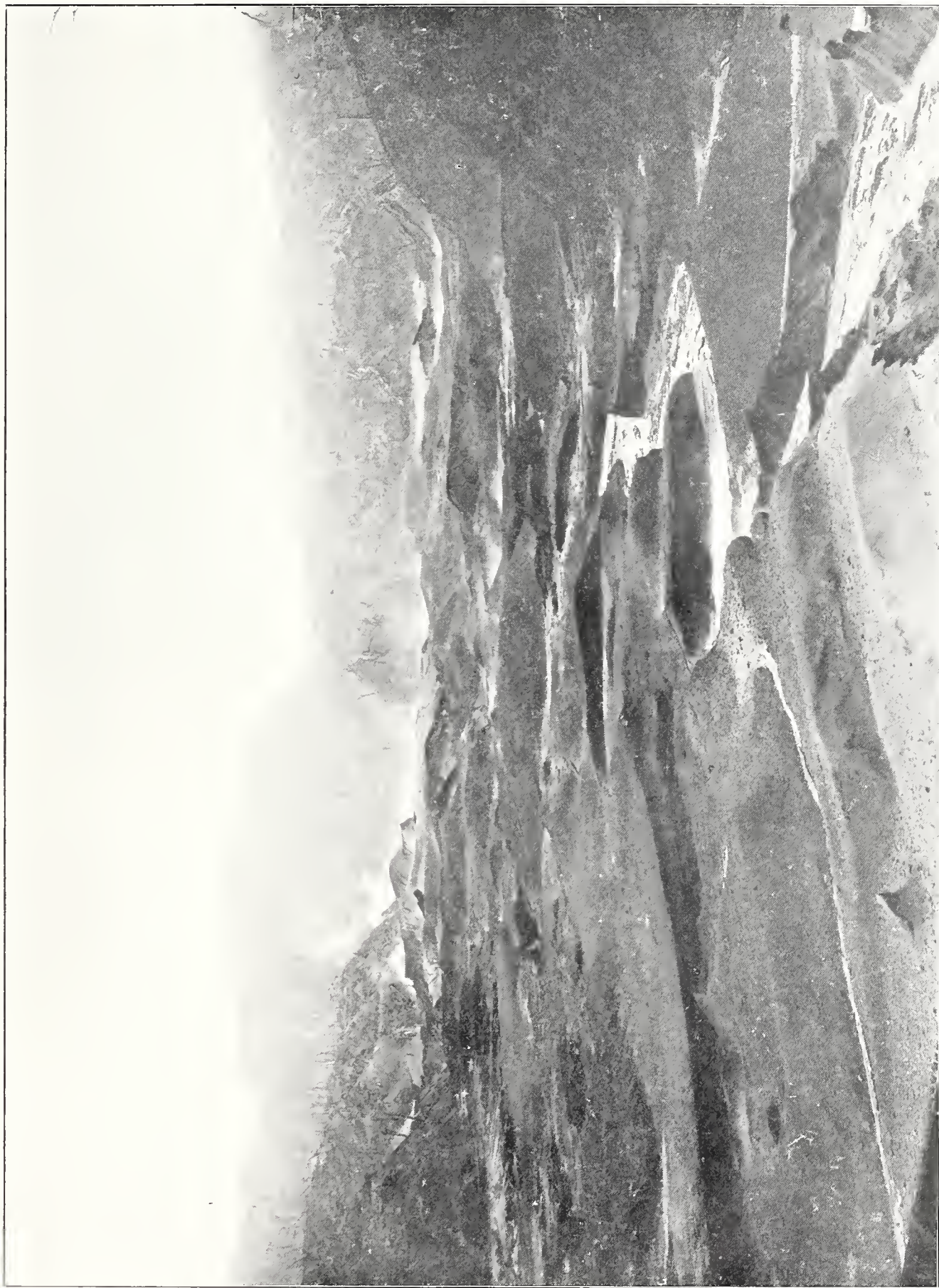
Fig. 2.—Ravine on slopes of Soufrière.



Fig. 1.—Carib Country, cane fields.



Fig. 2.—Higher Carib Country, and base of Soufrière.



Rabaka Dry River, upper valley.



Fig. 1.—Rabaka Dry River Valley, with crater bowls.



Fig. 2.—Nearer view of crater bowls.



Fig. 1.—Rabaka Dry River, lake in side valley.



Fig. 2.—Rabaka Dry River, mud lake in side valley.



Soufrière, windward slopes.



Fig. 1.—St. Pierre, Martinique; view in streets.



Fig. 2. — St. Pierre, view in streets.

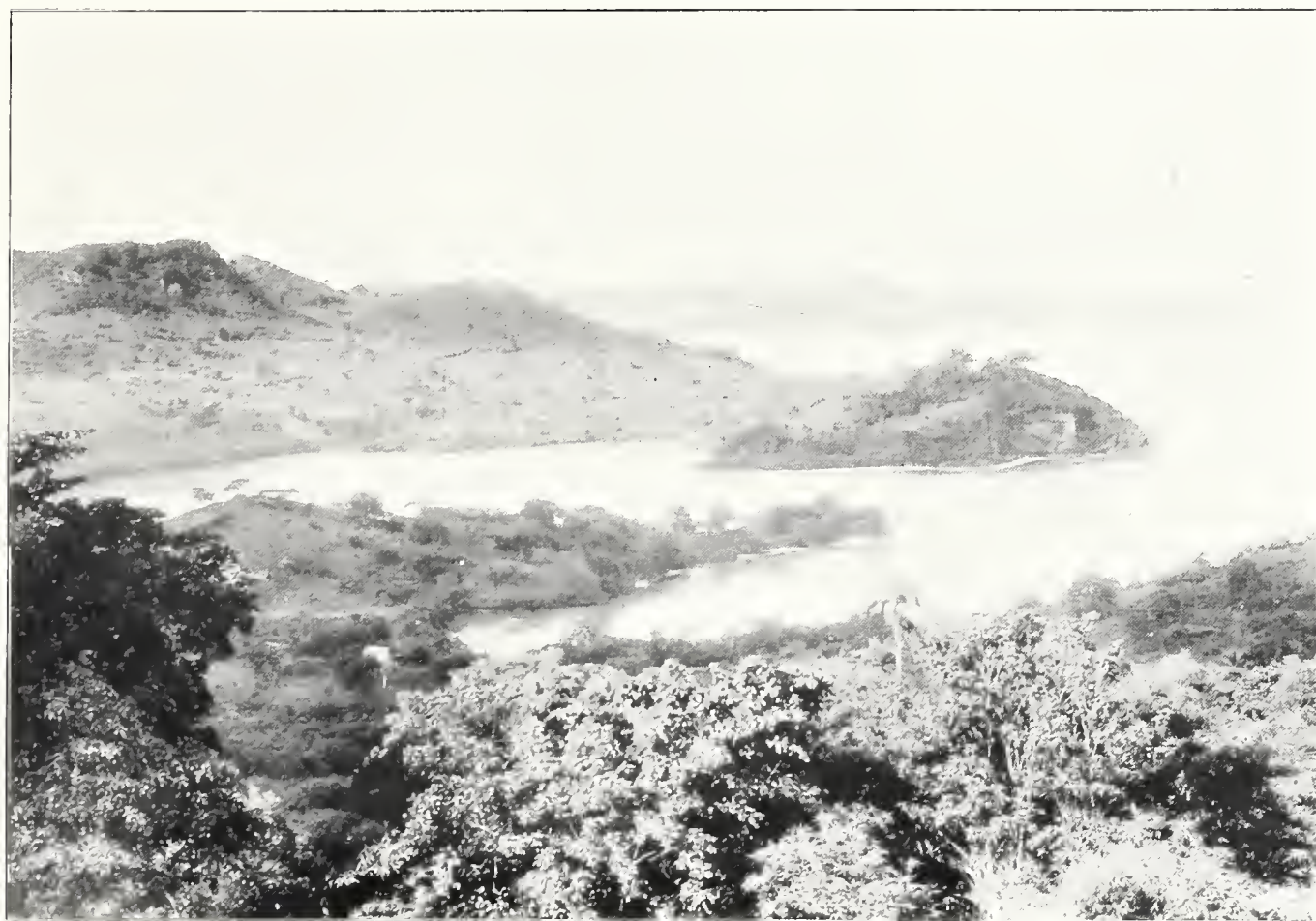


Fig. 1.—Grenada, lagoon and carenage.



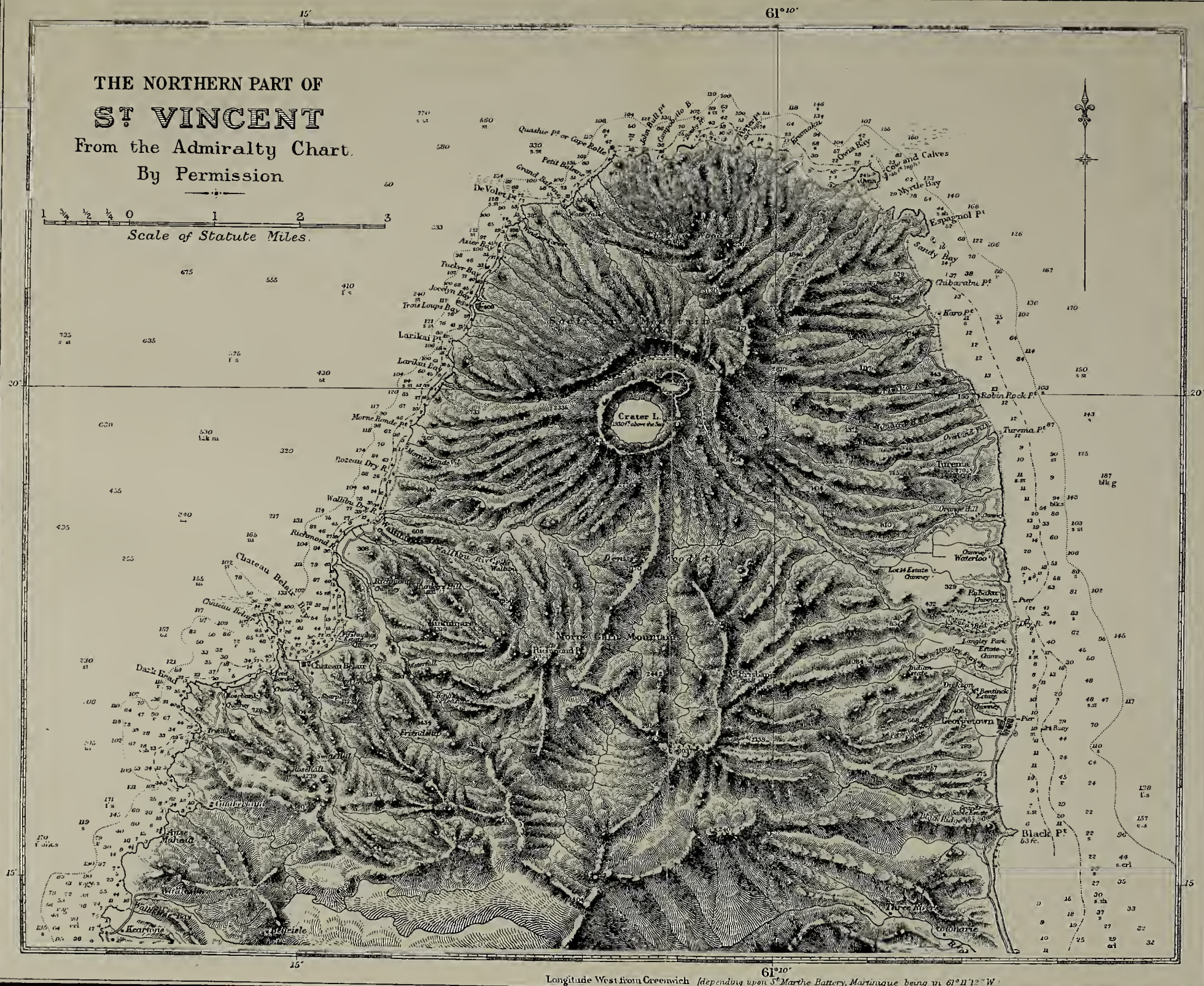
Fig. 2.—Soufrière crater lake before eruption.

(*Photo. by J. C. Wilson, Kingstown.*)



Montagne Pelée, July 9, 1902, evening.

Scale of Statute Miles.



Longitude West from Greenwich *(depending upon St. Marthe Battery, Martinique being in 61° 11' 12" W)*

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